Paramodulation with Built-In Abelian Groups*

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Abstract

A new technique is presented for superposition with first-order clauses with built-in abelian groups (AG). Compared with previous approaches, it is simpler, and no inferences with the AG axioms or abstraction rules are needed. Furthermore, AG-unification is used instead of the computationally more expensive unification modulo associativity and commutativity. Due to the simplicity and restrictiveness of our inference system, its compatibility with redundancy notions and constraints, and the fact that standard term orderings like RPO can be used, we believe that our technique will become the method of choice for practice, as well as a basis for new theoretical developments like logic-based complexity and decidability analysis.
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1 Introduction

It is crucial for the performance of a deduction system that it incorporates specialized techniques to work efficiently with standard algebraic theories, since a naïve handling of some axioms (like associativity and commutativity, AC) leads to an explosion of the search space. Paramodulation with built-in abelian groups (AG) has been investigated by many authors [6, 23, 11, 12, 10, 21, 22, 18]. This is not surprising since abelian groups are of course ubiquitous in many applications of (semi-)automated reasoning. But building-in AG is also attractive for at least two more reasons.

On the one hand, due to the fact that diophantine equation solving is easier in the integers than in the natural numbers, AG unification is easier than AC and AC1 unification. If all free symbols are constants, then there is one single most general AG unifier and the decision problem is polynomial, whereas for AC and AC1 the decision problems are NP-complete, and for AC there are exponentially many unifiers. Although with arbitrary free symbols the decision problem is NP-complete in all three cases, AG unification behaves better in practice. Also the number of unifiers is usually much smaller and not doubly exponential as for AC (see [1, 2] for surveys on these results).

Another aspect that makes building-in AG attractive is called *symmetrization* (e.g., by Le Chenadec in [6]): modulo abelian groups $(+,-,0)$, every ground equation can be written as $u + \ldots + u \simeq t$, where u is greater (w.r.t. the given term ordering $\succ$) than the summands in t. As we will see, this allows one to restrict inferences to this maximal summand and to avoid the prolific inferences with extended equations that appear in the AC case.

Symmetrisation is also exploited in Marché's framework for Knuth-Bendix completion of unit equations with built-in theories (ranging from AC to commutative rings) [11, 12]. His completion procedure decides the ground word problem modulo AG by building a finite convergent rewrite system. However, his procedure is not refutation complete for equations with variables: in many cases it fails since it cannot handle symmetrisation at the non-ground level.

Full first-order clauses are considered by Ganzinger and Waldmann in [10], where symmetrisation is also central. This work focusses not on AG, but on the more general theory of cancellative abelian monoids. Its completeness proofs and inference systems are extremely complex. It applies AC1 unification and *abstraction* rules, which, roughly, turn clauses like $C \vee f(s) \simeq t$ into $C \vee x \not\simeq s \vee f(x) \simeq t$, where x is a new variable; this of course increases the number of possible inferences on f. By specialising to torsion-free divisible abelian groups, AC-unification and inferences into variables can be avoided, but abstraction remains necessary [21, 22].

In Stuber's work on paramodulation for abelian groups represented as integer modules [18], symmetrisation is again crucial, but AG unification is not applied. Instead, AC unification is used, and hence paramodulation inferences with the AG axioms on the remaining clauses are needed. For example, refuting a clause like $f(-b+x+a) \not\simeq f(0)$

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requires inferences with the AG axioms, instead of directly finding the contradictory instance $b - a$ for x by AG-unification. Technically, even for the ground case, his inference rules and proofs are rather involved. In Stuber's PhD. Thesis [19], proofs for the ground case are given in a uniform framework for AG and several other commutative theories.

Here we apply a variant of Bachmair and Ganzinger's model generation technique [3], where the model is defined by rewriting, modulo associativity and commutativity of $+$, with the well-known convergent rewrite system R_{AG} for AG, plus a set of ground rewrite rules R that consists of symmetrised rules $u + \ldots^{n)} + u \rightarrow t$ and their *inverse* version $-u \rightarrow u + \ldots^{n-1)} + u - t$. Hence $\succ$ has to be an AC-compatible reduction ordering orienting these rules, which can be fulfilled by simple general-purpose orderings like RPO (this was already mentioned by Marché). This gives relatively simple completeness proofs for full first-order ground clauses. From our results it is easy to obtain a decision procedure for the satisfiability of arbitrary sets of ground clauses modulo AG.

For completely building-in AG at the non-ground level, and hence avoiding all inferences with the AG axioms by applying AG-unification, the main problem is: how to lift, to inferences on non-ground clauses C, the rewrite steps with $R \cup R_{AG}$ on ground instances $C\sigma$? The steps with R indeed become inferences, but for the steps with R_{AG} this is precisely what we want to avoid. The key ideas to our solution are roughly as follows. We keep non-ground clauses C fully simplified w.r.t. R_{AG} (which is a cheap and useful simplification anyway). Furthermore, in the completeness proofs we consider instances with simplified substitutions σ (extending some ideas from the *basic* superposition approach [15, 4]). Some steps with R_{AG} may then still be needed in $C\sigma$ at the frontier between C and σ. But a careful analysis of these steps reveals that they can be covered by considering inferences with AG-unification on adequate subterms.

Our AG-superposition inference rules have strong ordering restrictions implying that inferences only need to involve the maximal summands of the clause. This generalises standard superposition: summands play the role of terms.

Due to the simplicity and restrictiveness of our inference system, its compatibility with redundancy notions and constraints, and the fact that standard term orderings like RPO can be used, we believe that our techniques will become the method of choice for practice.

On the theoretical side, we expect that our techniques and results will also lead to logic-based decidability and complexity results, along the lines of, e.g., [5, 13, 14] [9, 8, 22].

2 Basic notions

We use the standard notation and terminology for terms and rewriting of [7], for inference rules, clauses and equality Herbrand models of [3, 17], and for constraints and constrained clauses of [15, 17].

We use the following terminology for positions p and q in a term t: we say that p is (strictly) *below* q if q is a (proper) prefix of p, and then q is *above* p. Similarly, p is *beside* q (or *disjoint* with q) if no one is a prefix of the other. We also say that p is *below* a function symbol f in t if $t|_q$ is headed by f for some q above p, and then p is *immediately below* f if p is $q.i$ for some natural number i.

The rewrite system R_{AG} consists of the following five rules:

$$\begin{array}{rcl} x + 0 & \rightarrow & x \\ -x + x & \rightarrow & 0 \\ -(-x) & \rightarrow & x \\ -0 & \rightarrow & 0 \\ -(x+y) & \rightarrow & (-x) + (-y) \end{array}$$

By AG we denote the set of seven equations consisting of these five rules (seen as equations) plus AC, the associativity and commutativity axioms for $+$. By $=_{AC}$ and $=_{AG}$ we denote the corresponding congruences on terms. In this paper, rewriting with a set of rules R is always considered *modulo* AC, that is, when writing $\rightarrow_R$, we mean the relation $=_{AC}\rightarrow_R=_{AC}$. By *free* function symbols we mean symbols different from $+$, $-$ and 0.

We sometimes write terms with $+$ in infix notation, without parenthesis. For example, $+(a, +(+(b, c), d))$ is written $a + b + c + d$. But we remark that this is only done when writing terms and that terms are not considered to be in *flattened* form. A *summand* is a term u headed by a free symbol. We write nu as a shorthand for $u + \ldots^{n)} + u$, and $-nu$ as a shorthand for $(-u) + \ldots^{n)} + (-u)$. A ground equation $nu \simeq n_1v_1 + \ldots + n_kv_k$ in normal form w.r.t. R_{AG} is said to be in *reductive form* if $n > 0$, the n_i are non-zero integers, and u and the v_i are summands with $u \succ v_i$. The (logically equivalent) *inverse reductive form* of this equation is $-u \simeq (n-1)u - n_1v_1 - \ldots - n_kv_k$.

For every equation $s \simeq t$, its reductive form can be obtained by normalising $s + (-t) \simeq 0$ w.r.t. R_{AG} into $n_1u_1 + \ldots + n_ku_k \simeq 0$ where, say, u_1 is the maximal summand, and then, if n_1 is positive, the reductive form is $n_1u_1 \simeq -n_2u_2 - \ldots - n_ku_k$; otherwise, it is $-n_1u_1 \simeq n_2u_2 + \ldots + n_ku_k$. Note that the unary minus operator $-$ is overloaded in our notation since it is applied as well to coefficients (but remember that coefficients are not part of our logical language but just a shorthand in our notation).

Example 1 *If* $a \succ b \succ c$ *then the equation* $(-a) + c + 0 + (-(-c)) + (-b) \simeq (-c) + a + b + 0$ *is equivalent to*

$(-a)+(-a)+c+c+c+(-b)+(-b) \simeq 0$, *written shortly* $-2a + 3c - 2b \simeq 0$, *and becomes in reductive form* $2a \simeq 3c - 2b$, *and in inverse reductive form* $-a \simeq a - 3c + 2b$. $\Box$

In this paper, we assume that $\succ$ is an AC-compatible reduction ordering that is total up to $=_{AC}$ on the set of ground terms, and that orients all rules of R_{AG} and all equations in reductive form and inverse reductive form from left to right.

Such an ordering can easily and efficiently be defined in practice. One way to do so is to define $s \succ t$ if *flat*$(s) \succ_{rpo}$ *flat*(t), where $\succ_{rpo}$ is a recursive path ordering (RPO) [7] with a total precedence $\succ_{\mathcal{F}}$ such that $f \succ_{\mathcal{F}} - \succ_{\mathcal{F}} + \succ_{\mathcal{F}} 0$ for all free symbols f and where all symbols have a lexicographic status, except $+$, whose status is multiset (note that, since $+$ is the only AC symbol, this is monotonic; in the presence of more AC symbols, other orderings would have to be used).

Example 2 *Equations in reductive form can adequately be used as terminating rewrite rules. Assume we have* $a \succ b \succ c$ *and the equation (in reductive form)* $3a \simeq -b+c$. *It can be applied either as it is, or in its inverse form* $-a \simeq 2a+b-c$.

For example, $4a$ *is AG-equivalent by this equation to* $-2a - 2b + 2c$. *Let us prove it by rewriting both terms into their respective normal forms. On the one hand, by simply applying the equation to three of its four* a*'s,* $4a$ *rewrites into the normal form* $a - b + c$. *On the other hand, by applying the inverse form,* $-2a - 2b + 2c$ *rewrites into* $-a - 2b + 2c + 2a + b - c$ *which simplifies with* R_{AG} *into* $a - b + c$.

Note that normal forms w.r.t. both ways of rewriting with such equations $nu \simeq v$ *will always have a positive number of* u*'s between* 0 *and* $n-1$, *and that the inverse kind of steps is not needed if* $n = 1$. *The ground inference rule of AG-superposition that is given below in fact simply corresponds to both these ways of rewriting.* $\Box$

3 Ground Horn Case

Here we first introduce part of our techniques on the simple subcase of ground Horn clauses. We assume all equations in clauses to be eagerly maintained in reductive form, and moreover we assume negative equations $0 \not\simeq 0$ to be removed eagerly from all clauses.

In the inference rule below, let the positive equation of the left premise in reductive form be $nu \simeq v$. Then $l \simeq r$ stands either for $nu \simeq v$ itself or for its inverse form $-u \simeq (n-1)u - v$. The second kind of inference will sometimes be called *inverse AG-superposition*, and is needed only if $n > 1$.

$$\textit{AG-superposition:} \qquad \frac{C \vee l \simeq r \qquad D[l]_p}{C \vee D[r]_p}$$

Here $D|_p$ refers to a subterm of D modulo AC (formally, s is such a subterm if $D =_{AC} D'$ and s is $D'_{p'}$ for some p'). The ordering restrictions of AG-superposition are such that inferences are needed only if they take place *with* the strictly maximal summand and *on* a maximal summand (that is strictly maximal if it occurs in a positive equation). Formally this becomes the following. If s is a summand and C is a clause, then by $s \succ C$ we denote the fact that $s \succ t$ for every summand t occuring in C. Then, these inferences are needed only if $u \succ C$ (and remind that, by expression in reductive form, also $l \succ r$), and furthermore, if D is $D' \vee ms \simeq t$ (in reductive form) with $D|_p$ in ms, then the inference is needed only if $s \succ D'$; if D is $D' \vee ms \not\simeq t$ (in reductive form) with $D|_p$ in ms, then the inference is needed only if $s \succeq D'$. Note that inverse AG-superposition is needed only on proper subterms of summmands s since by notation the term $-u$ cannot occur elsewhere.

3.1 Completeness for the ground Horn case

We now use multiset extensions for lifting the ordering $\succ$ on terms to orderings on ground equations (in reductive form) and clauses in the usual way. Let C be a ground clause, and let $emul(s \simeq t)$ be $\{s, t\}$ if $s \simeq t$ is a positive equation in C, and $\{s, s, t, t\}$ if it is negative. Then we define the ordering $\succ_e$ on (occurrences of) ground equations in a clause by $e \succ_e e'$ if $emul(e) \succ_{mul} emul(e')$. Similarly, $\succ_c$ on ground clauses is defined $C \succ_c D$ if $mse(C)\ (\succ_{mul})_{mul}\ mse(D)$, where $mse(C)$ is the multiset of all $emul(e)$ for ocurrences e of equations in C.

We now show how to construct a model for sets S of ground Horn clauses closed under AG-superposition and where $\Box \notin S$. As usual (see [3]), in order to construct the model we will generate a set of rewrite rules R by induction on $\succ_c$. But here the model will contain as well the rules of R_{AG}, and, as said, all rules will be applied modulo AC.

