Basic Paramodulation and Decidable Theories (Extended Abstract)*

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Abstract

We prove that for sets of Horn clauses saturated under basic paramodulation, *the word and unifiability problems are in NP, and the number of minimal unifiers is simply exponential (i). For Horn sets saturated wrt. a special ordering under the more restrictive inference rule of basic* superposition, *the word and unifiability problems are still decidable and unification is finitary (ii).*

We define standard theories, *which include and significantly extend* shallow *theories. Standard presentations can be finitely closed under superposition and result (ii) applies.*

Generalizing shallow theories to the Horn case, we obtain (two versions of) a language we call Catalog, *a natural extension of Datalog to include functions and equality. The closure under paramodulation is finite for Catalog sets, hence (i) applies. Since for shallow sets this closure is even polynomial, shallow unifiability is in NP, which is optimal: unifiability in ground theories is already NP-hard. We even go beyond: the shallow word problem is tractable and for Catalog sets S we prove decidability of the full first-order theory of $\mathcal{T}(\mathcal{F})/_{=_S}$.*

1. Introduction

Word and unification problems (entailment and answer computation) in equational Horn theories are the heart of most applications of logic in computer science: (constraint) logic and functional programming, automated deduction, knowledge-based systems, computational linguistics, etc. Hence the decidability and computational feasibility of these problems for diverse classes of formulae have been extensively studied.

Some of these classes are defined by a combination of semantic and syntactic restrictions on the language. For example, for the class of ground equations where some symbols are associative and commutative (AC) always a finite convergent rewrite system can be computed [NR91, Mar91] by which the word problem is efficiently decidable. In the same class, the unification problem is also decidable [NR93] (see also [Mar94] for decidability of word problems in ground presentations *modulo* several other theories different from AC).

Some well-known classes of first-order formulae have been proved decidable by means of model-theoretic methods, and satisfiability of their corresponding clause sets can also be decided by means of more practical methods like resolution and paramodulation. This is the case e.g. for the Ackermann class [FS93] or the *monadic* class, which is moreover equivalent to a type of *set constraints* [BGW93a, BGW93b].

An important purely syntactically defined class is the Datalog language of *flat* Horn clauses without equality, which is well-known from deductive databases. Another such class are the equational *Shallow theories*, the ones axiomatized by equations where all variables are *shallow*, i.e. they appear only at depth at most one in each side of the equation. This is a fundamental class with decidable word and unification problems (and even a decidable first-order theory) [CHJ94], subsuming much previous work on classes like *ground, permutative, compact* or *quasi-free* theories. The results on shallow theories were obtained, following [Kir86], by transforming shallow presentations into equivalent *(cycle-)syntactic* ones, for which complete and terminating unification rules were derived (cf. also the techniques for dealing with shallow equations and rewrite rules given in [Chr92, Dom93]).

Here we proceed in a completely different way, namely by a careful termination analysis of saturation under *basic* paramodulation [NR95, BGLS95], for Horn clauses with equality. Basicness means in this context that no inferences are needed on certain *blocked* terms of the clauses, like typically the terms created in unifiers of previous inferences, as in *basic narrowing*. Completeness results have been given for several basic strategies, in which normally a trade-off exists between the amount of positions that can be blocked and the ordering restrictions of the inference rules. The main idea behind these completeness re-

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sults is roughly to consider a particular rewrite strategy at the ground level, which is lifted to the non-ground level. This allows one to impose irreducibility restrictions on certain terms, which can hence be blocked for inferences. For example, if in an inference with a premise $l \simeq r$ the term r is placed in the conclusion and all inferences on r have already been considered in $l \simeq r$, then one can block the term r in the conclusion.

Basic paramodulation techniques turn out to be a surprisingly powerful uniform method for a wide class of union-closed languages. We prove the following results:

Section 3: for sets of Horn clauses saturated under basic *paramodulation* (i), the word and unifiability problems are in NP, and the number of minimal unifiers is simply exponential. For Horn sets S saturated under the more restrictive inference rule of basic *superposition* (ii), the word and unifiability problems are still decidable and unification is finitary, provided S fulfils some natural syntactic and ordering requirements.

In section 4 we define *linear standard theories*, generalized in section 5 to full *standard theories* which include shallow theories and in fact significantly extend them since in several common situations non-shallow variables are allowed. Standard presentations can be finitely closed under superposition fulfilling the requirements of result (ii), which hence applies.

In section 6 we generalize shallow theories to the Horn case. We obtain (two versions of) a language we call *Catalog*, a natural extension of Datalog to include functions and equality. The closure under paramodulation is finite for Catalog sets, hence (i) applies. Since for shallow sets this closure is even polynomial, shallow unifiability is in NP, which is optimal: unifiability in ground theories is already NP-hard. We even go beyond: the shallow word problem is tractable and for Catalog sets S we prove decidability of the full first-order theory of $\mathcal{T}(\mathcal{F})/_{=_S}$.

2. Basic Paramodulation and other Basic Notions

For missing notations and definitions on equations, orderings and rewriting, we refer to [DJ90], and for the ones on ordered deduction with first-order equality clauses to [BG94].

A variable x (or a constant a) *occurs at depth* 0 in the term x (resp. a) and occurs at depth $k+1$ in a term $f(t_1,\ldots,t_n)$ if it occurs at depth k in t_i for some $i \in 1\ldots n$. A variable is *shallow* in a term s if it occurs only at depth 0 or 1 in s. Non-shallow variables, the ones occurring at depth two or more, are sometimes called *deep* variables. A term (equation) is shallow if all its variables are shallow. The *depth* of a term s is defined $depth(s) = max\{k \mid some\ variable\ or\ constant\ occurs\ at\ depth\ k\ in\ s\}$. The *top symbol* $top(f(\ldots))$ of a term $f(\ldots)$ is f. A term s is *linear* if each variable in $\mathcal{V}ars(s)$ occurs only once in s. An equation $s \simeq t$ is linear if s and t are linear terms and is *collapsing* if at least one of s or t is a variable. A variable is *linear* in a term s if it occurs only once in s. Two terms *share* a variable x if x occurs in both terms. A linear equation is a *permuter* if it is of the form $f(x_1,\ldots,x_n) \simeq f(x_{\pi(1)}\ldots x_{\pi(n)})$ where π is a permutation and all x_i are variables for $i \in 1\ldots n$.

