Approximating Dependency Graphs without using Tree Automata Techniques

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1 Introduction

In the dependency pair method of Arts and Giesl [2], one of the most popular methods for automatically proving (innermost) termination of TRSs, a TRS is transformed into groups of ordering constraints such that (innermost) termination of the system is equivalent to the solvability of these groups. The number and size of these groups is determined, among others, by the approximation used to estimate the dependency graph. The nodes of the dependency graph $\mathsf{DG}(\mathcal{R})$ of a TRS $\mathcal{R}$ are the dependency pairs of $\mathcal{R}$ and there is an arrow from $s \to t$ to $u \to v$ if and only if there exist substitutions σ and τ such that $t\sigma \to^* u\tau$. In the innermost dependency graph $\mathsf{IDG}(\mathcal{R})$ the requirement $t\sigma \to^* u\tau$ is strengthened to (1) $t\sigma \xrightarrow{\mathsf{i}}{}^* u\tau$ and (2) $s\sigma$ and $u\tau$ are in normal form. Here $\xrightarrow{\mathsf{i}}$ denotes the innermost rewrite relation. Since the relations $\to^*$ and $\xrightarrow{\mathsf{i}}{}^*$ are not computable in general, $\mathsf{DG}(\mathcal{R})$ and $\mathsf{IDG}(\mathcal{R})$ have to be approximated in order to arrive at a mechanizable criterion for (innermost) termination.

In Section 2 we discuss approximations for the dependency graph and in Section 3 we do the same for the innermost dependency graph.

2 Dependency Graph

The following definition stems from [1].

Definition 1. *Let $\mathcal{R}$ be a TRS. The nodes of the* estimated *dependency graph* $\mathsf{EDG}(\mathcal{R})$ *are the dependency pairs of $\mathcal{R}$ and there is an arrow from $s \to t$ to $u \to v$ if and only if* $\mathsf{REN}(\mathsf{CAP}(t))$ *and u are unifiable. Here* CAP *replaces all outermost subterms with a defined root symbol by distinct fresh variables and* REN *replaces all occurrences of variables by distinct fresh variables.*

Middeldorp [3] showed that better approximations of the dependency graph are obtained by adopting tree automata techniques. These techniques are however computationally expensive. In a more recent paper Middeldorp [4] showed that the estimation of Definition 1 can be improved by symmetry considerations without incurring the overhead of tree automata techniques.

Definition 2. *Let $\mathcal{R}$ be a TRS over a signature $\mathcal{F}$ and let $\mathcal{S} \subseteq \mathcal{R}$. The result of replacing all outermost subterms of a term t with a root symbol in $\mathcal{D}_{\mathcal{S}}^{-1}$ by* *fresh variables is denoted by* $\mathsf{CAP}_{\mathcal{S}}^{-1}(t)$*. Here* $\mathcal{D}_{\mathcal{S}}^{-1} = \{\mathrm{root}(r) \mid l \to r \in \mathcal{S}\}$ *if* $\mathcal{S}$ *is non-collapsing and* $\mathcal{D}_{\mathcal{S}}^{-1} = \mathcal{F}$ *otherwise. The nodes of the* estimated* *dependency graph* $\mathsf{EDG}^*(\mathcal{R})$ *are the dependency pairs of* $\mathcal{R}$ *and there is an arrow from* $s \to t$ *to* $u \to v$ *if and only if both* $\mathsf{REN}(\mathsf{CAP}(t))$ *and* u *are unifiable, and* t *and* $\mathsf{REN}(\mathsf{CAP}_{\mathcal{R}}^{-1}(u))$ *are unifiable.*

For instance, with the new estimation proving the termination of Toyama's famous rule $\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x)$ becomes trivial as there are no cycles in the estimated* dependency graph.

A comparison between the estimation of Definition 2 and the tree automata based approximations described in [3] can be found in [4]. From the latter paper we recall the identity $\mathsf{EDG}(\mathcal{R}) = \mathsf{EDG}^*(\mathcal{R})$ for collapsing $\mathcal{R}$.

3 Innermost Dependency Graph

The new estimation in Definition 4 is inspired by the one of Definition 2.

Definition 3 ([1]). *Let* $\mathcal{R}$ *be a TRS. The nodes of the* estimated *innermost dependency graph* $\mathsf{EIDG}(\mathcal{R})$ *are the dependency pairs of* $\mathcal{R}$ *and there is an arrow from* $s \to t$ *to* $u \to v$ *if and only if* $\mathsf{CAP}_s(t)$ *and* u *are unifiable with mgu* σ *such that* $s\sigma$ *and* $t\sigma$ *are in normal form. Here* CAP_s *replaces all outermost subterms* different from s *with a defined root symbol by distinct fresh variables.*

Definition 4. *Let* $\mathcal{R}$ *be a TRS. The nodes of the* estimated* *innermost dependency graph* $\mathsf{EIDG}^*(\mathcal{R})$ *are the dependency pairs of* $\mathcal{R}$ *and there is an arrow from* $s \to t$ *to* $u \to v$ *if and only if both* $\mathsf{CAP}_s(t)$ *and* u *are unifiable with mgu* σ *such that* $s\sigma$ *and* $t\sigma$ *are in normal form, and* $\mathsf{REN}(\mathsf{CAP}_{\mathcal{U}(t)}^{-1}(u))$ *and* t *are unifiable with mgu* σ *such that* $s\sigma$ *and* $t\sigma$ *are in normal form. Here* $\mathcal{U}(t)$ *denotes the set of* usable rules *[1] for the term* t*.*

The following example shows that we cannot omit REN from $\mathsf{REN}(\mathsf{CAP}_{\mathcal{U}(t)}^{-1}(u))$ without violating the soundness condition $\mathsf{IDG}(\mathcal{R}) \subseteq \mathsf{EIDG}^*(\mathcal{R})$, which is essential for inferring innermost termination.