Definition 3 *Let* S *be a set of ground Horn clauses (in reductive form), and let* C *be a clause in* S *of the form* $C \vee nu \simeq r$. *Then* C generates *the rule* $nu \to r$ *if the following three conditions are satisfied:*

1. $(R_C \cup AG)^* \not\models C$
2. $u \succ r$ *and* $u \succ C$
3. nu *is irreducible by* R_C

where R_C *is the set of rules generated by clauses of* S *smaller than* C *w.r.t.* $\succ_c$. *Furthermore, if* C *generates* $nu \to r$ *with* $n > 1$, *in addition* C *generates its inverse form* $-u \to (n-1)u - r$. *The set of all rules generated by clauses in* S *is denoted by* R.

All rules in $R \cup R_{AG}$ are oriented w.r.t. $\succ$, and hence $R \cup R_{AG}$ is terminating. Also, for all rules $l \to r$ in R, the term l is irreducible by $R \setminus \{l \to r\}$. Furthermore, R_{AG} is well-known to be confluent. Finally, the (extended) critical pairs between R_{AG} and R are easily shown to be joinable. This gives us the following:

Theorem 4 *$R \cup R_{AG}$ is terminating and confluent modulo AC.*

Theorem 5 *AG-superposition is refutation complete for ground Horn clauses.*

Proof: Let S be a set of ground Horn clauses (whose equations are in reductive form) such that S is closed under AG-superposition and $\square \notin S$. We prove that then S is satisfiable by showing that the equality Herbrand interpretation I defined as the congruence generated by $R \cup AG$, that is, $I = (R \cup AG)^*$, is an AG-model for S. Note that, since $R \cup R_{AG}$ is terminating and confluent, $I \models s \simeq t$ if, and only if, $s \to^*_{R \cup R_{AG}} \leftarrow^*_{R \cup R_{AG}} t$. We proceed by induction on $\succ_c$, that is, we derive a contradiction from the existence of a minimal (w.r.t. $\succ_c$) clause D (in reductive form) of S such that $I \not\models D$.

Let s be the maximal summand in D. Then D is either of the form $D' \vee ms \simeq t$ with $s \succ D'$ (a), or else it is $D' \vee ms \not\simeq t$ with $s \succeq D'$ (b). We first show that in both cases ms is reducible by R.

(a) Since $I \not\models D$, it has generated no rule. This can only be because ms is reducible by R_D. (b) Since $I \not\models D$, we have $I \models ms \simeq t$. Therefore $ms \simeq t$ is joinable by $R \cup R_{AG}$, and since $ms \succ t$, the maximal side ms, which is in normal form w.r.t. R_{AG}, has to be reducible by R. The rule reducing ms has been generated by a clause of the form $C \vee nu \simeq v$, and there exists an inference by AG-superposition (or inverse AG-superposition)

$$\frac{C \vee l \simeq r \qquad D[l]_p}{C \vee D[r]_p}$$

where $I \not\models C \vee D[r]_p$ and D is larger w.r.t. $\succ_c$ than the reductive form of $C \vee D[r]_p$, contradicting the minimality of D. $\square$

3.2 Selecting negative literals

It is easy to see that our inference rules remain complete with *selection* of negative literals (see, e.g., [3]), where it is assumed that in each clause with a non-empty antecedent one of its negative equations has been *selected*. In the Horn case this leads to positive unit strategies (and in the non-Horn case to positive strategies): all left premises of AG-superpositions are positive unit clauses, and the only inferences involving non-unit clauses are AG-superpositions on the selected negative equation. The following result is a simple modification of the previous one; it is immediate if we define R such that only unit clauses generate rules:

Theorem 6 *AG-superposition with selection is refutation complete for ground Horn clauses.*

3.3 Deciding the satisfiability of sets of ground clauses

From our results it is easy to obtain a decision procedure for the satisfiability of arbitrary sets of ground clauses modulo AG.

For the Horn inference system with selection, clearly each inference of $l \simeq r$ on a clause D produces a smaller clause D'. Furthermore, D is a logical consequence (modulo AG) of the smaller clauses $l \simeq r$ and D', i.e., D has become *redundant*.

It is not difficult to see that in our procedure such redundant clauses can be removed without loss of completeness (redundant clauses never generate any rules, and in the proof of the compleness theorem, they are never the smallest counter example; see, e.g., [3] for details). Hence after each inference, the clause set decreases w.r.t. the multiset extension of the ordering and hence the process terminates, thus deciding satisfiability.

A decision procedure for the satisfiability of sets of arbitrary ground clauses modulo AG can be obtained by first transforming into Horn clauses (where $S \cup C \vee A_1 \vee \ldots \vee A_n$ is split into the disjunction of sets S_i of the form $S \cup C \vee A_i$; then S is satisfiable if some of the S_i is).

Theorem 7 *AG-superposition with selection decides the satisfiability of sets of ground clauses modulo AG.*

4 Clauses with variables

In this section, we adapt the inference rule in order to deal with equality constrained clauses with variables, where constraints are conjunctions of equalities $s = t$. As usual, the semantics of a constrained clause $C \mid T$ is the set of its ground instances, that is, the instances $C\sigma$ such that $T\sigma$ evaluates to true if $=$ is interpreted as $=_{AG}$. Then σ is called a solution for T. The empty clause with a constraint T is hence a contradiction, denoted simply by $\square$, if, and only if, $T\theta$ is true for some ground θ.

Roughly, the following is needed: if for clauses $C_1 \mid T_1$ and $C_2 \mid T_2$ there is a ground inference between instances $C_1\sigma$ and $C_2\sigma$ producing D, then there exists an inference by the non-ground version of the inference rule between $C_1 \mid T_1$ and $C_2 \mid T_2$, producing a conclusion $D' \mid T$ such that D is an instance of $D' \mid T$.

As we will see in Section 4.6, for completeness it suffices to be able to do this only for instances with σ of C_1 and C_2 that are in some sense irreducible with respect to R, where R is the set of rules generated in a way similar to the previous section (but now by ground instances of clauses), and where irreducibility is not required for some of the parts of σ that disappear under normalisation w.r.t. R_{AG} of $C\sigma$.

In the following, we assume that all equations in clauses are written as $e \simeq 0$ where e is in normal form w.r.t. R_{AG}. Note that each equation can be written like this in two ways, since $-e \simeq 0$ is equivalent to $e \simeq 0$. Also for all substitutions σ, we assume w.l.o.g. that $x\sigma$ is in normal form w.r.t. R_{AG} for all x.

In order to define the non-ground inference rules, we now analyse for each inference rule how their premises have to be expressed. For simplicity, we omit the constraints; let us only remark at this point that the amount of possible inferences can be further restricted in many different ways by checking their compatibility with the constraints.

4.1 Left premises of AG-superposition

Intuitively, our aim is the following. Let C be a clause with a positive equation $e \simeq 0$, and assume its ground instance $C\sigma$ generates a rule $nu \to r$ with $n > 0$, and $C\sigma$ is the left premise of an AG-superposition. Then for the non-ground case we have to be able to express $e \simeq 0$ as $s \simeq t$ such that the terms $s\sigma$ and $t\sigma$ have, respectively, nu and r as normal forms w.r.t. R_{AG}, and then perform the inference with AG-unification between s and the corresponding subterm of the right premise. *Orienting* $e \simeq 0$ as $s \simeq t$ in this way may require to *split* the variables of e into two parts:

Example 8 *Consider the clauses $a + 2x \simeq b$ and $f(4a) \not\simeq f(a + b - 2c)$, where $a \succ b \succ c$. For the instance where $x \mapsto a + c$, the equation $a + 2x \simeq b$ generates the rule $3a \to b - 2c$, and a ground inference produces $f(a + b - 2c) \not\simeq f(a+b-2c)$, which in reductive form becomes $0 \not\simeq 0$ and hence the empty clause.*

To cover this inference at the non-ground level, x has to be split into y (which, roughly, will contain the maximal summands in $x\sigma$) and z (for the remaining summands), and $a + 2x \simeq b$ is oriented as $a + 2y \simeq b - 2z$. Then the non-ground inference unifies $a + 2y$ with three of the a's in $f(4a)$ and produces $f(a + b - 2z) \not\simeq f(a + b - 2c)$, and AG-unification finds its reductive form $0 \not\simeq 0$. □

W.l.o.g., let e be of the form $n_1s_1 + \ldots + n_ps_p + m_1x_1 + \ldots + m_qx_q$ where the s_i are non-variable summands, the x_i are variables, and the n_i and m_i are non-zero integers. By splitting each x_i into two new variables y_i and z_i, and splitting the summands into two disjoint sets, the equation $e \simeq 0$ can be written as an equivalent equation $s \simeq t$ of the form

$$n_1s_1 + \ldots + n_ks_k + m_1y_1 + \ldots + m_qy_q \simeq$$
$$-n_{k+1}s_{k+1} - \ldots - n_ps_p - m_1z_1 - \ldots - m_qz_q$$

Definition 9 *In the following, we call each such an equation $s \simeq t$ an* orientation *for $e \simeq 0$ (or $-e \simeq 0$), and we call the corresponding constraint*

$$x_1 = y_1 + z_1 \wedge \ldots \wedge x_p = y_p + z_p$$

the splitting constraint *(which will usually be denoted by τ).*

It is not difficult to see that this notion of orientation fulfils what we wanted: if $e\sigma \simeq 0$ generates a rule $nu \to r$ then indeed for some orientation $s \simeq t$ of $e \simeq 0$ and some extension of σ in order to include the y_i and z_i, the terms $s\sigma$ and $t\sigma$ have, respectively, nu and r as normal forms w.r.t. R_{AG}. This we will see in detail in in the completeness proofs, but let us give here some intuition about why it is true. There is some subset $\{s_1, \ldots, s_k\}$ of $\{s_1 \ldots s_p\}$ such that for every summand s in $\{s_1, \ldots, s_k\}$ the normal form w.r.t. R_{AG} of $s\sigma$ is u. Furthermore, for each x_i in $x_1 \ldots x_p$, the normal form of $x_i\sigma$ is of the form $l_iu + t_i$ where l_i is an arbitrary integer, such that $n_1 + \ldots + n_k + m_1l_1 + \ldots + m_ql_q = n$. Clearly, if $y_i\sigma$ is l_iu and $z_i\sigma$ is t_i, the terms $s\sigma$ and $t\sigma$ have, respectively, nu and r as normal forms w.r.t. R_{AG}.

Of course, the fewer orientations have to be considered for a given equation $e \simeq 0$, the fewer inferences will be performed, which is better for efficiency in practice. Indeed, a little more careful analysis reveals that a large number of optimizations are possible. In Section 6 we will mention some of them. It is also important for efficiency to exploit the unifiability and ordering restrictions as the strongest possible filters to avoid redundant inferences with such orientations $s \simeq t$. For example, apart from the unification restrictions of the inference itself, where s is unified with a subterm of the right premise, in the above orientation we can add $s_1 = \ldots = s_k$ to the constraint. In Section 6 the problem of checking the ordering restrictions is addressed.

4.2 Left premises of inverse AG-superposition

Example 10 *Consider $a \succ b \succ c$ and the clauses $f(-a + b + c) \not\simeq f(a - c)$ and $2x \simeq b$. With the instance $x \mapsto a - c$, the second equation becomes $2a \simeq b + 2c$. At the ground level, there exists an inference with inverse AG-superposition which produces $f(a - c) \not\simeq f(a - c)$. At the non-ground level, x is split into $y + z$, and the inference is performed with $-y \simeq y + 2z - b$, and we obtain $f(a + c + 2z) \not\simeq f(a - c)$. From this, by AG-unification the instance $z \mapsto -c$ is found and the empty clause is obtained.* □

Definition 11 *Let e (or $-e$) be a term of the form $x_1 + \ldots + x_n + v$, where v contains only negative variables and (positive or negative) summands, and let e' be e but where every occurrence of x_i at top-level position has been replaced by $y_i - z_i$, where y_i and z_i are new variables. The splitting constraint τ is $x_1 = y_1 - z_1 \wedge \ldots \wedge x_n = y_n - z_n$. Hence e' is of the form $y_1 - z_1 + \ldots + y_n - z_n + v$.*

Then, if e' is of the form $s + e''$ where s is a positive summand, then $-s \simeq e''$ is an inverse orientation *for $e \simeq 0$ with splitting constraint τ*

Furthermore, if e' is of the form $w + e''$ where w is a variable (i.e., w is some y_i), then $-w_1 \simeq w_2 + e''$ is an inverse orientation *for $e \simeq 0$ with splitting constraint $\tau \wedge w = w_1 + w_2$.*

Finally, if e' is of the form $-w+e''$ where w is a variable, but none of the z_i, then $w_1 \simeq -w_2 + e''$ is an inverse orientation *for $e \simeq 0$ with splitting constraint $\tau \wedge w = w_1 + w_2$.*

The splitting of the variable w in the second case of inverse orientation is the one illustrated by the previous example. The splittings of the constraint τ are due to technical reasons that will become clear in the completeness proof.