The inference rules for *basic* paramodulation and superposition are expressed here by means of *equality constrained clauses* ([NR95]). A clause is a disjunction of equations and negated equations; non-equality atoms A are expressed by equations $A \simeq \top$. An equality constraint is a conjunction of equalities $s = t$, where s and t are terms in $\mathcal{T}(\mathcal{F},\mathcal{X})$ and where $=$ is interpreted as syntactic equality, i.e. a constraint is *satisfied* by a ground substitution σ if σ is a unifier of s and t for all $s = t$ in the conjunction. The semantics of a constrained clause $C\,[\![T]\!]$ is the set of all its *ground instances*: ground clauses $C\sigma$ such that σ satifies T. Hence a clause with an unsatisfiable constraint is a tautology.

The inference rules below are parameterized by a simplification ordering $\succ$ on ground terms, which, unless stated otherwise, will be be the *recursive path ordering* (RPO) (see e.g. [Der87])[1]. Assume given a *precedence* $\succ_p$ on the function symbols, and assume the equality $==$ to be modulo permutations of the direct subterms of any symbol. More precisely, for every permutation π and for every terms $t_1,\ldots,t_n$, $f(t_1,\ldots,t_n) == f(t_{\pi(1)},\ldots t_{\pi(n)})$. Furthermore, let $\succeq$ stand for $\succ\ \vee\ ==$. Then the RPO $\succ$ is defined: $s = f(s_1,\ldots,s_n) \succ g(t_1,\ldots,t_m) = t$ iff (i) $\exists i \in \{1,\ldots,n\}$, $s_i \succeq t$ or (ii) $f \succ_p g$ and $s \succ t_i$ for all $i = 1,\ldots,m$ or (iii) $f = g$ and $\{s_1,\ldots,s_n\} \gg \{t_1,\ldots,t_m\}$ where $\gg$ is the multiset extension of $\succ$.

The inference rules of *paramodulation* and *equality resolution* are, respectively:

$$\frac{C' \vee s \simeq t\,[\![T']\!] \qquad C\,[\![T]\!]}{C' \vee C[t]_p\,[\![T' \wedge T \wedge C|_p = s]\!]} \qquad\qquad \frac{C \vee s \not\simeq t\,[\![T]\!]}{C\,[\![T \wedge s = t]\!]}$$

In paramodulation, the term $C|_p$ is never a variable. The equality constraints express in a natural way the *basicness restriction*: no inferences take place on subterms introduced by unifiers of previous inferences.

Along this paper, assume that in every clause C having at least one negative literal, one such negative literal is *selected* [BGLS95], i.e. all inferences with C as a premise are paramodulations and equality resolutions on this literal. This implies for the Horn case

[1] Although RPO is not total on ground terms, it can be extended to a total ordering, which suffices for the completeness results of paramodulation and superposition.

that in all paramodulation inferences the left premise is a positive unit clause (hence the part C' of the left premise is empty). Furthermore, depending on the ordering restrictions with $\succ$, we obtain *basic paramodulation* or *basic superposition*. Paramodulations with $s \simeq t$ on a clause C are always restricted to the cases where $s\sigma \succ t\sigma$ for some σ satisfying the constraint of the conclusion. If only this restriction is imposed, then the inference rule is called *basic paramodulation*. In this case, the basicness restriction can be imposed also on the subterm t in the conclusion $C' \vee C[t]_p\, [\![T' \wedge T \wedge C|_p = s]\!]$ and hence the conclusion can be replaced by $C' \vee C[x]_p\, [\![T' \wedge T \wedge C|_p = s \wedge x = t]\!]$ where x is a new variable. If in addition also $s'\sigma \succ t'\sigma$ is required when $C|p$ is in the term s' of the (possibly negated) equation $s' \simeq t'$ in the right premise C, then we obtain *basic superposition*, in which the term t cannot be moved into the constraint.

From [NR95, BGLS95] we have the following results. Basic paramodulation (superposition), together with equality resolution is refutationally complete for Horn clauses (for the non-Horn case also a factoring rule is needed). It is compatible with selection of negative literals and *redundancy* criteria including e.g. simplification by rewriting with unconstrained equations or deletion of tautologies. *Saturation* is the process of closure under the inference rules up to redundancy. From [Nie95] we have completeness results for answer computation: e.g. if E is a set of Horn clauses, the saturation of $E \cup \{\neg s_1 \simeq t_1 \vee \ldots \vee \neg s_n \simeq t_n\}$ will produce an empty clause for every (simultaneous, irreducible) E-unifier σ s.t. $s_i\sigma =_E t_i\sigma$ for all i, and σ can be recovered from the accumulated instantiations of the variables of $s \simeq t$. These results are also compatible with *propagation* [KKR90] of (parts of) a constraint to the clause part: for example, $f(x) \simeq a\, [\![x = g(b)]\!]$ propagates into $f(g(y)) \simeq a\, [\![y = b]\!]$. Eager propagation of the whole constraint gives us normal ordered paramodulation and superposition. A set is saturated under basic paramodulation (resp. superposition) whenever it is saturated under normal paramodulation (resp. superposition).

3. Deciding Unifiability by Basic Inference Rules

In this section we observe that for some kinds of sets of clauses closed under the basic inference rules, unification (answer computation) by again applying these inference rules can be proved terminating.

Theorem 3.1 *Let S be a finite set of Horn clauses that is saturated under basic paramodulation with selection.*

Then the word and unifiability problems in S are in NP, and unification (answer computation) is finitary: the number of minimal unifiers is simply exponential.

Proof: Let ϕ be a formula of the form $s_1 \simeq t_1 \wedge \ldots \wedge s_n \simeq t_n$ where $\mathcal{V}ars(\phi) = \vec{x}$. Let C be the clause $\neg s_1 \simeq t_1 \vee \ldots \vee \neg s_n \simeq t_n$. Then $S \models \exists \vec{x} \phi$ iff $S \cup \{C\, [\![\top]\!]\}$ is inconsistent. Note that the word problem is a particular case in which C is Skolemized (ground). We proceed by basic paramodulation. S is closed (up to redundant inferences) under paramodulation, hence only steps on descendants $C'\, [\![T']\!]$ of $C\, [\![\top]\!]$ are performed. Since all non-unit clauses have a selected negative literal, the only applicable clauses of S are positive unit clauses. Clearly, from each $C'\, [\![T']\!]$ we get only finitely many new $C_i\, [\![T_i]\!]$ and in C_i the number of non-variable positions is strictly smaller than in C' (remind that in basic *paramodulation* the term inserted in the conclusion is replaced by a variable). We obtain a so-called narrowing tree whose depth is at most the number of nonvariable positions in ϕ and with a branching factor polynomial in the number of such positions in ϕ and the number of unit clauses in S. This shows membership in NP of unifiability: simply guess the right path in the tree. By computing this whole tree, *all* (irreducible, i.e. minimal) unifiers (answers) are computed [Nie95] (and can be extracted by recording the instantiations of the variables, like in logic programming), which implies that the number of unifiers is simply exponential. $\Box$

