The Confluence of Ground Term Rewrite Systems is Decidable in Polynomial Time

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Term Rewrite Systems (TRS)

Sets of rules for replacing (sub)expressions.

Example 1:

$$\begin{array}{rcl} 0 + x & \to & 0 \\ s(x) + y & \to & s(x + y) \\ 0 * x & \to & 0 \\ s(x) * y & \to & y + x * y \end{array}$$

defines $(N, +, *)$. We have:

$$\begin{array}{ll} \underline{s(s(0)) * s(s(0))} & \to \\ s(s(0)) + \underline{s(0) * s(s(0))} & \to \\ s(s(0)) + s(s(0)) + \underline{0 * s(s(0))} & \to^* \\ s(s(s(s(0)))) & \end{array}$$

Example 2:

$$\begin{array}{rcl} a & \to & q_a \\ g(q_a) & \to & q_g \\ g(q_g) & \to & q_g \\ f(q_a, q_a) & \to & q_f \\ f(q_g, q_f) & \to & q_{accept} \end{array}$$

is a tree automaton recognizing $f(g^+(a), f(a, a))$.

Rewriting Theory

Results from and for:

- formal language theory
- algebra
- programming languages:
  - semantics
  - implementation
  - static analysis
  - ...
- automated deduction:
  - decision procedures
  - combination procedures
  - constraint solving
  - symbolic computation
  - ...
- design/verification (e.g., hardware)
- ...

Two basic properties of TRS R:

Termination:
no infinite sequences $t_1 \to_R t_2 \to_R \ldots$

Confluence: (or Church-Rosser p.)
$t_1 \leftarrow_R^* s \to_R^* t_2$ implies $\exists u.\ t_1 \to_R^* u \leftarrow_R^* t_2$

Both are undecidable in general.

Ground Term Rewrite Systems (GTRS)

- GTRS include suffix (or, equivalently, prefix) string rewrite systems
- Word problem PTIME-complete (Kozen '77)
- Termination in PTIME (Plaisted '93)
- Confluence decidable (Dauchet et al '87, Oyamaguchi '87):
  1. build in PTIME tree automaton A_1 accepting $\langle t, u, v \rangle$ such that u and v are reachable from t
  2. A_2 accepting $\langle t, u, v \rangle$ such that u and v are joinable
  3. then R confluent iff $A_1 \subseteq A_2$ (DEXPTIME)

Exact complexity of deciding confluence of GTRS?

Open problem #12 on RTA-LOOP since its creation in '91.

Unknown also for string rewriting.

Between DEXPTIME and P

Important property:

R confluent iff confluent from every lhs l of R:

$$t_1 \leftarrow_R^* l \rightarrow_R^* t_2 \quad \text{implies} \quad \exists u.\ t_1 \rightarrow_R^* u \leftarrow_R^* t_2$$

Some of our failed attempts (since '98):

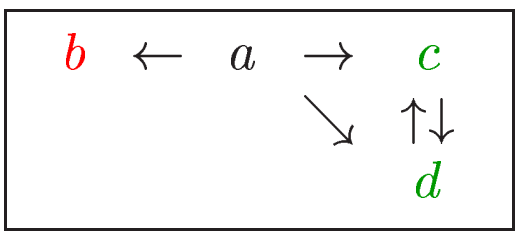
- prove hardness (EXPTIME, (co-)NP, ...)
- build special-purpose automata for the string case
- try to characterize confluence of SRS by properties based on stable prefixes:

  with $b \to ab$, $ab \to b$ and $b \to c$, we have:

  $$acb \to_R acab \to_R acaab \to_R acaac$$

  E.g.: R confluent from l iff cannot pump up two different stable prefixes from l?

This is false:



Last autumn: PTIME algorithm for SRS

- Extremely technical proof
- Based on analysis of stable prefixes. PTIME test for:

  can u become stable in uv by only rewriting on v?
- Apply Plaisted's idea: transform R into

$$\{s \to t \mid s \to_R^* t, \quad s,\ t \text{ subterms of R}\}$$

- then get increasing rewrite derivations: with $a \to b$ (1) and $ab \to ca$ (2)

  $\underline{ab} \to_2 c\underline{a} \to_1 b$ is increasing

  $a\underline{a} \to_1 \underline{ab} \to_2 ca$ is not.
- decide confluence from each l by enumerating enough words reachable from l in increasing derivations.

Now:
PTIME algorithm for full GTRS

First, apply three PTIME confluence-preserving transformations:

- into Curry form, with
  1 single binary symbol f + constants:
  $g(a, h(b), c)$ becomes
  $f(\ f(\ f(g, a),\ f(h, b)\ ),\ \ c\ )$
- into shallow, by adding new constants:
  $f(f(a, b)) \to c$ becomes
  $f(d) \to c, \quad f(a, b) \to d, \quad d \to f(a, b)$
- Plaisted's transformation for terms

Second, give PTIME algorithm for:

can f become stable in $f(s, t)$ by only rewriting on s and t?

(essentially, a reachability problem)

PTIME algorithm for full GTRS (cont.)

Third, check a number of necessary conditions for confluence (this simplifies analysis in fourth step).

Fourth, s and t are deeply joinable (dj) if

$$s \to_R^* s' \text{ and } t \to_R^* t' \quad \text{implies} \quad s' \downarrow t'$$

And note that:

R confluent iff each lhs is dj with itself

Fifth, PTIME test for dj between sides of R

PTIME algorithm for deep joinability

Define DJ-conditions of s and t: necessary for their deep joinability.

Fixpoint construction: (s,t) in some ndj_k if they are not dj

$$ndj_0 = \{\ (s,s') \mid \text{DJ-conditions}(s,s') = \text{false}\ \}$$

$$\begin{aligned} ndj_{i+1} &= ndj_i \\ &\cup \{\ (s,s') \mid l \in le(s),\ \ l' \in le(s'),\ \ (l,l') \in ndj_i\ \} \\ &\cup \{\ (s,s') \mid r \in ri(s),\ r' \in ri(s'),\ (r,r') \in ndj_i\ \} \end{aligned}$$

- where $le(s) \approx$ the set of $f(a,b)$ s.t.
  - $s \to_R^3 f(f(a,b),t)$ and
  - f can become stable in $f(a,b)$.
- similarly for $ri(s)$.

Exact complexity

We prove PTIME-hardness for strings (by LOGSPACE reduction from the alternating graph reachability problem).

Altogether this implies PTIME-completeness for strings and terms.

Conclusions

- picture closed for GTRS
- Surprising: deciding dj is in PTIME, but half deep joinability (hdj):

  given s, t, R, is any term reachable from s joinable with t?

  is EXPTIME-complete!
- confluence $\equiv$ hdj(l, t) for all lhs l and t reachable from l in one step.
  hdj is hard, but the conjunction is not!
- is there a non-trivial class of tree automata for which e.g. inclusion is in P?
- precise degree $|R|^5$
- eliminating transformation to shallow, it could be that:
  - co-NP for left-linear right-ground TRS
  - decidable for right-ground TRS