4.3 Right premises for AG-superposition

Example 12 *Consider $a \succ b \succ c$, the left premise $3a \simeq b$, and the right premise $f(2x, x) \not\simeq f(a + b + 2c, 2a + c)$. With the instance $\{x \mapsto 2a+c\}$, the right premise is $f(4a+2c, 2a+c) \not\simeq f(a+b+2c, 2a+c)$ which gives in one ground inference $f(a + b + 2c, 2a + c) \not\simeq f(a + b + 2c, 2a + c)$, which in reductive form is $0 \not\simeq 0$.*

Now the question is: how can we, at the nonground level, perform the inference into the term $2x$? (which is the term t in the definition below). By splitting x into only the two variables y and z, one gets $f(y + y + z + z, y + z) \not\simeq f(a+b+2c, 2a+c)$, for which the ground inference cannot be lifted: it is impossible to split $y + y + z + z$ into $t_1 + t_2$ such that $t_1\sigma$ is $3a$, and $t_2\sigma$ is $a + 2c$.

As we will see, by splitting x into three variables y, y', and z, lifting is always possible. In our example, then one gets $f(2y + 2y' + 2z, y + y' + z) \not\simeq f(a + b + 2c, 2a + c)$, where $2y + 2y' + 2z$ is split into $2y + y'$ and $y' + 2z$ (these are the terms t_1 and t_2 in the definition below). Then an AG-unifier of $3a$ and $2y + y'$ instantiates y and y' with a, and the conclusion of the non-ground inference is $f(a + b + 2z, 2a + z) \not\simeq f(a+b+2c, 2a+c)$, which by one more AG-unification, where z is instantiated with c, becomes $0 \not\simeq 0$. □

Definition 13 *Let t be a non-variable subterm of e in a literal $e \simeq 0$ or $e \not\simeq 0$ where t is not immediately below an AG-symbol and the head symbol of t is free or $+$. W.l.o.g., let t be of the form*

$$n_1 s_1 + \ldots + n_p s_p + m_1 x_1 + \ldots + m_q x_q + t'$$

where all s_i are summands, all x_i variables, all n_i and m_i are positive coefficients, and t' contains only negative summands and variables.

Then $t_1 + t_2$ is a splitting *for t if t_1 is a term whose head symbol is free or $+$ of the form*

$$k_1 s_1 + \ldots + k_p s_p + m_1 y_1 + \ldots + m_q y_q + l_1 y'_1 + \ldots + l_q y'_q$$

where $0 \leq k_i \leq n_i$ and $0 \leq l_i < m_i$, and t_2 is

$$(n_1 - k_1)s_1 + \ldots + (n_p - k_p)s_p + m_1 z_1 + \ldots + m_q z_q + l'_1 y'_1 + \ldots + l'_q y'_q + t'$$

where l'_i is 0 if l_i is 0 (i.e., then x_i is split only into two parts y_i and z_i), and l'_i is $m_i - l_i$ otherwise. Again we denote by τ the corresponding splitting constraint.

As before, other restrictions apply; for example it is also not necessary to consider t_1 of the form $y_i + y'_i$ (i.e., if m_i is 1).

4.4 Right premises for inverse AG-superposition

Definition 14 *Let t be a non-variable subterm of e in a literal $e \simeq 0$ or $e \not\simeq 0$ where t is not immediately below an AG-symbol.*

If t is of the form $-s+t'$, where s is a summand, then $t_1 + t_2$ is an inverse splitting *for t with empty splitting constraint τ if t_1 is $-s$ and t_2 is t'.*

If t is of the form $-x + t'$, where x is a variable, then $t_1 + t_2$ is an inverse splitting *for t if t_1 is $-y$ and t_2 is $-z + t'$, and the splitting constraint τ is $x = y + z$.*

4.5 Inference rules for clauses with variables

The AG-superposition rule below can be applied in two different ways as in the ground case. Either, (i) in the left premise, $C \vee l \simeq r$ denotes that for the clause $C \vee e \simeq 0$, the equation $l \simeq r$ is an orientation of $e \simeq 0$ and in the right premise, $D[t_1+t_2]_p$ denotes that $D|_p$ is a non-variable term t that is not immediately below an AG symbol with a splitting $t_1 + t_2$, or else (ii) they denote inverse orientations and splittings. In all cases, τ is the conjunction of the splitting constraints of the two premises. Let $\mathcal{H}$ denote the following two rules for constrained clauses:

AG-superposition:

$$\frac{C \vee l \simeq r \ \mid\ T \qquad D[t_1 + t_2]_p \ \mid\ T'}{C \vee D[r + t_2]_p \ \mid\ T \wedge T' \wedge l = t_1 \wedge \tau}$$

AG-zero-instance:

$$\frac{C \vee e \not\simeq 0 \ \mid\ T}{C \ \mid\ T \wedge e = 0}$$

The ordering restrictions of the superposition rules are the ones corresponding to the ground rules. More precisely, a superposition with premises $C_1 \mid T_1$ and $C_2 \mid T_2$ and conclusion $D \mid T$ is needed if, for some (ground) solution θ of T, there is a ground inference (satisfying the ordering requirements) between the reductive forms of $C_1\theta$ and $C_2\theta$, and with conclusion $D\theta$. The AG-zero-instance rules can be restricted to maximal equations of the clause.

4.6 Completeness

As said before, we will deal with instances with ground substitutions σ of clauses C that are in some sense irreducible with respect to R, where R is the set of rules generated in a way similar to how it was done in the previous section, and where irreducibility is not required for some of the parts of σ that disappear under normalisation w.r.t. R_{AG} of $C\sigma$. We now formally define when a pair (C, σ) is irreducible in this sense[1].

We first define irreducibility for pairs (s, σ) where s is a term and σ a substitution, both in normal form w.r.t. R_{AG}. Then still $s\sigma$ needs not be in normal form w.r.t. R_{AG}, because the following two kinds of steps may be applicable: (i) if x is a variable occurring imediately below a $-$ in s and $x\sigma$ is headed by $+$, then this $-$ is "moved inwards"; (ii) after this, some "complementary" pairs u and $-u'$ below the same $+$ are cancelled if u and u' are summands with $u =_{AG} u'$.

Below we model the cancellation of such complementary summands by *cancellation* sequences $(p_1, q_1), \ldots, (p_n, q_n)$ of pairs of positions where they are eliminated. Furthermore, instead of just eliminating the complementary summands, we replace both of them by a 0 in order to preserve the meaning of positions during the process. Once this is done, the irreducibility of (s, σ) (with respect to the given sequence) amounts to the irreducibility of some of the terms at positions which in s were variables.

Definition 15 *Let s be a term. The* minus inwards normal form *of s is its normal form with respect to $-(-x) \to x$, $-0 \to 0$, and $-(x + y) \to (-x) + (-y)$.*

The term s is in zero normal form *if it is in minus inwards normal form and rewrites into its AG-normal form by eliminating all its zeros by the rules $0 + x \to x$, $x + 0 \to x$ and $-0 \to 0$.*

In both cases we refer to standard rewriting (i.e. not modulo AC).

Definition 16 *Let s be a term and let t and u be occurrences in s of subterms of s. Then this occurrence of t is an* AG-context in s *if it is not immediately below a $+$ or a $-$. It is the* AG-context of u in s *if it is the smallest AG-context containing u. u is a* top-level summand (variable) *in s if it is a summand (variable) and its AG-context is s itself. An occurrence of a summand or variable in s is* negative *if it is immediately below a $-$ and* positive *otherwise. We sometimes use these notions as well for positions in s; e.g., if $s|_p$ is an AG-context in s then p is called an AG-context position in s.*

[1] The expert reader will note that the necessary result of Lemma 25 below does not hold with more standard notions of irreducibility: large rules could reduce the substitution part of smaller ones, if this substitution part disappears under normalisation w.r.t. R_{AG}.

Definition 17 *Let Seq be a sequence $(p_1, q_1), \ldots, (p_n, q_n)$ of pairs of positions and let s be a ground term. Then Seq is a* cancellation sequence for s *if there exist ground terms $s_0, \ldots, s_n$ such that:*

- *s_0 is the minus inwards normal form of s*
- *for all i in $\{1 \ldots n\}$*
  - *p_i and q_i are positions in the same AG-context of s_{i-1} and p_i is not immediately below a $-$*
  - *$s_{i-1}|_{p_i}$ is a summand and $-s_{i-1}|_{p_i} =_{AG} s_{i-1}|_{q_i}$*
  - *s_i is $s_{i-1}[0]_{p_i}[0]_{q_i}$*

The term s_n is called the cancellation of s by Seq. *The subsequence s_{i-1}, s_i is named a* cancellation step. *The set of all cancelation sequences for s is noted by $\mathcal{S}(s)$. A sequence Seq in a set S is called* maximal *in S if it is not a proper prefix of any other sequence in S.*

Definition 18 *Let R be a rewrite system, let s be a ground term and let p be a position in s whose AG-context position is q. Then u is the* maximal summand subject to rewriting in s at position p *if u is the maximal (w.r.t. $\succ$) term such that the AG-normal form of $s|_q$ can be written as $nu + t$ where nu is reducible by R.*

Definition 19 *Let R be a rewrite system, let v be a ground summand, let s be a ground term, and let Seq be sequence $(p_1, q_1), \ldots, (p_n, q_n)$ in $\mathcal{S}(s)$.*

For a given position p in s, let u_p be v if p is a top-level position, and let u_p be the maximal summand subject to rewriting in s at p otherwise.

Then Seq is is said to be v-restricted to R in s, *if, for all i in $\{1 \ldots n\}$, $\mathrm{nf}_{R_{AG}}(s|_{p_i}) \succeq u_{p_i}$.*

If, in particular, v is the maximal summand in the AG-normal form of s then Seq is simply called restricted to R in s.

The set of cancellation sequences for s restricted to R is denoted by $\mathcal{S}(s, R)$, and the v-restricted one is denoted by $\mathcal{S}(s, R, v)$.

In the following, if s is a ground term and R is a rewrite system, by $R_{\prec s}$ (resp. $R_{\preceq s}$) we denote the set $\{l \to r \mid l \to r \in R \text{ and } s \succ l \text{ (resp. } s \succeq l)\}$.

If, moreover, s is a summand, by $R_{/s}$ we denote the set $R \setminus \{ns \to r | n > 0 \wedge (ns|tor) \in R\}$.

Definition 20 *Let s be a term in zero normal form. Let σ be a substitution in AG-normal form. Let R be a rewrite system. Let v be a ground summand. Let Seq be a sequence in $\mathcal{S}(s\sigma, R, v)$. Let s_1 be the cancellation of $s\sigma$ by Seq.*

For a given position p in s, let u be the maximal summand subject to rewriting in s at p if p is not a top-level position and let u be v otherwise.

Furthermore, let $R^{\preceq}_{\overline{p}}$ be $R_{\preceq-u}$ if the AG-context of p is λ, or p is a variable position in s, and $(R_{\preceq-u})_{/u}$ otherwise. Let $R^{\prec}_{p}$ be $R_{\prec u}$ if the AG-context of p is λ, and $R^{\preceq}_{\overline{p}}$ otherwise. Let $R^{\preceq/}_{\overline{p}}$ be $R_{\preceq-u}$ if p is a variable position, and $(R_{\preceq-u})_{/u}$ otherwise.

Then (s,σ) is $(v\succeq)$-irreducible *(resp.* $(v\succ)$-irreducible *or* $(v\succeq/)$-irreducible*) w.r.t. R by Seq if:*

- *for each positive variable position p in s such that p is also a position in s_1, the AG-normal form of $s_1|_p$ is irreducible w.r.t. $R^{\preceq}_{\overline{p}}$ (resp. $R^{\prec}_{p}$ or $R^{\preceq/}_{\overline{p}}$)*
- *for each position p in s such that $s|_p$ is of the form $-x$ for some variable x, the AG-normal form of $-s_1|_p$ is irreducible w.r.t. $R^{\preceq}_{\overline{p}}$ (resp. $R^{\prec}_{p}$ or $R^{\preceq/}_{\overline{p}}$).*

The pair (s,σ) is irreducible *w.r.t. R by Seq if it is $(v\succeq)$-irreducible w.r.t. R by Seq where v is the maximal summand in the AG-normal form of $s\sigma$.*

Furthermore, the pair (s,σ) is irreducible *(or* $(v\succ)$-irreducible, *or* $(v\succeq)$-irreducible, *or* $(v\succeq/)$-irreducible*) w.r.t. R if it is by some Seq.*

Definition 21 *Let R be a rewrite system, and let C be a clause and σ a substitution, both in normal form w.r.t. R_{AG}. Then (C,σ)* is irreducible *w.r.t. R iff for all its literals $e\simeq 0$ or $e\not\simeq 0$, the pair (e,σ) is irreducible w.r.t. R.*

4.7 Model generation

We now adapt the notion of rule generation to the non-ground case. Instead of by ground clauses in reductive form, now the rules are generated by the reductive forms of instances $C\sigma$ of clauses $C \mid T$ of S, where (C,σ) is irreducible:

Definition 22 *Let S be a set of constrained Horn clauses, let $C \mid T$ be a clause in S with a ground instance $C\sigma$, and let D be the reductive form of $C\sigma$, where D is of the form $D'\vee nu\simeq r$. Then D* generates *the rule $nu\to r$ if the following four conditions are satisfied:*

1. *$(R_D\cup AG)^*\not\models D$*
2. *$u\succ r$ and $u\succ D$*
3. *nu is irreducible by R_D*
4. *(C,σ) is irreducible w.r.t. R_D.*

where R_D is the set of rules generated by reductive forms of instances of clauses of S that are smaller than D w.r.t. $\succ_c$. Furthermore, for each generated rule $nu\to r$ with $n>1$, in addition the rule $-u\to(n-1)u-r$ is generated. The set of all rules generated by clauses in S is denoted by R.