Later on, we wil show that for example shallow theories can be closed in polynomial time under paramodulation. The result of membership in NP of shallow unifiability is indeed optimal. For example, the commutativity axiom (C) is shallow and the C-unifiability problem is already NP-hard. Also ground theories are shallow, and the following is well-known:

Property 3.2 *The unifiability problem in ground equational theories is NP-hard.*

Proof: Reduce for example 3-sat: $(x_1 \vee \overline{x_2} \vee x_3) \wedge \ldots$ is expressed by an equation
$and(or(x_1, not(x_2), x_3), and(\ldots)\ldots) \simeq t$, to be unified in the theory defined by the ground equations
$not(t) \simeq f, \quad not(f) \simeq t,$
$and(f, f) \simeq f, \ldots, and(t, t) \simeq t,$
$or(f, f, f) \simeq f, \ldots, or(t, t, t) \simeq t.$ $\Box$

Definition 3.3 *Let $\succ$ be an RPO ordering based on a total precedence $\succ_p$. Let σ_0 be the substitution mapping all variables to the smallest constant wrt. $\succ$. The ordering $\succ_u$ (the "u" stands for unification) is defined by $s \succ_u t$ iff $s\sigma_0 \succ t\sigma_0$. It is well-founded and monotonic, but not stable under substitutions. Its multiset extension $\gg_u$ is well-founded on clauses (seen as multisets of terms).*

Theorem 3.4 *Let S be a finite set of Horn clauses that is closed under basic superposition with selection and where all positive unit clauses are unconstrained* *permuters or equations of the form $s \simeq t\,[\![T]\!]$ where $top(s) = f$ and $f(x \ldots x) \succ_u t$ for a variable x and $s\sigma \succ t\sigma$ for all its ground instances $s\sigma \simeq t\sigma$. Then the word and unifiability problems in S are decidable, and unification is finitary.*

Proof: Similar to the previous theorem, except that we apply basic superposition. From each $C'\,[\![T']\!]$, again only finitely many new descendants $C_i\,[\![T_i]\!]$ are obtained with $C' \gg_u C_i$, except in the case of innermost superpositions with permuters, which do not modify C'. But only finitely many of such permuting steps can be done consecutively without cycling. Hence, if we avoid cycling, termination follows again by König's lemma. □

4. Linear Standard Theories

Before studying in the next section full standard theories, we first deal with the following linear version:

Definition 4.1 *An equation $s \simeq t$ is* linear standard *iff s and t are linear terms sharing only shallow variables. A* linear standard theory *is a theory axiomatisable by a set of linear standard equations.*

Our aim will be, for every linear standard presentation E, to show that it can be finitely saturated under superposition. For this purpose, a first interesting property (straightforward to show) is that the language of linear standard equations is closed under superposition. Our termination proof is based on the fact that only finitely many different equations $s \simeq t$ exist (up to variable renamings) if s and t belong to $\mathcal{T}(\mathcal{F}, \mathcal{X})$ for some finite $\mathcal{F}$ and $depth(s) < k$ and $depth(t) < k$ for some k. Unfortunately, although inferences at top position between linear standard equations are non-depth increasing, this is not the case for inferences at other positions.

Example 4.2 From the linear standard equations $f(g(x,y),z) \simeq h(z)$ and $g(x',y') \simeq i(x', f(f(z')))$, by one paramodulation step on $g(x,y)$ we obtain $f(i(x, f(f(z'))), z) \simeq h(z)$ which has a greater depth than the premises.

A solution to the problem appearing in the following example is to eagerly *split* each equation:

Definition 4.3 *Let $s \simeq t$ be a non-collapsing linear standard equation which is not already of the form $s \simeq h(x_1, \ldots, x_n)$ where $\{x_1, \ldots, x_n\}$ is a subset of the set of (shallow) variables of s. Then $s \simeq t$ is* split *by replacing it by two new ones $s \simeq h(x_1, \ldots, x_n)$ and $t \simeq h(x_1, \ldots, x_n)$, where h is a new function symbol and $\{x_1, \ldots, x_n\}$ is the set of (shallow) variables shared by s and t.*

Lemma 4.4 *For all sets of equations E over $\mathcal{T}(\mathcal{F}, \mathcal{X})$, the theory of the set E' over $\mathcal{T}(\mathcal{F} \cup \{h\}, \mathcal{X})$, obtained from E by splitting some equation in E is a conservative extension, i.e. for all terms s and t in $\mathcal{T}(\mathcal{F}, \mathcal{X})$, $E \models s \simeq t$ iff $E' \models s \simeq t$.*

Moreover all ground terms $h(t_1, \ldots, t_n)$ are congruent under E' with some term in $\mathcal{T}(\mathcal{F})$, hence the initial models $\mathcal{T}(\mathcal{F})/=_E$ and $\mathcal{T}(\mathcal{F} \cup \{h\})/=_{E'}$ are isomorphic.

Note that by splitting an equation $s \simeq t$ the new symbol h introduced has a strictly smaller arity than each of the top symbols of s and t. In the following, we will eagerly split all equations (the initial ones and the ones generated by inferences) for which this is possible (remind that splitting is only defined for equations which are not already of the required form). In the context of saturation, splitting a conclusion $s \simeq t$ of an inference into $s \simeq h(\vec{x})$ and $t \simeq h(\vec{x})$ preserves completeness since these two new equations make $s \simeq t$ redundant if the symbol h is small in the ordering. It is not difficult to see that, for equations of this form, superposition inferences are indeed non depth increasing.

Lemma 4.5 *Let $s \simeq t$ be a linear standard equation where t is either a variable occurring in s or a term $h(x_1, \ldots, x_n)$ where all x_i are variables occurring in s and where $top(s) \succ_p h$. In every inference among linear standard equations by superposition wrt. $\succ$ with left premise $s \simeq t$, the depth of the conclusion, which is again a linear standard equation, is smaller or equal than the depth of the premises.*

Proof: Case analysis using the fact that if s and t are linear terms and $\sigma = mgu(s,t)$ then $depth(s\sigma) = depth(t\sigma) = max(depth(s), depth(t))$. □

However, splitting only postpones the termination proof, because if the number of new symbols introduced by splittings is infinite, the bounded depth argument does not suffice for proving termination of the closure process:

Example 4.6 Let t be a linear term not containing the variables x and y. By one inference at top position between (i) $f(x,y,t) \simeq h(x,y)$ and (ii) $f(x', a, y') \simeq h'(x', y')$ we obtain $h(x,a) \simeq h'(x,t)$, which has to be split into $h(x,a) \simeq h''(x)$ and $h'(x,t) \simeq h''(x)$ for some new symbol h''.