Example 5. Consider the TRS $\mathcal{R} = \{\mathsf{f}(x, x) \to \mathsf{f}(\mathsf{g}(x), x), \mathsf{g}(\mathsf{h}(x)) \to \mathsf{h}(x)\}$. There are two dependency pairs: (1): $\mathsf{F}(x, x) \to \mathsf{F}(\mathsf{g}(x), x)$ and (2): $\mathsf{F}(x, x) \to \mathsf{G}(x)$. Since $\mathsf{F}(\mathsf{g}(\mathsf{h}(x)), \mathsf{h}(x)) \xrightarrow{\mathsf{i}} \mathsf{F}(\mathsf{h}(x), \mathsf{h}(x))$, $\mathsf{IDG}(\mathcal{R})$ contains arrows from (1) to (1) and (2). However, $\mathsf{CAP}_{\mathcal{U}(\mathsf{f}(\mathsf{g}(x),x))}^{-1}(\mathsf{F}(\mathsf{g}(x), x)) = \mathsf{CAP}_{\mathcal{R}}^{-1}(\mathsf{F}(\mathsf{g}(x), x)) = \mathsf{F}(\mathsf{g}(x), x)$ does not unify with $\mathsf{F}(x', x')$.

Note that in the above example $\xrightarrow{\mathsf{i}}$ differs from $(\xleftarrow{\mathsf{i}})^{-1}$. It is also not difficult to show that replacing CAP^{-1} by CAP_v^{-1} (or CAP_s^{-1}) would make Definition 4 unsound.

4 Comparison

The following theorem summarizes the relationships between the various approximations.

Table 1. Dependency graph statistics.

TRS	DPs	EDG arrows	EDG* arrows	EDG SCCs	EDG* SCCs	EDG cycles	EDG* cycles	EIDG arrows	EIDG* arrows	EIDG SCCs	EIDG* SCCs	EIDG cycles	EIDG* cycles
[2]:3.23	2	4	2	1	1	3	1	4	2	1	1	3	1
[2]:3.44	4	4	0	2	0	2	0	4	0	2	0	2	0
[2]:3.45	4	5	3	3	2	3	2	5	3	3	2	3	2
[2]:3.48	6	17	12	2	2	8	4	17	12	2	2	8	4
[2]:4.20a	3	3	1	2	1	2	1	2	0	1	0	1	0
[2]:4.20b	4	7	5	2	1	4	3	5	3	2	1	2	1
[2]:4.21	6	12	8	2	2	6	4	6	2	2	0	2	0
[2]:4.37b	4	6	3	3	2	3	2	2	2	2	2	2	2
[5]:2.8	8	24	24	3	3	7	7	19	18	3	3	3	3
[5]:2.51	3	8	7	1	1	6	5	8	7	1	1	6	5
[5]:2.52	9	36	35	4	4	17	16	36	35	4	4	17	16
[5]:4.31	3	4	4	2	2	2	2	4	2	2	1	2	1
[5]:4.44	4	4	0	2	0	2	0	4	0	2	0	2	0
[5]:4.59	6	12	4	3	2	5	2	12	4	3	2	5	2

Theorem 6. *For any TRS $\mathcal{R}$, the following inclusions hold:*

$$\begin{array}{ccccc} \mathsf{DG}(\mathcal{R}) & \subseteq & \mathsf{EDG}^*(\mathcal{R}) & \subseteq & \mathsf{EDG}(\mathcal{R}) \\ \cup\!\mid & & \cup\!\mid & & \cup\!\mid \\ \mathsf{IDG}(\mathcal{R}) & \subseteq & \mathsf{EIDG}^*(\mathcal{R}) & \subseteq & \mathsf{EIDG}(\mathcal{R}) \end{array}$$

Unlike the inclusion $\mathsf{EDG}^*(\mathcal{R}) \subseteq \mathsf{EDG}(\mathcal{R})$, the inclusion $\mathsf{EIDG}^*(\mathcal{R}) \subseteq \mathsf{EIDG}(\mathcal{R})$ need not become an equality for collapsing $\mathcal{R}$, due to the use of usable rules in the second part of Definition 4. An (artificial) example illustrating this is provided by the TRS consisting of the rules $\mathsf{f}(\mathsf{a},\mathsf{b}) \to \mathsf{f}(\mathsf{f}(\mathsf{a},\mathsf{a}),\mathsf{f}(\mathsf{b},\mathsf{b}))$ and $\mathsf{g}(x) \to x$.

Table 1 lists all examples in [2] and Sections 3 and 4 of [5] where the new estimations make a difference.

References

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