Definition 23 *Let Seq be a sequence $(p_1,q_1),\dots,(p_n,q_n)$, and let p be a position.*

- *p is* preserved *in Seq if it is below no p_i or q_i.*
- *p is* untouched *in Seq if it is preserved and in adition no p_i or q_i is below it.*
- *p is* independent *in Seq iff for all i in $\{1,\dots,n\}$ either both p_i and q_i are below p, or both are beside p.*

Theorem 24 *$\mathcal{H}$ is refutation complete for constrained Horn clauses.*

Proof: We proceed in a top-down fashion, i.e., useful lemmas are given after their need has been motivated (see `www.lsi.upc.es/~roberto` for missing proofs). Let S be the closure under $\mathcal{H}$ of a set of Horn clauses S_0 without constraints, and assume $\square\notin S$. Again we prove that then $I=(R\cup AG)^*$ is a model for S. But now this is done in two steps. Let $Ir_R(S)$ denote the set of ground instances $C\sigma$ of $C \mid T$ in S where (C,σ) is irreducible w.r.t. R.

1. First, it is proved that $I\models Ir_R(S)$, in a very similar way as for the ground case, by deriving a contradiction from the existence of such a $C\sigma$ whose reductive form is minimal w.r.t. $\succ_c$. This is done in detail below.
2. Second, if $I\models Ir_R(S)$, then, since $S_0\subseteq S$, $I\models Ir_R(S_0)$, which implies $I\models S_0$, since $I\models R$ and by Lemma 33 $Ir_R(S_0)$ includes all instances $C\sigma$ such that $x\sigma$ is irreducible w.r.t. R for all variables x of C, and S_0 has no constraints. Finally, since S is a logical consequence of S_0, this gives $I\models S$.

We now prove the first part. Let $Dred$ be the minimal, w.r.t $\succ_c$, reductive form of some $D\sigma$ in $Ir_R(S)$ that is an instance of a clause $D \mid T_D$ such that $I\not\models Dred$.

If $Dred$ is a disjunction of literals of the form $0\not\simeq 0$, then an inference by AG-zero-instance applies to any one of these literals, eliminating it. Otherwise, as in the ground case, $Dred$ is of the form $Dred'\vee t\simeq 0$ or $Dred'\vee t\not\simeq 0$, and t is reducible at position p by a rule of the form $nu\to r'$ or $-u\to(n-1)u-r'$. Moreover, p is in the maximal summand of $Dred$, and w.l.o.g. we can assume that, if p' is the AG-context position of p, then the maximal summand subject to rewriting in $Dred|_{p'}$ is u. Therefore, D is of the form $D'\vee e\simeq 0$ or $D'\vee e\not\simeq 0$, where t is a normal form of $e\sigma$ w.r.t. R_{AG}.

The rule reducing t_p has been generated by the reductive form $Cred$ of an instance $C\sigma$ of a clause $C \mid T_C$. Let C be of the form $C'\vee e\simeq 0$. Now, we distinguish two cases:

- If the rule reducing $t|_p$ is $nu\to r'$, then, by lemma 26, there exists an orientation $l\simeq r$ of $e\simeq 0$ such that $nf_{R_{AG}}(l\sigma)$ is nu and $nf_{R_{AG}}(r\sigma)$ is r'. Moreover,

(r, σ) is $(u \succ)$-irreducible w.r.t. R, and u is not at a top-level position in $r\sigma$. By lemma 28, there exists a splitting $s[t_1 + t_2]_q$ at some AG-context position q of s. Moreover, $(s[r + t_2]_q, \sigma)$ is irreducible w.r.t. R, and $nf_{R_{AG}}(s[r + t_2]_q\sigma)$ is $nf_{R_{AG}}(t[r']_p)$. Therefore, the following inference exists

$$\frac{C' \vee l \simeq r \ \mid \ T_C \qquad D' \vee s[t_1 + t_2]_q \ \mid \ T_D}{C \vee D[r + t_2]_p \ \mid \ T_C \wedge T_D \wedge l = t_1 \wedge \tau}$$

Its conclusion belongs to S, since S is closed under $\mathcal{H}$, and it has an instance with σ contradicting the minimality of $Dred$.

- If the rule reducing $t|_p$ is $-u \to (n-1)u - r'$, then, the contradiction of the minimality of $Dred$ follows, now, from lemmas 27, and 29.

□

Lemma 25 *If, as in the definition of R, the reductive form $Cred$ of $C\sigma$ generates the rules $nu \to r$ and $-u \to (n-1)u - r$, then (C, σ) is irreducible not only w.r.t. R_{Cred}, but w.r.t. $R \setminus \{nu \to r, \ \ -u \to (n-1)u - r\}$. Moreover, if $e \simeq 0$ is a negative equation of C, then (e, σ) is irreducible w.r.t. R.*

Proof: Let $e \simeq 0$ be an equation of C. Let R^{Cred} be the set of rules generated by reductive forms bigger than $Cred$ w.r.t. $\succ_c$. Then, R^{Cred} is of the form $\bigcup_{i \in I}\{n_i u_i \to r_i, \ \ -u_i \to (n_i - 1)u_i - r_i\}$. All these u_i's are larger than the maximal summand of $nf_{R_{AG}}(e\sigma)$. Moreover, if $e \simeq 0$ is a negative equation, u is larger than the maximal summand $nf_{R_{AG}}(e\sigma)$. By applying lemma 32 with R^{Cred} and $R^{Cred} \cup \{nu \to r, -u \to (n-1)u - r\}$ for negative equations, the lemma follows. □

Lemma 26 *Let the reductive form $Cred$ of $C\sigma$ generate the rule $nu \to r'$.*

Then there exists an orientation $l \simeq r$ of the positive equation $e \simeq 0$ of C such that $\mathrm{nf}_{R_{AG}}(l\sigma)$ *is nu,* $\mathrm{nf}_{R_{AG}}(r\sigma)$ *is r', and (r, σ) is $(u \succ)$-irreducible w.r.t. R. Moreover, u is not at top-level position in $r\sigma$.*

Proof: By lemma 25, (e, σ) is irreducible (in fact $(u \succeq)$-irreducible) w.r.t. $R \setminus \{nu \to r, -u \to (n-1)u - r\}$. By property $a.$ of proposition 30, AC-changes in the substitution do not affect irreducibility. Hence we can suppose that $x_i\sigma$ is of the form (i) $l_i u$, or (ii) $l_i u + v_i$, or (iii) v_i, for all variable x_i in e, where v_i has no ocurrencies of u. Let e' be the result of replacing each ocurrence of x_i at top-level position by $y_i + z_i$, where y_i and z_i are new variables. Let σ be extended such that $y_i\sigma$ is $l_i u$ (in cases i and ii) or 0 (in case iii), and $z_i\sigma$ is 0 (case i) and v_i otherwise. By lemma 36, (e', σ) is irreducible w.r.t. $R \setminus \{nu \to r, -u \to (n-1)u - r\}$. Now, by AC-changes, a new term e'' of the form $l + l'$ can be obtained from e', such that l contains all the y_i, and all the summands at top-level position p in e' such that $nf_{R_{AG}}(e'|_p\sigma)$ is u; and l' contains all the z_i, and the rest of the summands. By lemma 34, (l', σ) is $(u \succeq)$-irreducible w.r.t. $R\setminus\{nu \to r, -u \to (n-1)u - r\}$. Clearly, $nf_{R_{AG}}(l\sigma)$ is nu, and $nf_{R_{AG}}(l'\sigma)$ is $nf_{R_{AG}}(-r')$, and all the summands in $nf_{R_{AG}}(l'\sigma)$ are strictly smaller than u w.r.t. $\succ$. By lemma 32, (l', σ) is $(u \succ)$-irreducible w.r.t. R. Taking r as $nf_{R_{AG}}(-l')$, we have that $nf_{R_{AG}}(r\sigma)$ is r', and, by property $b.$ of proposition 30, (r, σ) is $(u \succ)$-irreducible w.r.t. R. □

Lemma 27 *Let the reductive form $Cred$ of $C\sigma$ generate the rule $-u \to (n-1)u - r'$.*

Then there exists an orientation $l \simeq r$ of the positive equation $e \simeq 0$ of C such that $\mathrm{nf}_{R_{AG}}(l\sigma)$ *is $-u$,* $\mathrm{nf}_{R_{AG}}(r\sigma)$ *is $(n-1)u - r'$, and (r, σ) is $(u \succ)$-irreducible w.r.t. R. Moreover, for all x at a top-level positive variable position in r, $x\sigma$ has only positive summands (i.e. there are not top-level negative positions, in $x\sigma$).*

Proof: By lemma 25, (e, σ) is irreducible (in fact $(u \succeq)$-irreducible) w.r.t. $R \setminus \{nu \to r, -u \to (n-1)u - r\}$. Let us consideer now a certain variable x that appears in e at top-level positive variable position. By property $a.$ of proposition 30, AC-changes in the substitution do not affect irreducibility. Hence we can suppose that $x\sigma$ is of the form (i) v, or (ii) $v + w$, or (iii) w, where v (respectively w) contains only positive (respectively negative) summands at their top-level positions. Let e' be the result of replacing each ocurrence of x at top-level positive variable position by $y + z'$, where y and z' are new variables. Let σ be extended such that $y\sigma$ is v (in cases i and ii) or 0 (in case iii), and $z'\sigma$ is 0 (case i) and w otherwise. By lemma 36, (e', σ) is $(u \succeq)$-irreducible w.r.t. $R \setminus \{nu \to r, -u \to (n-1)u - r\}$ by some sequence Seq. Let e'' be the result of replacing each occurrence of z' by $-z$, where z is a new variable; and let σ be extended such that $z\sigma$ is the minus inwards normal form of $-z'\sigma$. Observe that moving minus inwards in $e'\sigma$ and $e''\sigma$ gives the same term. Therefore, if t is the cancellation of $e'\sigma$ by Seq, then t is also the cancellation of $e''\sigma$ by Seq. This justifies that (e'', σ) is also $(u \succeq)$-irreducible w.r.t. $R\setminus\{nu \to r, -u \to (n-1)u - r\}$ by Seq. Furthermore, z is not at top-level maximal positive variable position in e', and $y\sigma$ has no top-level negative positions. If we repeat this process, iteratively, for all the variables that appear at maximal positive variable position in e, we will obtain a term e_1 such that $e\sigma =_{AG} e_1\sigma$, and for all variable x at top-level variable positive position in e_1, $x\sigma$ does not have top-level negative positions. Moreover, (e_1, σ) is $(u \succeq)$-irreducible w.r.t. $R \setminus \{nu \to r, -u \to (n-1)u - r\}$ by some sequence Seq_1.

Since $nf_{R_{AG}}(e_1\sigma)$ is $nu - r$, then, in the cancellation by Seq_1, a positive u remains at top-level position. It can come from a positive summand or a variable; but not from one of the new added variables z.

If $e_1|_p$ is a top-level summand that gives the remaining u, then, p is independent in Seq_1. By lemma 38, by AC-changes, e_1 can be transformed in a term of the form $s + d$, such that s is a summand where $nf_{R_{AG}}(s\sigma)$ is u; and, moreover, there is a sequence Seq_2 such that $(s + d, \sigma)$ is $(v \succeq)$-irreducible w.r.t. $R \setminus \{nu \to r, -u \to (n-1)u - r\}$ by Seq'', and 1 is an independent position in Seq_2. By lemma 34, (d, σ) is $(u \succeq)$-irreducible w.r.t. $R \setminus \{nu \to r, -u \to (n-1)u - r\}$. Observe that $d\sigma$ is $(n-1)u - r$. By lemma 32, (d, σ) is $(u \succ)$-irreducible. Therefore, $-s \simeq d$ is an orientation that satisfy the required conditions.

If $e_1|_p$ is a top-level variable that gives the remaining u. We suppose that p is a positive position. The case where p is negative is similarly justified. Therefore, $x\sigma$ contains some u at maximal positive position that persists after cancellation by Seq_1. The case where $x\sigma$ is u is solved as in the previous case. Then, we suppose the opposite. By lemma 35, there exists a substitution σ' and a sequence Seq_2 such that σ' coincides with σ except in x, where $x\sigma'$ is of the form $u + v$, for a certain term v; and (e_1, σ') is $(v \succeq)$-irreducible w.r.t. $R \setminus \{nu \to r, -u \to (n-1)u - r\}$ by Seq_2. Moreover, $p.1$ is an untouched position in Seq_2. By lemma 34 if we obtain a new term e_2 by replacing this occurrence of x by $w_1 + w_2$, where w_1 and w_2 are new variables, and σ is extended such that $w_1\sigma$ is u, and $w_2\sigma$ is v, then, there exists a sequence Seq_3 such that (e_2, σ) is $(u \succeq)$-irreducible w.r.t. $R \setminus \{nu \to r, -u \to (n-1)u - r\}$ by Seq_3, and $p.1$ is and untouched position in Seq_3. The rest is identical to the previous case. □

Lemma 28 *Let $nu \to r'$ be a rule of R. Let (s, σ) be irreducible w.r.t. R. Let t be an AG-normal form of $s\sigma$. Let $t|_p$ be nu.*

Then, there exists a splitting $s[t_1 + t_2]_q$ at some AG-context position q of s, where $\mathrm{nf}_{R_{AG}}(t_1\sigma)$ is nu; and, if r is any term, $\mathrm{nf}_{R_{AG}}(t[r\sigma]_p)$ is $\mathrm{nf}_{R_{AG}}(s[r + t_2]_q\sigma)$.