Assume some inference with $l \simeq r$ on some position p inside the term t of (i) produces (iii) $f(x,y,t[r]_p)\sigma \simeq h(x,y)\sigma$ where x and y are not in the domain of σ. A new inference at top position between (ii) and (iii) now gives us $h(x,a) \simeq h'(x, t[r]_p\sigma)$, which is split by making both sides equal to some $h'''(x)$.

If e.g. h occurs inside t, then this never terminates because h can be replaced in (i) by $h''(\ldots)$, which leads to a new inference with (ii) generating $h(x,a) \simeq h'''(\ldots)$ and then in (i) h can also be replaced by $h'''(\ldots)$, etc.

However, by a careful analysis of standard equations, the following interesting property (with a clearly *basic* flavour) can be shown: Let e and e' be two equations not generated in a superposition step at non-topmost position on some other equation. and let E and E' be the sets of all descendants by zero or more inferences at non-topmost positions on e and e' respectively. Then, out of all possible equations that can be obtained by superposing at topmost position equations of E on equations of E', *only one* of them is needed.

Example 4.7 Look again at Example 4.6. The new equation (iii) in fact has exactly the form $f(x,y,t') \simeq h(x,y)$ for some t', because due to the variable restrictions of standard equations, the inference between (i) and (ii) only involves t (only the variables of t are in the domain of σ). The conclusion of the inference at top position between (i) and (iii) has therefore the form $h(x,a) \simeq h'(x,t')$, which can also be obtained by doing the same inference on the first conclusion $h(x,a) \simeq h'(x,t)$.

Definition 4.8 *Let e_1 and e_2 be linear standard equations of the form $f(s_1,\ldots,s_n) \simeq g(\vec{x})$ and $f(t_1,\ldots,t_n) \simeq h(\vec{y})$ where $\vec{x} \subseteq \{s_1,\ldots,s_n\}$ and $\vec{y} \subseteq \{t_1,\ldots,t_n\}$, and let σ be the substitution with domain $\vec{x} \cup \vec{y}$ defined by $s_i\sigma = t_i$ and $t_i\sigma = t_i$ if $s_i \in \vec{x}$ and $t_i\sigma = s_i$ otherwise. Then $g(\vec{x})\sigma \simeq h(\vec{y})\sigma$ is called the* needed *conclusion of e_1 and e_2.*

Note that the needed conclusion of e_1 and e_2 is equal up to variable renamings to the needed conclusion of e_2 and e_1, and if there is any topmost inference possible between e_1 and e_2, then its conclusion is exactly the needed conclusion.

Lemma 4.9 *Let E be a set of split linear standard equations, and let e_1 and e_2 be two equations with a needed conclusion e. Let D_1 and D_2 be the sets of all descendants by zero or more inferences at non-topmost positions with equations of E on e_1 and e_2 respectively, and let D be the set of all conclusions that can be obtained by superposing at topmost position an equation of D_1 on an equation of D_2.*

If $D \neq \emptyset$ then $e \in D$. Furthermore, all equations in D can be obtained by inferences at non-topmost positions with equations of E on e.

Proof: Let e_1 and e_2 be of the form $f(s_1,\ldots,s_n) \simeq g(\vec{x})$ and $f(t_1,\ldots,t_n) \simeq h(\vec{y})$ where $\vec{x} \subseteq \{s_1,\ldots,s_n\}$ and $\vec{y} \subseteq \{t_1,\ldots,t_n\}$. All equations in D_1 will have the form $f(s'_1,\ldots,s'_n) \simeq g(\vec{x})$ and the ones in D_2 will be of the form $f(t'_1,\ldots,t'_n) \simeq h(\vec{y})$.

If no topmost inference is possible between e_1 and e_2 this is because for some i the non-variable linear terms t_i and s_i are not unifiable. If some inference between descendants of e_1 and e_2 is possible, then this is because the corresponding s'_i and t'_i have become unifiable. By performing non-topmost inferences only on such positions i, we can clearly obtain the needed conclusion, since we leave unchanged the terms s_j and t_j where $s_j \in \vec{x}$ or $t_j \in \vec{y}$ which determine how the conclusion is going to be.

If some topmost inference with a substitution σ' between descendants of e_1 and e_2 produces a conclusion e' with $e' \neq e$, e.g. because $x\sigma' = s'_i \neq s_i$ for some $x \in \vec{x}$, then, since s'_i has been obtained from s_i by non-topmost steps on e_1, we can apply the same steps to obtain e' from e. □

Definition 4.10 *An* almost permuter *is a linear equation of the form $f(x_1,\ldots,x_n) \simeq g(x_{\pi(1)}\ldots x_{\pi(n)})$ where π is a permutation, all x_i are variables and where $f \neq g$.*

The closure process is parameterized by a recursive path ordering (RPO) $\succ$ on terms where the precedence $\succ_p$ is compatible with arities, i.e. if $arity(f) > arity(g)$ then $f \succ_p g$. We always eagerly split initial equations and conclusions of inferences. Note that this implies that all equations in E are orientable as rewrite rules wrt. $\succ$ except permuters. For efficiency reasons, almost permuters could also be applied as simplification rules to eagerly eliminate everywhere all occurrences of f by rewriting. After this, no inferences with it will exist. Note that a particular case of this is rewriting with equations between constants. In the closure of E we have the following **case analysis** of all possible kinds of inferences:

1. A superposition of some other equation on a position inside the linear term s of an equation $f(\ldots,s,\ldots) \simeq t$, where t is a variable or a term $h(\vec{x})$: in this case we obtain a conclusion $f(\ldots,s',\ldots) \simeq t$, i.e. only s is replaced by some other linear term s' with smaller or equal depth. No splitting of the conclusion is needed.

2. A superposition at topmost position between a collapsing equation $f(\ldots,x,\ldots) \simeq x$, and another (possibly collapsing) equation $s \simeq t$. This may produce conclusions: (i) a tautology $x \simeq x$; (ii) $x \simeq t$ where $x \notin \mathcal{V}ars(t)$ (i.e. an inconsistency); (iii) a new collapsing equation $t\sigma \simeq x$ for which splitting is not needed, or (iv) an equation $s \simeq t$ where s and t are linear terms not sharing any variables, which will be split into $s \simeq a$ and $t \simeq a$ where a is a new constant.

3. A superposition at topmost position between two non-collapsing equations $s \simeq h(\vec{x})$ and $t \simeq h'(\vec{y})$.

The conclusion will be a non-collapsing equation $h(\vec{x})\sigma \simeq h'(\vec{y})\sigma$ for which splitting may be needed. Since $\succ$ is compatible with arities, inferences between equations with a top symbol with arity i will only produce conclusions with top symbols with arity smaller or equal to i. If splitting of such a conclusion is needed, the new symbol introduced has arity strictly smaller than i.