Moreover, $(\mathrm{nf}_{R_{AG}}(s[r + t_2]_q), \sigma)$ is irreducible w.r.t. R if in addition (i) if the AG-context position of p is λ, then u is the maximal summand in t, and otherwise u is the maximal summand subject to rewriting in the AG-context position of p in t, and (ii) $\mathrm{nf}_{R_{AG}}(r\sigma)$ is r' and (r, σ) is $(u \succ)$-irreducible w.r.t. R, and (iii) there are no u's at top-level position in $r\sigma$.

Proof: Let p' be the AG-context position of p in t. Then, $t|_{p'}$ is of the form $nu + v$ for a certain ground term v. We continue now supposing that p' is not λ, since the case where p' is λ is similar. By lemma 42 and property $d.$ of proposition 30, there exists an AG-context position q in s different from λ such that $nf_{R_{AG}}(s|_q\sigma)$ is $nu + v$, and for all term h, $nf_{R_{AG}}(s[h]_q\sigma)$ is $nf_{R_{AG}}(t[h\sigma]_{p'})$. Moreover, if $t|_{p'} \succ nf_{R_{AG}}(h\sigma)$, and u is greater or equal than the maximal summand subject to rewritting in $h\sigma$, and (h, σ) is $(u \succeq /)$-irreducible w.r.t. R; then, $(nf_{R_{AG}}(s[h]_q), \sigma)$ is irreducible w.r.t. R.

Now, we want to find the splitting, transforming first the term $s|_q$.

Since (s, σ) is irreducible w.r.t. R, we have that $(s|_q, \sigma)$ is $(u' \succeq /)$-irreducible w.r.t. R, where u' is the maximal summand subject to rewritting in $s|_q\sigma$ (and, clearly, $u' \succeq u$), by a certain sequence $Seq \in \mathcal{S}(s|_q\sigma, R, u')$. Let s_1 be the cancellation of $s|_q\sigma$ by Seq. Let s_2 the cancellation of s_1 by a sequence $Seq' \in \mathcal{S}_{max}(s_1)$, i.e. Seq' is maximal. The zero elimination of s_2 is a normal form of $s|_q\sigma$, and then, of the form $nu + v$. This means that there exists n top-level positive positions $q_1, \ldots, q_n$ in the minus inwards normal form of $s|_q\sigma$ such that are preserved in Seq and in Seq', and $s_2|_{q_i} =_{AC0} u$. This positions q_i are not below a position in s containing $-x$, for some variable x; since, otherwise, it would be a contradiction with the $(u' \succeq /)$-irreducibility hypothesis w.r.t. R by Seq.

Let us consider now a certain variable x that appears in $s|_q$, at the top-level positive variable positions $p_1, \ldots, p_m$. Assume l of the n preserved u's are from positions below some of these x. By lemma 39, there exists a substitution σ' and a sequence Seq_1 such that:

- σ' coincides with σ except in the variable x, where $x\sigma' =_{AC} x\sigma$, and, additionally, if $l \bmod m$ is 0, then, $x\sigma'$ is $((l \operatorname{div} m)u) + v'$, and, otherwise, $x\sigma'$ is $((l \operatorname{div} m)u) + (u + v')$, for a certain ground term v'.
- $(s|_q, \sigma')$ is $(u' \succeq /)$-irreducible w.r.t. R by Seq_1.
- $p_i.1$ is untouched in Seq_1, and, $p_i.2.1$ is untouched for all i with $1 \leq i \leq (l \bmod m)$.
- For all position not below any of the p_i, and untouched (independent) in Seq, then, it is also untouched (independent) in Seq_1.

Now, if the term s' is $s|_q$, but splitting these x's into $y+z$, and σ' is extended such that $y\sigma'$ is $(l \operatorname{div} m)u$, and $z\sigma'$ contains the rest of $x\sigma'$, by lemma 36, there exists a sequence Seq_2 such that:

- (s', σ') is $(u' \succeq /)$-irreducible w.r.t. R by Seq_2.
- $p_i.1$ is untouched in Seq_2.
- $p_i.2.1$ is untouched in Seq_2 for all i with $1 \leq i \leq (l \bmod m)$.
- A position disjoint with the p_i's, and untouched (independent) in Seq_1 is also untouched (independent) in Seq_2.

In the case where l mod $m > 0$, we split, in s', all the appearances of the new variable z into $y' + z'$. If we name s'' to the new obtained term, and σ' is extended such that $y'\sigma'$ is u, and $z'\sigma'$ is v', then, by lemma 36, there exists a sequence Seq_3 such that:

- (s'', σ') is $(u' \succeq /)$-irreducible w.r.t. R by Seq_3.
- $p_i.1$ is untouched in Seq_3.
- $p_i.2.1$ is untouched in Seq_3, for all i with $1 \leq i \leq (l \text{ mod } m)$.
- A position disjoint with the p_i's, and untouched (independent) in Seq_2 is also untouched (independent) in Seq_3.

Now, if we extend σ in the same way that σ' (i.e. both coincide except in x, where $x\sigma =_{AC} x\sigma'$), by property a. of proposition 30, there are sequences Seq_4 and Seq_5 such that (s', σ) and (s'', σ) are $(u' \succeq /)$-irreducible w.r.t. R by Seq_4 and Seq_5, respectively. Moreover, $p_i.1$ is untouched in Seq_4 and Seq_5, and $p_i.2.1$ is untouched in Seq_5, for all i with $1 \leq i \leq (l \text{ mod } m)$. Furthermore, a position of $s|_q$ disjoint with the p_i's, and untouched in Seq is also untouched in Seq_4 and Seq_5.

With the new obtained term s' or s'', and with the new sequence Seq_4 or Seq_5, we can repeat the same process, but taking now an other variable at maximal positive variable position in $s|_q$. Let these variables be $x_1, \ldots, x_k$, let l_i be the number of the apaerances of the n preserved u's below the variable x_i, and let m_i the number of apaerances of the variable x_i in $s|_q$ at maximal position. If we do, iteratively, the previous process for all these variables we will have split every x_i into $y_i + z_i$ or into $y_i + y'_i + z'_i$, obtaining a term that we name h'. And we will have obtained a sequence S' that left all the y_i's, and l_i mod m_i of the y'_i untouched; and (h', σ) is $(u' \succeq /)$-irreducible w.r.t. R by S'. The term h' will have n independent positions $q'_1, \ldots, q'_n$ such that include some top-level summand positions, and the y_i's, and only some of the y'_i's, and such that $(h'|_{q'_1} + \ldots + h'|_{q'_k})\sigma =_{AG} nu$. Thanks to lemma 38, there exists a new term h'' of the form $t_1 + t_2$, such that $h'' =_{AC} h'$, t_1 contain some summands, all the y_i's, and part of the y'_i's, and $t_1\sigma =_A Gnu$, and there exists a sequence S such that (h'', σ) is $(u' \succeq /)$-irreducible w.r.t. R by S, and 1 is an independent position in S.

Observe that $s[h'']_q$ is the splitting that we want: given a term r, by lemma 43, if h is $r + t_2$, then, $nf_{R_{AG}}(h)$ is $nf_{R_{AG}}(r\sigma + v)$. As said before, from lemma 42, $nf_{R_{AG}}(s[r + t_2]_q\sigma)$ is $nf_{R_{AG}}(t[(r + t_2)\sigma]_{p'})$. Therefore, $s[r + t_2]_q\sigma =_{AG} t[(r + t_2)\sigma]_{p'} =_{AG} t[r\sigma + v]_{p'} =_{AG} t[r\sigma]_p$. Thus, $nf_{R_{AG}}(t[r\sigma]_p)$ is $nf_{R_{AG}}(r + t_2)_q\sigma$.

Suppose now that u is the maximal summand subject to rewritting at position p' in t (i.e. u' is u), and suppose that $nf_{R_{AG}}(r\sigma)$ is r', (r, σ) is $(u \succ)$-irreducible w.r.t. R and there are no u's at top-level position in $r\sigma$.

Then, we have that the maximal summand subject to rewritting in $t_2\sigma$ is not greater than u w.r.t. $\succ$; and, by lemma 34, (t_2, σ) is $(u \succeq /)$-irreducible w.r.t. R. Moreover, since $nf_{R_{AG}}((t_1 + t_2)\sigma)$ is $nu + v$, then, $nf_{R_{AG}}(t_2\sigma)$, that is v, cannot contain u just below $-$ at top-level position.

Therefore, by lemma 45, $(nf_{R_{AG}}(r + t_2), \sigma)$ is $(u \succeq /)$-irreducible w.r.t. R. Observe that $t_p = nu \succ r' = nf_{R_{AG}}(r\sigma)$. Thus $t_{p'} \succ nf_{R_{AG}}((r + t_2)\sigma)$. As said before, by lemma 42, $(nf_{R_{AG}}(s[r + t_2]_q), \sigma)$ is irreducible w.r.t. R. $\square$

Lemma 29 *Let $n > 1$ and $-u \to (n-1)u - r'$ be a rule of R. Let (s, σ) be irreducible w.r.t. R. Let t be a normal form of $s\sigma$ w.r.t. R. Let $t|_p$ be $-u$. Let the AG-context position of p be not λ.*

Then there exists a splitting $s[t_1 + t_2]_q$ at some AG-context position q of s diferent from λ, and an extension of σ to the new variables in the splitting, such that $\mathrm{nf}_{R_{AG}}(t_1\sigma)$ is $-u$; and, if r is any term, $\mathrm{nf}_{R_{AG}}(t[r\sigma]_p)$ is $\mathrm{nf}_{R_{AG}}(s[r + t_2]_q\sigma)$.

Moreover, $(\mathrm{nf}_{R_{AG}}(s[r+t_2]_q), \sigma)$ is irreducible w.r.t. R if in addition (i) u is the maximal summand subject to rewriting in the AG-context position of p, and (ii) $\mathrm{nf}_{R_{AG}}(r\sigma)$ is $(n-1)u - r'$ and (r, σ) is $(u \succ)$-irreducible w.r.t. R, and (iii) $x\sigma$ has only positive summands, for each variable x that appears in r at top-level positive variable position.

Proof: Let p' be the AG-context position of p in t. Then, $t|_{p'}$ is of the form $-u + v$ for a certain ground term v. By lemma 42 there exists an AG-context position q in s different from λ such that $nf_{R_{AG}}(s|_q\sigma)$ is $-u + v$, and for all term h, $nf_{R_{AG}}(s[h]_q\sigma)$ is $nf_{R_{AG}}(t[h\sigma]_{p'})$. Moreover, if $t|_{p'} \succ nf_{R_{AG}}(h\sigma)$, u'' is the maximal summand subject to rewritting in $nf_{R_{AG}}(h\sigma)$, and (h, σ) is $(u'' \succeq /)$-irreducible w.r.t. R; then, $(nf_{R_{AG}}(s[h]_q), \sigma)$ is irreducible w.r.t. R.

Now, we want to find the splitting, transforming first the term $s|_q$.

Since (s, σ) is irreducible w.r.t. R, we have that $(s|_q, \sigma)$ is $(u' \succeq /)$-irreducible w.r.t. R, where u' is the maximal summand subject to rewritting in $s|_q\sigma$ (and, clearly, $u' \succeq u$), by a certain sequence $Seq \in \mathcal{S}(s|_q\sigma, R, u')$. Let s_1 be the cancellation of $s|_q\sigma$ by Seq. Let s_2 be the cancellation of s_1 by a sequence $Seq' \in \mathcal{S}_{max}(s_1)$, i.e. Seq' is maximal. The zero elimination of s_2 is an AG-normal form of $s|_q\sigma$, and then, of the form $-u + v$. This means that there exists a position q' such that $s_2|_{q'} =_{AC0} -u$, q' is preserved in Seq and Seq', and $q'.1$ is a top-level position in s_1.

Now, we distinguish two cases:

- Let q' be a position in $s|_q$ such that $s|_{q.q'}$ is not of the form x nor $-x$, for any variable x. Then, by lemma 38, by AC-changes, $s|_q$ can be transformed in a term of the form $-l+d$,such that l is a summand where $nf_{R_{AG}}(l\sigma)$ is u; and, moreover, there is a sequence Seq_2 such that $(-l+d,\sigma)$ is $(u' \succeq /)$-irreducible w.r.t. R, and 1 is an independent position in Seq_2.

We show that $s[-l+d]_q$ is the splitting satisfying the required conditions. Given a term r, by lemma 43, if h is $r+d$, then, $nf_{R_{AG}}(h)$ is $nf_{R_{AG}}(r\sigma+v)$. As said before, from lemma 42, $nf_{R_{AG}}(s[r+d]_q\sigma)$ is $nf_{R_{AG}}(t[(r+d)\sigma]_{p'})$. Therefore, $s[r+d]_q\sigma =_{AG} t[(r+d)\sigma]_{p'} =_{AG} t[r\sigma+v]_{p'} = AGt[r\sigma]_p$. Thus, $nf_{R_{AG}}(t[r\sigma]_p)$ is $nf_{R_{AG}}(r+d)_q\sigma$.

Suppose now that u is the maximal summand subject to rewritting at position p' in t (i.e. u' is u), and suppose that $nf_{R_{AG}}(r\sigma)$ is $(n-1)u-r'$, (r,σ) is $(u \succ)$-irreducible w.r.t. R and $x\sigma$ has only positive summands, for each variable x that appears in r at top-level positive variable position.

Then, we have that the maximal summand subject to rewritting in $d\sigma$ is not greater than u w.r.t. $\succ$; and, by lemma 34, (d,σ) is $(u \succeq /)$-irreducible w.r.t. R. Moreover, since $nf_{R_{AG}}((-l+d)\sigma)$ is $-u+v$, then, $nf_{R_{AG}}(d\sigma)$, that is v, cannot contain u at top-level positive position.

Therefore, by lemma 47, $(nf_{R_{AG}}(r+d),\sigma)$ is $(u'' \succeq /)$-irreducible w.r.t. R, where u'' is the maximal summand subject to rewritting in $(nf_{R_{AG}}(r+d),\sigma)$. Observe that $t_p = -u \succ (n-1)u-r' = nf_{R_{AG}}(r\sigma)$. Thus $t_{p'} \succ nf_{R_{AG}}((r+d)\sigma)$. As said before, by lemma 42, $(nf_{R_{AG}}(s[r+d]_q),\sigma)$ is irreducible w.r.t. R.

- Otherwise, there exists a position q'' such that $s|_{q.q''}$ is x not inmediately below $-$, or $-x$, for some variable x; and $q'' < q$. The case x is not possible, since it would be a contradiction with the irreducibility hypothesis w.r.t. R by Seq. Therefore, the situation is that $s|_{q.q''}$ is $-x$. By lemma 35, there exists a substitution σ' and a sequence Seq_2 such that σ' coincides with σ except in x, where $x\sigma =_{AC} x\sigma'$, and $x\sigma$ is $u+w$ for a certain term w. Moreover, $(s|_q,\sigma')$ is $(u' \succeq /)$-irreducible w.r.t. R' by Seq_2, and $q.q''.1$ is an untouched position in Seq_2. Now, we obtain a new term s' from $s|_q$ by replacing this $-x$ at position q'' by $(-x_1)+(-x_2)$, where x_1 and x_2 are new variables. We extend σ such that $x_1\sigma$ is u, and $x_2\sigma$ is w. By lemma 36, there exists a sequence Seq_3 such that (s',σ') is $(u' \succeq /)$-irreducible w.r.t. R by Seq_3, and $q.q''.1$ is an untouched position in Seq_3. By property a. of proposition 30, there exists a sequence Seq_4 such that (s',σ) is $(u' \succeq /)$-irreducible w.r.t. R by Seq_4, and $q.q''.1$ is an untouched position in Seq_4. By lemma 38, by AC-changes, s' can be transformed into a term of the form $-x_1+d$, and there exists a sequence Seq_5 such that $(-x_1+d,\sigma)$ is $(u' \succeq /)$-irreducible w.r.t. R by Seq_5. Moreover, 1 is an untouched position in Seq_5. The rest is as in the previous case.

□

Proposition 30 *Let Seq be a sequence $(p_1,q_1),\ldots,(p_n,q_n)$, and let (s,σ) be $(u \succeq)$-irreducible (resp. $(u \succ)$-irreducible and $(u \succeq /)$-irreducible) w.r.t. R by Seq.*

Then the following properties are satisfied:

- **a.** *If σ' is a ground substitution with $x\sigma =_{AC} x\sigma'$ for all variable x in s, then (s,σ') is $(u \succeq)$-irreducible (resp. $(u \succ)$-irreducible and $(u \succeq /)$-irreducible) w.r.t. R by a certain sequence Seq'; such that, if p is a, non-variable or variable not below $-$, position of s and untouched (independent) in Seq, then, p is untouched (independent) in Seq'.*
- **b.** *$(nf_{R_{AG}}(-s),\sigma)$ is $(u \succeq)$-irreducible (resp. $(u \succ)$-irreducible and nothing can be said in the case where (s,σ) is $(u \succeq /)$-irreducible) w.r.t. R (note that here only the rules $-(x+y) \to (-x)+(-y)$ and $-(-x) \to x$ are needed).*
- **c.** *If Seq is not maximal, it can always be extended into a maximal sequence Seq' such that (s,σ) is $(u \succeq)$-irreducible (resp. $(u \succ)$-irreducible and $(u \succeq /)$-irreducible) w.r.t. R by Seq'.*
- **d.** *If a ground term v satisfy that $u \succ v$, then, (s,σ) is $(v \succeq)$-irreducible (resp. $(v \succeq)$-irreducible, i.e. coincidence in this case, and $(v \succeq /)$-irreducible) w.r.t. R by Seq.*

Lemma 31 *Let s be a ground term and let u be the maximal summand in $\mathrm{nf}_{R_{AG}}(s)$. Let R' be a rewrite system. Let v be a ground summand in AG-normal form such that $v \succeq u$. Let t be the cancellation of s by a sequence $Seq \in \mathcal{S}_{max}(s,R',v)$. Let p be a position of a summand in t.*

Then $v \succeq \mathrm{nf}_{R_{AG}}(t|_p)$ if the AG-context of p is λ, and $v \succ \mathrm{nf}_{R_{AG}}(t|_p)$, otherwise.

Furthermore, if q is an AG-context position in t then, v is greater than or equal to the maximal summand in $\mathrm{nf}_{R_{AG}}(t|_q)$ (strictly greater if q is not λ).

Proof: We only prove the first statement since the second part is a direct consequence. Let $t|_p$ be the summand in t whose AG-normal form is maximal (w.r.t. $\succ$). Let t' be the cancellation of t by a maximal sequence Seq' in $\mathcal{S}(t)$. The position p may be preserved in Seq' or not. In case it is preserved, since t' gives the normal form w.r.t. R_{AG} after eliminating 0's, we have that $nf_{R_{AG}}(t|_p)$ is a subterm of $nf_{R_{AG}}(s)$. Therefore, $u \succeq nf_{R_{AG}}(t|_p)$ (strict if p is not a top-level position), and $v \succeq nf_{R_{AG}}(t|_p)$. In case it is not preserved, there exists a prefix p' of p, and another position p'' in t, both in the same AG-context, such that $t|_{p'} =_{AG} -t|_{p''}$. Let us choose p' as the maximal prefix of p satisfying this condition, with its corresponding p''. Now, p' may be in the same AG-context of p (i.e. p' is p, and it is not inmediately below $-$, or p is inmediately below $-$ and p is $p'.1$). In this case, since the removal of the pair with p' and p'' in the maximal sequence Seq was not done, either the AG-context of p' is λ, and $v \succ nf_{R_{AG}}(t|_{p'})$, or the AG-context of p' is not λ, and in the normal form of this context there is a summand bigger than $nf_{R_{AG}}(t|_p)$. Consequently, there is a position q of a summand in t such that $nf_{R_{AG}}(t|_q) \succ nf_{R_{AG}}(t_p)$, which is a contradiction with the initial assumption. Consider now the case where p' is not in the AG-context of p. Then, p' or $p'.1$ is the position of a summand in t. This summand, in its normal form w.r.t. R_{AG}, contains $nf_{R_{AG}}(t|_p)$ as a proper prefix, since no subterm at a position between p' and p is cancelled. Therefore, p' or $p'.1$ is the position of a summand whose AG-normal form is bigger than $nf_{R_{AG}}(t|_p)$, which contradicts the initial hypothesis. $\square$

Lemma 32 *Let u be the maximal summand in $\mathrm{nf}_{R_{AG}}(s\sigma)$, let R' be a rewrite system and let R'' be a rewrite system with left hand sides of the form nw or $-w$, where $n > 0$ and w is a summand. Let v be a ground summand in AG-normal form such that $v \succeq u$ and (s, σ) is $(v \succeq)$-irreducible w.r.t. R'.*

If every of the w satisfy that $w \succ v$, then, (s, σ) is $(v \succeq)$-irreducible w.r.t. $R' \cup R''$.

If every of the w satisfy that $w \succeq v$, then, (s, σ) is $(v \succ)$-irreducible w.r.t. $R' \cup R''$.

Proof: We only prove the first statement (the second one is identical). The pair (s, σ) is $(v \succeq)$-irreducible w.r.t. R' by some sequence Seq v-restricted to R', that, by property $c.$ of proposition 30 can be taken to be maximal. A new sequence Seq' can be obtained by eliminating from Seq each pair (p, q) whose AG-context is not a preserved position (in a sense, these pairs (p, q) are useless). Hence (s, σ) is also $(v \succeq)$-irreducible w.r.t. R' by Seq', which is also maximal. All the pairs in Seq' belong to AG-contexts that are preserved positions. By the second statement of Lemma 31, for all these AG-contexts, there is no summand bigger than v in their AG-normal forms. Therefore, the maximal summand subject to rewriting at each one of these AG-contexts that is at non-top-level continues being the same by $R' \cup R''$. Thus, the sequence Seq' is also a sequence v-restricted to $R' \cup R''$, and by the first statement of lemma 31, in the cancellation of $s\sigma$ by Seq' there is no summand of a left hand side of a rule of R''. Thus (s, σ) is $(v \succeq)$-irreducible w.r.t. $R' \cup R''$ by Seq'. $\square$

Lemma 33 *Let σ be irreducible w.r.t. R. Then, (s, σ) irreducible w.r.t. R.*

Proof: If Seq is the empty sequence, (s, σ) is irreducible w.r.t. R by Seq. $\square$

Lemma 34 *Let s be a term of the form $s_1 + s_2$. Let (s, σ) be $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R' by a sequence Seq, for a given summand v. If the positions 1 and 2 are independent in Seq then (s_1, σ) and (s_2, σ) are $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R'.*

Lemma 35 *Let R' be a rewrite system. Let (s, σ) be $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R' by a sequence Seq. Let s_0 be the minus inwards normal form of $s\sigma$. Let u be a ground term, and let p be a position in s such that $s|_p$ is x (not inmediately below $-$), or $-x$. Let q be a top-level (i) positive or (ii) negative position in $s_0|_p$ and not λ such that $s_0|_{p.q}$ is u, and $p.q$ is an untouched position in Seq.*

Then, there exists a substitution σ' and a sequence Seq' such that σ' coincides with σ except in x, where $x\sigma =_{AC} x\sigma'$; and, if s'_0 is the minus inwards normal form of $s\sigma'$, then, $s'_0|_{p.1}$ is (i) u or (ii) $-u$, and (s, σ') is $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R' by Seq'. Moreover, $p.1$ is an untouched position in Seq'.

Lemma 36 *Let (s, σ) be $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R' by Seq, for a given summand v. Let x be a variable in s that appears at top-level position, and let x_1 and x_2 be variables and not in s. Let we have one of the following situations: (i) $x\sigma$ is t_1, $x_1\sigma$ is t_1 and $x_2\sigma$ is 0, or (ii) $x\sigma$ is $t_1 + t_2$, $x_1\sigma$ is t_1 and $x_2\sigma$ is t_2, or (iii) $x\sigma$ is t_2, $x_1\sigma$ is 0 and $x_2\sigma$ is t_2. If x appears at top-level positive position in s, let s' be obtained from s by replacing some of its x at top-level position by $x_1 + x_2$. If x appears at top-level negative position in s, let s' be obtained from s by replacing some of its $-x$ with AG-context λ by $-x_1 - x_2$.*

Then, there exists a sequence Seq' satisfying the following conditions:

- *(s', σ) is $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R' by Seq'.*
- *If p is one of the replaced positions, and, in case (i), p is untouched in Seq, or in case (ii), $p.1$ is untouched in Seq', then, $p.1$ is untouched in Seq'.*
- *Let q be a position such that, for all replaced position p, in case (i) and (iii), q is disjoint with p, or in case (ii), q is disjoint with $p.1$. If q is untouched (independent) in Seq, then, q is untouched (independent) in Seq'.*

Lemma 37 *Let (s,σ) be $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R', and $s' =_{AC} s$.*

Then (s',σ) is $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R'.

Lemma 38 *Let (s,σ) be $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R' by Seq, for a given summand v. Let $s =_{AC} (s_1 + \ldots + s_n) + v$. Suppose that there exist $p_1, \ldots, p_n$ disjoint positions with AG-context λ in s such that p_i is an independent (untouched) position in Seq, and $s|_{p_i} =_{AC} s_i$.*

Then, there exists a sequence Seq' such that $((s_1 + \ldots + s_n) + v, \sigma)$ is $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R' by Seq', and 1 is an independent (untouched) position in Seq'.

Lemma 39 *Let nu be a left-hand-side of a rule from R. Let (s,σ) be $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R by Seq, for a given summand v. Let x be a variable that appears in s at maximal positive variable position. Let $p_1, \ldots, p_m$ be the maximal positive variable positions of s such that $s|_{p_i}$ is x. Let s_1 be the minus inwards normal form of $s\sigma$.*

Let there are l maximal positions $q_1, \ldots, q_l$ in s_1 such that q_i is untouched in Seq, a prefix of q_i in s headed by x (i.e., q_i is below x), and $s_1|_{q_i}$ is u.

Then, there exists a substitution σ' and a sequence Seq' such that:

- *σ' coincides with σ except in the variable x, where $x\sigma' =_{AC} x\sigma$, and, additionally, if l mod m is 0, then, $x\sigma'$ is $((l \text{ div } m)u) + v$, and, otherwise, $x\sigma'$ is $((l \text{ div } m)u) + (u + v)$, for a certain ground term v.*
- *(s,σ') is $(v \succeq)$-irreducible (resp. $(v \succeq /)$-irreducible) w.r.t. R by Seq'.*
- *$p_i.1$ is untouched in Seq', and, $p_i.2.1$ is untouched for all i with $1 \leq i \leq (l \text{ mod } m)$.*
- *For all position p that has no a prefix p' with $s|_{p'}$ being x, if p is independent (untouched) in Seq, then, p is independent (untouched) in Seq'.*

Lemma 40 *Let s be a term in zero normal form.*

Then, (s,σ) is irreducible w.r.t. R if, and only if, $(\text{nf}_{R_{AG}}(s),\sigma)$ is irreducible w.r.t. R.