No splitting is required for superpositions with (almost) permuters. If at least one of the right hand sides is a constant, then also the conclusion will have a constant side and splitting is not needed.

Definition 4.11 *Let $s \simeq t$ be a (split) linear standard equation. We define $degree(s \simeq t) = k$ if $arity(top(s)) > |\mathcal{V}ars(t)| = k$; otherwise $s \simeq t$ is an (almost) permuter and then $degree(s \simeq t) = 0$. For non-empty sets of standard equations E we define $degree(E) = max\{k \mid s \simeq t \in E$ and $degree(s \simeq t) = k\}$.*

Theorem 4.12 *Every set of linear standard equations can be finitely closed under superposition.*

Proof: The initial splitting of such a set E is clearly finite, so we can assume E to be closed under splitting. We proceed by induction on $degree(E)$. Note that topmost superposition between two equations of degree n produces conclusions of a strictly smaller degree, and that the degree of a set E does not increase under superpositions.

If $degree(E) = 0$ then there are only (almost) permuters and equations with constants as right hand sides. According to the previous case analysis, the language of degree 0 equations is closed under superposition inferences and no new symbols will be ever introduced. Since the depth of equations is not increased, the closure of E will be finite.

For the induction step, let $degree(E) = n$ and let $E_0, E_1, \ldots$ denote a *closure derivation* where E_0 is E and each E_i is obtained from the previous set E_{i-1} by adding a new conclusion (or its splittings) not in E_i under superposition. Let E_i^n be the set of all equations in E_i with degree n. All equations in E_i^n have the form $s \simeq h(x_1, \ldots, x_n)$ where $arity(top(s)) > n$. Assume that E_0 is closed under inferences with all (almost) permuters (this is clearly a finite process). Note that no new permuters can appear that can be superposed at topmost position on an equation in E_i^n, because these can only be generated by inferences between equations of degree greater than n, which do not exist. Also, no new equation with degree n can appear which is not a descendant of an equation in E_0^n by superpositions at non-topmost positions.

We show that $E_0, E_1, \ldots$ is finite, i.e. there exists some k such that E_k is closed under superposition. By Lemma 4.9, only the *needed* inferences between equations in E_i^n are necessary and there are at most $|E_0^n|^2$ of them. But there is no infinite subsequence of the derivation in which no such needed inferences are performed: by induction hypothesis, the closure of each set $E_i \setminus E_i^n$ is finite, and the only remaining steps are non-topmost inferences with or on equations in E_i^n. But these steps introduce no new function symbols and are non-depth increasing, hence there can only be finitely many of them. □

5. Standard Theories

Definition 5.1 *A* standard signature *$\mathcal{F}$ is a signature where every function symbol f with arity n in $\mathcal{F}$ has an associated set of* shallow *positions $sh(f)$ and a set of* linear *positions $lin(f)$, such that $sh(f) \cap lin(f) = \emptyset$ and $sh(f) \cup lin(f) = \{1 \ldots n\}$.*

A depth-one argument s_i of $f(s_1, \ldots, s_n)$ is a linear position argument *of it if $i \in lin(f)$ and a* shallow position *argument if $i \in sh(f)$.*

Definition 5.2 *A term s is is a* standard *term iff it is a variable or a term of the form $f(s_1, \ldots, s_n)$ where if $i \in sh(f)$ then s_i is a variable or a ground term and if $i \in lin(f)$ then all variables in s_i are linear in s.*

Note that acording to the previous definition, all ground terms are standard, but not all linear terms are, because at shallow positions no terms with deep variables are allowed. Furthermore, the only non-linear variables of a standard term s are shallow variables at shallow positions.

Definition 5.3 *An equation $s \simeq t$ is* standard *iff*
(i) s is linear and t is ground or
(ii) s is a standard term $f(\ldots, g(t), \ldots)$ and t is a variable or
(iii) s and t are standard terms sharing only shallow variables and no variable x is both a shallow position argument and a linear position argument in $s \simeq t$.
A standard presentation *is a set of standard equations and a* standard theory *is a theory axiomatizable by a standard presentation.*

Note that for standard equations of type (i) s is not required to be a standard term. The standard equations of type (ii) are the only ones in which both sides of the equation share a non-shallow variable. Note however that this variable has strong restrictions: it must appear at depth two below a unary symbol.

In what follows we will consider that all terms are built over a standard signature $\mathcal{F}$ (in which sufficiently many new symbols for splitting are available). Let us also remark that for monadic symbols wlog. we can choose its unique position to be linear.

Clearly, shallow presentations are standard: just define $lin(f) = \emptyset$ for all function symbols f; on the other hand, linear standard presentations are also standard: take $sh(f) = \emptyset$ for all f. The union of two standard presentations over the same standard signature is again standard.

Example 5.4
$f(x, x, g(y)) \simeq h(x, g(z))$, $succ(pred(x)) \simeq x$ and $not(not(x)) \simeq x$ are standard equations which are not shallow nor linear standard.

Example 5.5 The presentation
$\{ \ f(g(y), x) \simeq h(x), \ \ f(x, x) \simeq g(x) \ \}$
is *not* standard because the first position of f cannot be linear and standard at the same time. It is clear why this kind of equations cannot be allowed: by one superposition inference we obtain $g(g(x)) \simeq h(g(x))$ sharing non-shallow variables; and indeed if $g \succ_p h$ in the closure under superposition the infinite set of equations schematized by $g(h^n(g(x))) \simeq h^{n+1}(g(x))$ for $n \geq 0$ is generated.

For standard theories, we will proceed like we did for linear standard ones. Inferences are still non-depth increasing, if splitting is done eagerly. For standard equations splitting has to be defined more precisely than it was done for linear standard ones:

Definition 5.6 *A non-collapsing standard equation $s \simeq t$ which is not already of the form $s \simeq h(x_1, \ldots, x_n)$ where $\{x_1, \ldots, x_n\}$ is a subset of the set of (shallow) variables of s, is* split *by replacing it by two new ones $s \simeq h(x_1, \ldots, x_n)$ and $t \simeq h(x_1, \ldots, x_n)$, where h is a new function symbol and $\{x_1, \ldots, x_n\}$ is the set of (shallow) variables shared by s and t.*

The new symbol h will have linear and shallow positions such that a variable x will be at a linear position of h iff it was at linear positions in $s \simeq t$.

Lemma 5.7 *Let s be a standard term and let t be either a linear term or a standard term. If $\sigma = mgu(s, t)$ then $depth(s\sigma) = depth(t\sigma) = max(depth(s), depth(t))$. Furthermore, if t is linear then $t\sigma$ is also linear.*

Due to the richer language (e.g. the collapsing equations of case (ii) of the definition) the case analysis of all possible inferences is more tedious. Below we analize all cases according to the type of standard equation (i),(ii) or (iii) of Definition 5.3. Note that due to eager splitting, in all equations $s \simeq t$ of type (i) the ground term t is in fact a constant.

Non-topmost inferences:

1. A superposition of some other equation of any type on a position inside the linear term s of an equation $f(\ldots, s, \ldots) \simeq t$ of type (i), (ii) or (iii) where $t \notin \mathcal{V}ars(s)$. In this case we obtain a conclusion $f(\ldots, s', \ldots) \simeq t$, i.e. only s is replaced by some other linear term s' with smaller or equal depth. No splitting of the conclusion is needed.
2. A superposition of an equation $g(s) \simeq t$ of type (i) on $f(\ldots, g(x), \ldots) \simeq x$ of type (ii). In this case we obtain a conclusion $f(\ldots, t, \ldots) \simeq s$ with smaller or equal depth than the premises, and where both sides share no variables, i.e. which is split into two type (i) equations.
3. A superposition of an equation $g(h(x)) \simeq x$ of type (ii) on $f(\ldots, g(y), \ldots) \simeq y$ of type (ii), producing a conclusion $f(\ldots, x, \ldots) \simeq h(x)$, which is a non-splittable standard equation of type (iii) with smaller depth than the second premise.
4. A superposition of an equation $g(x) \simeq x$ of type (iii) on $f(\ldots, g(y), \ldots) \simeq y$ of type (ii), producing a conclusion $f(\ldots, x, \ldots) \simeq x$, which is a non-splittable standard equation of type (iii) with smaller depth than the second premise.
5. A superposition of an equation $g(x) \simeq h(x)$ of type (iii) on $f(\ldots, g(y), \ldots) \simeq y$ of type (ii). In this case we obtain a conclusion $f(\ldots, h(x), \ldots) \simeq x$, which is a non-splittable standard equation of type (ii) with depth equal to the one of the second premise.

Topmost inferences:

1. An equation $s \simeq a$ of type (i) on another equation $s' \simeq t$ of type (i), (ii) or (iii). This creates $t\sigma \simeq a$ of type (i) where $t\sigma$ is linear with $depth(t\sigma) \leq max(depth(s), depth(s'))$.
2. Two equations $f(\ldots, g(x), \ldots) \simeq x$ and $f(\ldots, h(y), \ldots) \simeq y$ of type (ii). If $g(x)$ and $h(y)$ are at the same argument position of f then $f = g$ and we obtain the tautology $x \simeq x$. Otherwise, they are on different linear positions and hence either the conclusion is an inconsistency $x \simeq t$ where $x \notin \mathcal{V}ars(t)$ or else we get $s \simeq t$ with smaller depth and where s and t do not share any variables, i.e. we split with a new constant.
3. An equation $f(\ldots, g(x), \ldots) \simeq x$ of type (ii) and another collapsing one $f(s_1, \ldots, s_n) \simeq y$ of type (iii). If $g(x)$ and y are at the same linear argument position of f then we obtain $g(x) \simeq x$. If $g(x)$ and y are at different positions of f then we get either an inconsistency or $s \simeq t$ with smaller depth and where s and t do not share any variables, i.e. we split with a new constant.
4. An equation $f(\ldots, g(x), \ldots) \simeq x$ of type (ii) and another collapsing one $f(s_1, \ldots, s_n) \simeq h(\vec{y})$ of

type (iii). If $g(x)$ and some variable y_i of $\vec{y}$ are at the same linear argument position of f then we obtain the type (ii) conclusion $h(\vec{y})\sigma \simeq x$, where $y_i\sigma = g(x)$. No splittin is needed and the depth has not increased wrt. at least one of the premises.

5. A superposition at topmost position between two collapsing type (iii) equations $f(s_1, \ldots, s_n) \simeq x$ and $f(t_1, \ldots, t_n) \simeq y$.
   - If x occurs at a linear position i in $f(s_1, \ldots, s_n)$ then either $y = t_i$ and we obtain $x \simeq x$ or else we obtain a linear equation $s \simeq t$ with smaller depth where s and t do not share any variables.
   - If x occurs at one or more shallow positions in $f(s_1, \ldots, s_n)$ then either $y\sigma = x\sigma$ (because y also occurs at one of these positions, or because a chain of unifications between shallow positions occurs) and we obtain a tautology, or else we obtain a linear equation $x\sigma \simeq y\sigma$ with smaller depth and where both sides do not share any variables.

6. A superposition at topmost position between a collapsing equation $f(s_1, \ldots, s_n) \simeq x$, of type (iii) and a non-collapsing type (iii) equation $f(t_1, \ldots, t_n) \simeq h(\vec{y})$.
   - If x occurs at a linear position i in $f(s_1, \ldots, s_n)$ then either $t_i \in \vec{y}$ and we obtain a new collapsing type (iii) equation with depth smaller or equal $h(\vec{y})\sigma \simeq x$ or else we obtain a linear equation $s \simeq t$ with smaller depth where s and t do not share any variables.
   - If x occurs at one or more shallow positions in $f(s_1, \ldots, s_n)$ then either $x\sigma = y\sigma$ for one or more variables $y \in \vec{y}$ and we get a new type (iii) equation $h(\vec{y})\sigma \simeq x$ where $x\sigma = y$ or $x\sigma$ is a ground term in which case it has to be split, or else $x\sigma \neq y\sigma$ for any $y \in \vec{y}$ and then we obtain a linear equation $s \simeq t$ with smaller depth where s and t do not share any variables.

7. A superposition at topmost position between two non-collapsing equations otf type (iii) $f(s_1, \ldots, s_n) \simeq h(\vec{x})$ and $f(t_1, \ldots, t_n) \simeq h(\vec{y})$. In this case we obtain $h(\vec{x})\sigma \simeq h(\vec{y})\sigma$. For each $s_i \in \vec{x}$ with $i \in lin(f)$ we can have:
   - $t_i \in \vec{y}$; then $s_i\sigma = t_i\sigma$ is a shared shallow variable in $h(\vec{x})\sigma \simeq h(\vec{y})\sigma$.
   - $t_i \notin \vec{y}$; then $s_i\sigma = t_i$ is linear term.

   Each $s_i \in \vec{x}$ with $i \in sh(f)$ only can get instantiated with a variable in $\vec{y}$ or a ground term, since no shallow arguments get unified with linear ones. The conclusion $h(\vec{x})\sigma \simeq h(\vec{y})\sigma$ is a standard equation with depth smaller or equal which might require splitting using a new symbol of arity smaller than n.