Lemma 41 *Let s be in zero normal form, and $\text{nf}|_{R_{AG}}(s|_p) \succ \text{nf}|_{R_{AG}}(r)$. Then, $\text{nf}|_{R_{AG}}(s) \succ \text{nf}|_{R_{AG}}(s[r]_p)$.*

Lemma 42 *Let (s,σ) be irreducible w.r.t. R by Seq. Let t be an AG-normal form of $s\sigma$. Let p be an AG-context position in t, different from λ, such that $t|_p$ is reducible w.r.t. R.*

Then, there exists a context position q in s different from λ, such that $s\sigma|_q =_{AG} t|_p$, and for all term r, $\text{nf}_{R_{AG}}(s[r]_q\sigma) = \text{nf}_{R_{AG}}(t[r\sigma]_p)$.

Moreover. Let $t|_p \succ \text{nf}_{R_{AG}}(r\sigma)$. Let v be the maximal summand subject to rewritting in $r\sigma$. Let (r,σ) be $(v \succeq /)$-irreducible w.r.t. R,

Then, $(\text{nf}_{R_{AG}}(s[r]_q),\sigma)$ is irreducible w.r.t. R.

Proof: Let s_1 be the cancellation of $s\sigma$ by Seq. Let s_2 be the cancellation of s_1 by a sequence $Seq' \in \mathcal{S}_{max}(s_1)$; i.e. Seq' is maximal. Let s_3 be the zero elimination of s_2. Then, s_3 is an AG-normal form of $s\sigma$. Therefore, there exists a context position p' in s_3 diferent from λ such that $s_3|_{p'} =_{AC} t|_p$, and for all term r, $s_3[r]_{p'} =_{AC} t[r]_p$. Therefore, there exists a context position q in s_2 diferent from λ such that $s_2|_q =_{AC0} t|_p$, and for all term r, $s_2[r]_q =_{AC0} t[r]_p$. This AG-position q cannot be below a non-variable position in s: otherwise, we would have a contradiction with the irreducibility hypothesis of (s,σ) w.r.t. R by Seq. Therefore, q is a non-variable AG-position in s. Observe that $s\sigma[r]_q =_{AG} s_2[r]_q$, since we can obtain the seccond from the first by applying minus inward steps, and the AG-steps indicated by the sequences Seq and Seq', excepting that ones applyed below the context q. Thus, $nf_{R_{AG}}(s[r]_q\sigma) = nf_{R_{AG}}(t[r\sigma]_p)$ for all term r.

Now, let s_0 be the minus inwards normal form of $s\sigma$. In order to justify the seccond part, we need to see, previously, that, for all the AG-context positions q' in s_0 such that q' is disjoint with q, or $q' < q$, we have that the maximal summand subject to rewritting in $s_0|_{q'}$ is greater or equal than the maximal summand subject to rewritting in $s_0[r\sigma]_q|_{q'}$. In other words, after introducing $r\sigma$, the maximal summand subject to rewritting in each of these AG-contexts do not increases. This is easy to justify for the positions q' disjoint with q, since its AG-normal form continues being the same. Then, suppose $q' < q$, and let q'' be the position of the summand in the AG-context q' such that $q'' < q$ (note that q'' can be below a minus in s_0). Observe that, since the position q'' persists in Seq and Seq', $nf_{R_{AG}}(s_0|_{q'})$ has $nf_{R_{AG}}(s_0|_{q''})$ as subterm. But by the same argument, $nf_{R_{AG}}(s_0|_{q''})$ has $nf_{R_{AG}}(s_0|_q) = t|_p$ as subterm. Therefore, since $t|_p$ is reducible, $nf_{R_{AG}}(s_0|_{q''})$ is a reducible summand, and then, the maximal summand subject to rewritting in $s_0|_{q'}$ is greater or equal w.r.t. $\succ$ than $nf_{R_{AG}}(s_0|_{q''})$. Since the summand $s_0|_{q''}$ is replaced by $s_0[r\sigma]_q|_{q''}$, if we justify that this new summand, in normal form w.r.t. R_{AG}, is smaller than $nf_{R_{AG}}(s_0|_{q''})$ w.r.t. $\succ$, it will be clear that the maximal summand subject to rewritting does not increase. This is equivalent to see that $nf_{R_{AG}}(s_2|_{q''}) \succ nf_{R_{AG}}(s_2[r\sigma]_q|_{q''}$, since $s_0|_{q''} =_{AG} s_2|_{q''}$, and $s_0[r\sigma]_q|_{q''} =_{AG} s_2[r\sigma]_q|_{q''}$. But this is a direct consequence of lemma 41.

Observe that we have additionally seen that for all the AG-context positions $q' > q$, and the corresponding q'', the maximal summand subject to rewritting in $s_0|_{q'}$ is greater or equal than $nf_{R_{AG}}(s_0|_{q''})$.

Now, we try to prove that $(nf_{R_{AG}}(s[r]_q), \sigma)$ is irreducible w.r.t. R. By lemma 40, this is equivalent to prove that if s' is the zero normal form of $s[r]_q$, (s', σ) is irreducible w.r.t. R. But since s and r are in zero normal form, s' can only be either (i) $s[r]_q$ or (ii) $s[0]_{q''}[0]_l$ for some positions q'' and l with the same AG-context position q', and such that $q'' < q$, $(s[r]_q)|_{q''} =_{AG} -s[r]|_l$, and. We show now that, for each case, (s', σ) is irreducible w.r.t. R.

- **(i)** By hypothesis, (r, σ) is $(v \succeq /)$-irreducible w.r.t. R by some sequence Seq''. A new sequence S can be made by taking from Seq all the pairs that are not below position q, and taking all the pairs from Seq'' but adding the prefix q to each of their positions. The new sequence is in $\mathcal{S}(s[r]_q\sigma)$: all the pairs below q are clearly correct, and the pairs taken from Seq are also correct, since the maximal summand subject to rewritting on the context where this pairs are applyed is smaller or equal than the summands eliminated by the pair. Once the cancellation by S is obtained, the terms find below each variable position coincide with the ones found when justifying irreducibility of (s, σ) and $(v \succeq /)$-irreducibility of (r, σ); and the rules allowed to justify irreducibility are not more than before, since the maximal summand subject to rewritting at each AG-context position does not increases when introducing r instead of $s|_q$.

- **(ii)** The case $s[0]_{q''}[0]_l$ is similar. First, we show that l is a preserved position in Seq. The summand, or negative summand at position l in s_0, in normal form w.r.t. R_{AG}, is strictly smaller than $nf_{R_{AG}}(s_0|_{q''})$; and the maximal summand subject to rewritting in the AG-context $s_0|'_q$ is greater than or equal to this one. Therefore, if q' is λ, the maximal summand in $nf_{R_{AG}}(s_0)$ is greater than $s_0|_l$ w.r.t. $\succ$; and, if q' is not λ, the maximal summand subject to rewritting in s_0 at position q' is greater than $s_0|_l$ w.r.t. $succ$. Thus, l is a preserved position in Seq.

  Now, the sequence S for justifying irreducibility, needs only to have the pairs from Seq such that their positions are not below q'' or l. The rest is as in case (i).

$\square$

Lemma 43 *Let $nf_{R_{AG}}(s_1 + s_2)$ be $t_1 + t_2$. Let $nf_{R_{AG}}(s_1)$ be t_1.*

Then, given a term r, $nf_{R_{AG}}(r + s_2)$ is $nf_{R_{AG}}(r + t_2)$.

Proof: We have that $s_2 =_{AG} -s_1 + s_1 + s_2 =_{AG} -t_1 + t_1 + t_2 =_{AG} t_2$, and then, $r + s_2 =_{AG} r + t_2$. $\square$

Lemma 44 *Let s be a term such that $nf_{R_{AG}}(s\sigma)$ is irreducible w.r.t. R. Then, (s, σ) is irreducible w.r.t. R.*

Proof: Let $Seq \in \mathcal{S}_{max}(s\sigma)$, and let s' be the cancellation of $s\sigma$ by Seq (s' is in zero normal form). Now, we will use a similar trick to the one in the proof of lemma 32. A new sequence Seq' can be obtained by eliminating from Seq all the pairs (p, q) such that their context is not preserved position. Then, s' is also the cancellation of $s\sigma$ by Seq', and it is also maximal. All the pairs in Seq' belong to contexts that are preserved positions. But, since s' is in zero normal form, i.e. by zero elimination we obtain $nf_{R_{AG}}(s\sigma)$, all of their context positions have no any summand subject to rewritting. Therefore, the sequence Seq' satisfy the restriction of no having pairs of summands smaller w.r.t. $\succ$ than the maximal summand subject to rewritting in their AG-context. Thus, (s, σ) is irreducible w.r.t. R by Seq'. $\square$

Lemma 45 *Let $nu \to r'$ be a rule of R.*

Let $nf_{R_{AG}}(r\sigma)$ be r'.

Let (r, σ) be $(u \succ)$-irreducible w.r.t. R.

Let (t, σ) be $(u \succeq /)$-irreducible w.r.t. R.

Suppose that the maximal summand subject to rewritting in $nf_{R_{AG}}(t\sigma)$ is not greater than u w.r.t. $\succ$.

Suppose that there are no u's inmediately below $-$ at top-level position in $nf_{R_{AG}}(t\sigma)$.

Suppose that there are no u's at top-level position in $r\sigma$.

Suppose that r and t have no common variables.

Then, $(nf_{R_{AG}}(r + t), \sigma)$ is $(u \succeq /)$-irreducible w.r.t. R.

Proof: By hypothesis, there exist sequences Seq_1 and Seq_2 such that (r, σ) is $(u \succ)$-irreducible w.r.t. R by Seq_1, and (t, σ) is $(u \succeq /)$-irreducible w.r.t. R by Seq_2.

Let $s_0 = r_0 + t_0$ be the zero normal form of $r + t$. Observe that, since r and t do not have variables in common, only ground summands have been replaced by zeros. Now, by lemma 40, it suffices to prove that $(r_0 + t_0, \sigma)$ is $(u \succeq /)$-irreducible w.r.t. R. In order to do it, we have only to create the corresponding sequence $Seq \in \mathcal{S}(s\sigma, R, u)$. First, let $s_1 = r_1 + t_1$ be the minus inwards normal form of $(r_0 + t_0)\sigma$. Now, we create Seq by the following steps:

- *1.* Let t_- and r_- be the minus inwards normal form of $t\sigma$ and $r\sigma$, respectively.

  Now, we take from Seq_2 all the pairs (p, q) such that, if p' is the prefix of p at top-level position, then, $t_-|_{p'} =_{AG} u$. We add this pairs to Seq, but prefixed by 2. Observe than no u nor $-u$ appears in r as an occurrence with AG-context λ; and then, this positions $2.p'$ (and $2.q$ if p' is p) have not been replaced by zeros when obtaining $r_0 + t_0$ from $r + t$. Therefore, the sequence Seq constructed until now is a correct cancellation sequence for s_1, and u-restricted to R.

- *2.* Let s_2 be the cancellation of s_1 by the sequence Seq constructed until now. We extend Seq by iterating, when possible, the process of finding pairs of top-level positions p and q such that, $s_2|_p$ is not below $-$, $nf_{R_{AG}}(s_2|_p) =_{AG} v$ and $nf_{R_{AG}}(s_2|_q) =_{AG} -v$ for a certain summand v with $v \succ u$; and adding the pair (p, q) to Seq. After a pair (p, q) has been considered, neither p nor q are considered again. Note that this process can be done in diferent ways. Since only summands at top-level position, and greater than u, are considered, the sequence Seq constructed until now is u-restricted to R.

- *3.* Let s_3 be the cancellation of s_1 by the sequence Seq constructed until now.

  Here, we want to justify that, if a top-level position p in s_3 satisfy that $nf_{R_{AG}}(s_3|_p) = v$, for a given summand $v \succ u$, then, v is irreducible w.r.t. R. Observe that, the cancellation of s_3 by a sequence Seq' in $\mathcal{S}_{max}(s_3)$ is a zero normal form for s_3, and by lemma 40, it is $AC0$-equal to the AG-normal form of s_3. But due to our previous process, in Seq' there are no pairs of the form (p', q') such that p' is a top-level position in s_3, and $nf_{R_{AG}}(s_3|_{p'}) = v'$, for a given summand $v' \succ u$. Therefore, the considered positions p are preserved in Seq', and then, satisfy that $nfR_{AG}(s_3|_p)$ is a subterm of $nfR_{AG}(s_3)$, that is $nfR_{AG}((r+t)\sigma)$. But, by our hypothesis, $nfR_{AG}((r+t)\sigma)$ is irreducible by rules with left-hand-sides greater than $-u$. Therefore, the considered positions p satisfy the required condition.

  Given one of these positions p that is not below variable in s_0, the lemma 44 applies for the term $s_0|_p$, with σ. Therefore, $(s_0|_p, \sigma)$ is irreducible w.r.t. R by a certain sequence Seq_p. We add the pairs of Seq_p to Seq, but prefixed by p.

- *4.* Consideer now, all the top-level positions p such that $nf_{R_{AG}}(s_2|_p) =_{AG} v$, for a certain term v with $u \succ v$. Observe that this position p is of the form $1.p'$, or $2.p'$, where p' is a top-level position in the minus inwards normal form of $r\sigma$, or of $t\sigma$. Deppending on the case, p' is a preserved and independent position in Seq_1 or Seq_2 (remember that both sequences are u-restricted). Then, we take from Seq_1 or Seq_2 the pairs below p' and add it to Seq, but prefixed by 1, or 2.