Theorem 5.8 *Every standard presentation E can be finitely closed under superposition satisfying the conditions of Theorem 3.4. Hence the word problem in E is decidable and unification in E is decidable and finitary.*

6. Catalog

Datalog is a well-known language of Horn clauses in which all literals are flat, i.e. its atoms are predicates applied to variables or constants. Here we extend this language in a natural way to include equalities and function symbols. We first revisit shallow *equations*:

Theorem 6.1 *Every finite set E of shallow equations can be closed under paramodulation in polynomial time.*

Proof: Wlog. assume E is a set of *flat* equations: at depth one there are only variables and constants: otherwise, replace non-constant ground arguments t everywhere with a new constant a and add a new shallow equation $a \simeq t$ (iterate this if the arguments of t are again non-constant). This process generates a conservative extension [CHJ94]. It only adds a linear number of new constants and equations.

Now close E under paramodulation wrt. an RPO $\succ$ compatible with arities. This means that constants are small in the precedence, and hence all non-topmost inferences are replacements of constants by constants. The language of flat equations is closed under paramodulation: if σ is the most general unifier of two non-variable flat terms, then $x\sigma$ is either a variable or a constant for all x in the domain of σ. No unifications take place between a variable and another term, because no maximal side of an equation is a variable (except for equations $x \simeq t$ where $x \notin \mathcal{V}ars(t)$, but then the closure can be stopped with the —inconsistent— outcome $x = y$, where x and y are distinct variables). This language of shallow equations has a polynomial cardinality for a given signature, which means that only polynomially many inferences are considered in the closure. $\square$

Definition 6.2 *A* shallow atom *is an atomic formula of the form $P(t_1, \ldots, t_n)$ where P is a non-equality predicate symbol and where each t_i for $i \in 1 \ldots m$ is a ground term or a variable.*
A Horn clause of the form $\neg A_1 \vee \ldots \vee \neg A_n \vee A_0$ where $n \geq 0$, is a Catalog-1 clause *iff all A_i for $i \in 1 \ldots n$ are shallow atoms and A_0 is a shallow equation or a*

shallow atom. It is a Catalog-2 clause *iff each A_i for $i \in 0 \ldots n$, is either a shallow atom or a non-collapsing shallow equation.*

Note that in the case of Catalog-2 clauses, the non-collapsing requirement is essential, since otherwise arbitrary equations (for which obviously all these probems are undecidable) could be expressed. For example, an equation like $f(g(x), g(x)) \simeq h(x)$ could be expressed by the logically equivalent Horn clause $\neg y \simeq g(x) \vee f(y,y) \simeq h(x)$.

Lemma 6.3 *Let S be a finite set of only Catalog-1 clauses or only Catalog-2 clauses. Then S can be finitely saturated under paramodulation.*

Proof: Wlog. as above assume S is a set of *flat* Catalog-1 (or -2) clauses. Shallow atoms A are treated as equations $A \simeq \top$ where $\top$ is a new constant. The languages of flat Catalog-1 (or -2) clauses are closed under paramodulation and equality resolution steps[2]. Finiteness of the closure follows from finiteness of the number of different flat clauses that can appear (due to the unit strategy the number of literals per clause does not increase). $\Box$

Theorem 6.4 *Let S be a finite set of only Catalog-1 clauses or only Catalog-2 clauses. Let E be a set of shallow equations.*
(i) The word problem in E is tractable, E-unifiability is in NP, and the number of minimal E-unifiers is simply exponential.
(ii) The word and unifiability problems in S are decidable and unification (answer computation) is finitary.
(iii) The first-order theory of $\mathcal{T}(\mathcal{F})/_{=_S}$ is decidable.

Proof: For deciding $E \models s \simeq t$ in polynomial time, let $\overline{s} \simeq \overline{t}$ be the ground Skolemization of $s \simeq t$. Let E' be the (polynomial size) closure under paramodulation of the shallow set $E \cup \{\ \overline{s} \simeq a,\ \overline{t} \simeq b\ \}$ where a and b are new constants. The refutation by basic paramodulation (like in Theorem 3.1) of $E' \cup \{a \not\simeq b\}$ is linear in E'.

For (i) and (ii), E and S can be saturated as above; then apply Theorem 3.1. For (iii), let E be the set of unit clauses of the saturation. It is not difficult to prove that $\mathcal{T}(\mathcal{F})/_{=_S}$ and $\mathcal{T}(\mathcal{F})/_{=_E}$ are the same algebra. Since E is a set of shallow equations, from [CHJ94] we have that the first-order theory of $\mathcal{T}(\mathcal{F})/_{=_E}$ is decidable. $\Box$

Let us conclude this section with a comment on the complexity of deduction and unification in Catalog. Deduction in the well-known language of Datalog is said to be tractable. But this is only true for a *constant* set of non-unit deduction rules (in databases they are considered part of the —fixed— intensional database). If we consider the word problem for flat Horn clauses without equality, in which we have as instance a Datalog program P and a ground query atom Q and the question "does Q follow from P?", then we can easily encode 3-sat problems:

Property 6.5 *The word problem for flat Horn clauses without equality is NP-hard.*

Proof: Encoding 3-sat, using four predicates P_0, P_1, P_2, P_3, where each P_i is used for a 3-sat clause with i negative atoms. These predicates are defined by the unit clauses

$$\begin{array}{lll} P_0(t,x,y) & P_0(x,t,y) & P_0(x,y,t) \\ P_1(t,x,y) & P_1(x,t,y) & P_1(x,y,f) \\ P_2(t,x,y) & P_2(x,f,y) & P_2(x,y,f) \\ P_3(f,x,y) & P_3(x,f,y) & P_3(x,y,f) \end{array}$$

Clearly, a 3-sat problem like

$$(x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_4} \vee \overline{x_5} \vee \overline{x_6}) \wedge \ldots$$

is satisfiable iff the atom *Sat* follows from the Datalog program consisting of the unit clauses for the P_i plus the flat Horn clause:

$$\neg P_1(x_1, x_2, x_3) \vee \neg P_3(x_4, x_5, x_6) \vee \ldots \vee Sat \qquad \Box$$

Since Catalog includes Datalog, deduction in Catalog can be tractable only with the same assumptions as for databases. Indeed, in that case for Catalog we obtain the same results as in the previous theorem for shallow equations.

7. Further Work

Several interesting open problems appear as a consequence of this work. A first problem (which we have started working on) is decidability of the full first-order theory of standard presentations. On the other hand, it would be interesting to determine the exact complexity of the word and unification problems in standard theories.

Regarding the identification of more general classes, undecidability seems to be close:
(i) For equations $s \simeq t$ where only t is required to be shallow, these probems are undecidable even when s has depth at most two and there are only unary function symbols (and hence at most one occurence of a variable per term): word equations $ab = c$ can be written like $a(b(x)) \simeq c(x)$, and it is undecidable whether an equality $a = b$ follows from a set of word equations $a_1 \ldots a_n = b_1 \ldots b_m$ even when $n \leq 2$ and $m = 1$: replace too long words abw by cw and add a word equation $ab = c$ where c is a new symbol[3].

[2] Note that collapsing negative literals $s \simeq x$ are not created. Equality resolution would instantiate x by s and create non-shallow literals if x occurs elsewhere at depth 1.

[3] This idea subsumes the undecidability proof (reducing Turing machines) given in [FS93] for the more general class $\mathcal{T}^{=}$ in which both sides of equations may have depth two.

(ii) For collapsing equations $s \simeq x$ where x is a variable, even when s has depth at most three and there are only unary function symbols (and hence at most one occurence of a variable per term) we can encode $a(b(x)) \simeq c(x)$ by $d(a(b(x)) \simeq x$ and $c(d(y)) \simeq y$ with a new symbol d.
(iii) With (non-linear) collapsing presentations E of depth two, we can encode $a(b(x)) \simeq c(x)$ by $g(b(x), y, a(y)) \simeq x$ and $c(g(x, x, y)) \simeq y$.
(iv) The word problem is undecidable in theories presented by equations with one ground side [Oya90].

However, in several other interesting directions there is still room. The theorems of section 3 might be applicable to other languages. Standard theories include equations $s \simeq t$ where s is linear and t is ground. The linearity requirement on s cannot be dropped (see (iv) above), but can it be weakened in some sense? How much progress can be made towards the whole class $\{s \simeq t \mid s \textit{ and } t \textit{ share only shallow variables}\}$, which is also known to be undecidable by (iv)? The only standard equations where s and t share non-shallow variables are the collapsing equations of case (ii) of the definition; can they be generalized? Is it necessary that the deep variable occurs only at depth two and below a unary symbol? Finally, it should also be possible to define an extension to Horn clauses of standard theories in the same way as shallow equations extend to Catalog.

References

[BG94] Leo Bachmair and Harald Ganzinger. Rewrite-based equational theorem proving with selection and simplification. *Journal of Logic and Computation*, 4(3):217–247, 1994.

[BGLS95] L. Bachmair, H. Ganzinger, Chr. Lynch, and W. Snyder. Basic paramodulation. *Information and Computation*, 121(2):172–192, 1995.

[BGW93a] Leo Bachmair, Harald Ganzinger, and Uwe Waldmann. Set constraints are the monadic class. In *Eighth Annual IEEE Symposium on Logic in Computer Science*, pages 75–83, Montreal, canada, June 19–23, 1993. IEEE Computer Society Press.

[BGW93b] Leo Bachmair, Harald Ganzinger, and Uwe Waldmann. Superposition with simplification as a decision procedure for the monadic class with equality. In *3rd Kurt Gödel Colloquium: Computational Logic and Proof Theory*, 1993. To appear.

[CHJ94] Hubert Comon, Marianne Haberstrau, and Jean-Pierre Jouannaud. Syntacticness, Cycle-Syntacticness and Shallow Theories. *Information and Computation*, 111(1):154–191, 1994.

[Chr92] Jim Christian. Some termination criteria for narrowing and E-narrowing. In Deepak Kapur, editor, *11th International Conference on Automated Deduction*, LNAI 607, pages 582–588, Saratoga Springs, New York, USA, June 15–18, 1992. Springer-Verlag.

[Der87] Nachum Dershowitz. Termination of rewriting. *Journal of Symbolic Computation*, 3:69–116, 1987.

[DJ90] Nachum Dershowitz and Jean-Pierre Jouannaud. Rewrite systems. In Jan van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B: Formal Models and Semantics, chapter 6, pages 244–320. Elsevier Science Publishers B.V., Amsterdam, New York, Oxford, Tokyo, 1990.

[Dom93] Eric Domenjoud. Shallow AC Theories. In *Abstracts of the CCL Workshop*, La Escala, Spain, 1993. Also INRIA Nancy (France) research report.

[FS93] Christian Fermüller and Gernot Salzer. Ordered Paramodulation and Resolution as a Decision Procedure. In A. Voronkov, editor, *Logic Programming and Automated Reasoning, Int. Conf.*, LNAI 698, pages 122–133. Springer-Verlag, 1993.

[Kir86] Claude Kirchner. Computing unification algorithms. In *1st IEEE Symposium on Logic in Computer Science*, pages 206–216, Cambridge, Mass, USA, June 4–7, 1986.

[KKR90] Claude Kirchner, Hélène Kirchner, and Michaël Rusinowitch. Deduction with symbolic constraints. *Revue Française d'Intelligence Artificielle*, 4(3):9–52, 1990.

[Mar91] Claude Marché. On ground AC-completion. In Ronald V. Book, editor, *Rewriting Techniques and Applications, 4th International Conference*, LNCS 488, pages 411–422, Como, Italy, April 10–12, 1991. Springer-Verlag.

[Mar94] Claude Marche. Normalised Rewriting and Normalised Completion. In *Ninth Annual IEEE Symposium on Logic in Computer Science*, Paris, France, July 1994. IEEE Computer Society Press, Los Alamitos, CA, USA.

[Nie95] Robert Nieuwenhuis. On Narrowing, Refutation Proofs and Constraints. In J. Hsiang, editor, *6th International Conference on Rewriting Techniques and Applications*, LNCS 914, pages 56–70, Kaiserslautern, Germany, April 4–7, 1995. Springer-Verlag.

[NR91] Paliath Narendran and Michael Rusinowitch. Any ground associative commutative theory has a finite canonical system. In *Fourth int. conf. on Rewriting Techniques and Applications*, LNCS 488, pages 423–434, Como, Italy, April 1991. Springer-Verlag.

[NR93] Paliath Narendran and Michael Rusinowitch. The Unifiability Problem in Ground AC Theories. In *Eighth Annual IEEE Symposium on Logic in Computer Science*, pages 364–370, Montreal, canada, June 19–23, 1993. IEEE Computer Society Press.

[NR95] Robert Nieuwenhuis and Albert Rubio. Theorem Proving with Ordering and Equality Constrained Clauses. *J. of Symbolic Computation*, 19(4):321–351, April 1995.

[Oya90] Michio Oyamaguchi. On the word problem for right-ground term-rewriting systems. *The Transactions of the IEICE*, 73(5):718–723, 1990.