By construction, Seq is a sequence u-restricted to R. Now, we want to see that $(r_0 + t_0, \sigma)$ is $(u \succeq)$-irreducible w.r.t. R by Seq. Let s_4 be the cancellation of s_1 by Seq. Consideer a certain position p in s_0 that is preserved in Seq, and such that $s_0|_p$ is (i) x not inmediately below $-$, or (ii) $-x$, for some variable x.

- *a.* Suppose first that the AG-context of p is not λ. Let v be the maximal summand subject to rewritting in the AG-context of p. Let p' be the top-level position in s_0 that is prefix of p. If $nf_{R_{AG}}(s_1|_{p'})$ is u, then, Point 1. ensures that, (i) $s_4|_p$ or (ii) $nf_{R_{AG}}(-s_4|_p)$, is irreducible w.r.t. rules of R with left-hand-side smaller than or equal to $-v$, and, in case (ii), diferent from mv, with $m > 0$. Analogous justification are given by points 2. and 3., when $nf_{R_{AG}}(s_1|_{p'}) \succ u$; and by point 4., $u \succ nf_{R_{AG}}(s_1|_{p'})$.

- *b.* Suppose now that the AG-context of p is λ. This position p is of the form $1.p'$ or $2.p'$. We assume that it is of the form $2.p'$. The case $1.p'$ is justified analogously. Let t' be the cancellation of t_- by Seq_1. Observe that, in the cancellation process of t_-, no summand at top-level position in $t_-|_{p'}$, and smaller than u w.r.t. $\succ$, is replaced by zero, since Seq_2 is u-restricted. The same happens in the cancellation of s_1 by Seq. Moreover, the terms smaller than or equal to u below p in s_4, and below p' in t' coincide: observe that, by step 1., the terms u replaced by zero coincide below p in s_4, and below p' in t'. Therefore, since neither (i) $t_c|_{p'}$ nor (ii) $nf_{R_{AG}}(t'|_{p'})$ is reduced by a rule from R with left-hand-side smaller than or equal to $-u$, and, in case (ii), diferent from mv, with $m > 0$, then, no one of these rules reduces (i) $s_4|_p$ nor (ii) $nf_{R_{AG}}(s_4|_p)$.

$\square$

Lemma 46 *Suppose the same assumptions as in lemma 45, but now, (t, σ) is $(u \succeq)$-irreducible w.r.t. R, and $nf_{R_{AG}}(t\sigma)$ has no any summand greater than u w.r.t. $\succ$.*

Then, $(nf_{R_{AG}}(r + t), \sigma)$ is irreducible w.r.t. R.

Proof: By an identical proof to the one in lemma 45, it can be seen that $(nf_{R_{AG}}(r+t), \sigma)$ is $(u \succeq)$-irreducible w.r.t. R. But observe that the maximal summand in $nf_{R_{AG}}(r + t)$ is not greater than u. Therefore, by using property d. of proposition 30 the lemma follows. $\square$

Lemma 47 *Let $n > 1$, and $-u \to (n-1)u - r'$ be a rule of R.*

Let $nf_{R_{AG}}(r\sigma)$ be $(n-1)u - r'$.

Let $x\sigma$ has only positive summands, for all variable x that appears in r at top-level positive variable position.

Let (r, σ) be $(u \succ)$-irreducible w.r.t. R.

Let (t, σ) be $(u \succeq /)$-irreducible w.r.t. R.

Suppose that the maximal summand subject to rewritting in $\mathrm{nf}_{R_{AG}}(t\sigma)$ *is not greater than u w.r.t. $\succ$.*

Suppose that if some u is at maximal position in $\mathrm{nf}_{R_{AG}}(t\sigma)$*, then it is inmediately below $-$.*

Suppose that r and t have no common variables.

Let u' be the maximal summand subject to rewritting in the AG-normal form of $(r + t)\sigma$.

Then, $(nf_{R_{AG}}(r+t), \sigma)$ is $(u' \succeq /)$-irreducible w.r.t. R.

Proof: We proceed in a similar way as in lemma 45. But there are some diferences.

First, by property c. of proposition 30, Seq_1 and Seq_2 can be considered maximal; i.e. they are in $\mathcal{S}_{max}(r\sigma, R, u)$ and $\mathcal{S}_{max}(t\sigma, R, u)$, respectively.

Seccond, the construction of Seq is diferent in point $1.$. This point is modified as follows.

- *1.1.* Consideer all the positions p with AG-context λ in $r_1 + t_1$ such that $nf_{R_{AG}}(s|_p)$ is $-u$, and p is below positive variable position in s_0. By one of our hypothesis, this position has to be in the part of t_1, i.e. p is of the form $2.p'$, Since (t, σ) is $(u \succeq /)$-irreducible w.r.t. R by Seq_2, the position p' is replaced by a zero in a pair of Seq_2, in conjunction with other position q'. If this position q' is below variable in s_0, or $s_1|_q$ is not replaced by zero when obtaining $r_0 + t_0$ from $r + t$, then, the pair $(2.q', p)$ can be added to Seq, and therefore, the position p will be replaced by zero in the cancellation by Seq. But consideer now that q has been replaced by zero when obtaining $r_0 + t_0$ from $r + t$, in conjunction with a position l. Then, l is a position in the part of r_0, i.e. l is of the form $1.l'$, and l' is a position in r such that $r|_{l'}$ is $-u$. Since $nf_{R_{AG}}(r\sigma)$ is $(n-1)u - r'$ and Seq_1 is maximal, the position l' is replaced by a zero in a pair of Seq_1 of the form (k', l'). Observe that $r_-|_{k'} =_{AG} u$, and k' is not below $-$ in r_-. Therefore, since there are $-u$'s ground in the AG-context λ of r, and r is in AG-normal form, then, k' cannot be a position of r such that $r|_{k'}$ is ground. Thus, if k' is a position in r (not only in r_-), then, $r|'_k$ has not been replaced by zero when obtaining $r_0 + t_0$ from $r + t$. Consequently, we have that $1.k'$ is a position in s_1, such that $s_1|_{1.k'} =_{AG} u$; and the pair $(1.k', p)$ can be added to Seq.

  At this point, all the $-u$'s at top-level position, and below positive variable, are replaced by zero in the sequence Seq.

- *1.2.*

  Let s_2 be the cancellation of s_1 by the sequence Seq constructed until now. We proceed in a similar way to the step 2. of the previous proof; but now, u's and $-u$'s are cancelled, instead of v's and $-v$'s with $v \succ u$. We extend Seq by iterating, when possible, the process of finding pairs of top-level positions p and q such that, $s_2|_p$ is not below $-$, $nf_{R_{AG}}(s_2|_p) =_{AG} u$ and $nf_{R_{AG}}(s_2|_q) =_{AG} -u$; and adding the pair (p, q) to Seq.

- *1.3.*

  Let s_3 be the cancellation of s_1 by the sequence Seq constructed until now. We proceed in a similar way to the step $3.$ of the previous proof; but now, positions with u are considered; instead of positions with $v \succ u$. Let p be a top-level position in s_3 such that $s_3|_p =_{AG} u$ and p is not below variable in s_0. The lemma 44 applies for the term $s_0|_p$, with σ. Therefore, $(s_0|_p, \sigma)$ is irreducible w.r.t. R by a certain sequence Seq_p. We add the pairs of Seq_p to Seq, but prefixed by p.

The rest of the construction of Seq is as in the previous proof.

Third, as before, we want to see that $(r_0 + t_0, \sigma)$ is $(u' \succeq /)$-irreducible w.r.t. R by Seq. Observe that, by our hypothesis, $u \succeq u'$, and that a sequence u-restricted to R is u'-restricted to R. Let s_4 be the cancellation of s_1 by Seq. Consideer a certain position p in s_0 that is preserved in Seq, and such that $s_0|_p$ is (i) x not inmediately below $-$, or (ii) $-x$, for some variable x. The case $a.$, where the AG-context of p is not λ, is as before. In case $b.$, where the AG-context of p is λ, there are diferences. By the same reason as before, rules with left-hand-side smaller than u does not reduce (i) $s_4|_p$ or (ii) $nf_{R_{AG}}(-s_4|_p)$. Automatically, if $u \succ u'$, the result follows. Therefore, suppose that u' is u. By our hypothesis, there are no positive u's at top-level position in the AG-normal form of $t\sigma$; i.e. $nf_{R_{AG}}(t\sigma)$ is of the form $-lu + w$ or w, where $l > 0$ and w does not contain u at top-level position. Thus, the AG-normal form of $(r+t)\sigma$ either contain positive or zero u's, but no more than $n-1$, or it contain negative u's. But the positive or zero case is not possible, since, otherwise, u is not the maximal summand subject to rewritting in the AG-normal form of $(r+t)\sigma$. Observe that, thanks to step $1.2.$, at top level-position, all the possible u's and $-u$'s are cancelled. Therefore, in s_4 there are no positive u's at top-level position. It left to justify irreducibility of (i) $s_4|_p$ w.r.t. $nu \to r'$ and $-u \to (n-1)u - r'$, or of (ii) $nf_{R_{AG}}(-s_4|_p)$ w.r.t. $-u \to (n-1)u - r'$.

Consideer first the case (i). the rule $nu \to r'$ does not reduce $s_4|_p$, since s_4 does not contain positive u's. The rule $-u \to (n-1)u - r'$ cannot be aplied thanks to step $1.1.$ of the generation of Seq.

Consideer now the case (ii). Since $s_4|_p$ does not contain positive u's, then, $nf_{R_{AG}}(-s_4|_p)$ does not contain negative u's, and the rule $-u \to (n-1)u - r'$ cannot reduce it.

□

5 General Clauses

The inference system is extended to non-Horn clauses in the standard way, with (equality) factoring, which in the ground case is:

$$\textit{AG-factoring:} \qquad \frac{C \vee nu \simeq r \vee nu \simeq r'}{C \vee r \not\simeq r' \vee nu \simeq r'}$$

with the ordering restrictions that u is the maximal summand in the clause, which does not appear in a negative equation, and where $nu \simeq r$ is maximal w.r.t. $\succ_e$.

For the non-ground case, the two equations involved have to be oriented as the left premises of AG-superposition (note that if both orientations require to split a certain variable x, then it needs to be split only once). Let us denote by $\mathcal{I}$ the rules of $\mathcal{H}$ (with the same ordering restrictions as the factoring rule) plus this additional rule. By a relatively standard adaptation of the rule generation with respect to the Horn case (i.e., as for standard superposition, see [3]), we obtain the following:

Theorem 48 *The inference system $\mathcal{I}$ is refutation complete for general clauses.*

6 Extensions

We now very briefly comment on a few aspects that have not been treated yet in this paper.

Our completeness proofs are compatible with the notions for redundancy and saturation as in the *basic* framework and as in [15, 4]. Note that, by dealing with constrained clauses, no AG-unifiers are computed. Instead, the unification problems are stored in the constraints and a constrained clause $C \mid T$ is redundant if T is unsatisfiable. Apart from the well-known basicness restriction, an additional advantage is that only one conclusion is generated, instead of one conclusion for each AG-unifier[20, 16].

Checking the ordering restrictions in our framework is different from the usual situation. Instead of checking whether, say, for given terms s and t, there exists some ground σ such that $s\sigma \succ_{rpo} t\sigma$, we need to check whether this holds after normalising both sides by R_{AG}, that is, whether $nf_{R_{AG}}(s\sigma) \succ_{rpo} nf_{R_{AG}}(t\sigma)$. More work is needed for deciding such constraints, but it is not difficult to find sufficient conditions for ruling out redundant inferences (note that in practice, for efficiency reasons, such approximations are used as well for standard superposition, and that neither soundness nor completeness require to actually decide ordering constraints).

Example 49 *Suppose s is $f(f(0)-x)$ and t is x. It is easy to see that $s\sigma \succ_{rpo} t\sigma$ for all σ. But if σ is $\{\to f(0)\}$, both terms normalise w.r.t. R_{AG} into $f(0)$.* □

The fact that ordering restrictions are checked after normalisation w.r.t. R_{AG} complicates optimisations related to the analysis of the so-called *shielded* variables of a clause C, that is, variables that occur below a free symbol in C.

Example 50 *In the context of [10, 18], shieldedness of variables like x in the clause $f(x - f(a)) \not\simeq 0 \vee 2x \simeq b$ allow one to conclude that $2x$ cannot contain the maximal summand of $C\sigma$ for any σ and hence $2x$ need not be used as left premise in any inference. In our case, the instance where $x\sigma$ is $f(a)$ may generate the rule $2f(a) \to b$, and hence we can rule out the inferences only for other instances. Similar optimisations apply to right premises.* □

Also other shieldedness-related optimisations can be used. For example, let $e \simeq 0$ be an equation of a clause C where e is of the form $s + n_1x_1 + \ldots + n_kx_k \simeq 0$ and the distinct variables x_i do not occur elsewhere in s or in C. If $n_i = 1$ (or $n_i = -1$) for some i, then such an equation $e \simeq 0$ collapses the theory: $s + x \simeq 0$ implies $s + (-s + t) \simeq 0$ and hence $t \simeq 0$ for every t. Hence one can assume that any such a clause $C \vee s + x \simeq 0$ is eagerly replaced by C. This can be combined with the fact that $e \simeq 0$ is logically equivalent modulo AG to $s + nz \simeq 0$, where $n = gcd(n_1, \ldots, n_k)$ and z is a new variable.

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