Induction = I-Axiomatization + First-Order Consistency

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Abstract

In the early 80's, there was a number of papers on what should be called *proofs by consistency.* They describe how to perform inductive proofs, without using an explicit induction scheme, in the context of equational specifications and ground-convergent rewrite systems. The method was explicitly stated as a first-order consistency proof in case of pure equational, constructor based, specifications.

In this paper, we show how, in general, inductive proofs can be reduced to first-order consistency and hence be performed by a first-order theorem prover. Moreover, we extend previous methods, allowing non-equational specifications (even non-Horn specifications), designing some specific strategies. Finally, we also show how to drop the ground convergence requirement (which is called *saturatedness* for general clauses).

1 Inductive proofs

First-order specifications are ubiquitous in virtually all areas of computer science. In many cases, the intended meaning of a specification $\mathcal{E}$ is not its standard first-order semantics, i.e., the class of all its models, but rather a more specific class $\mathcal{M}$. Perhaps the best known example is the *initial* or *minimal Herbrand model* semantics, where $\mathcal{M}$ consists of the unique minimal Herbrand model of a set of Horn clauses with or without equality $\mathcal{E}$ (an algebraic specification, a logic program, a deductive data base, ...). Other interesting semantics that are used in practice as well include

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final semantics, or, for possibly non-Horn specifications $\mathcal{E}$, *perfect model* semantics or the class of all (minimal) Herbrand models.

In this paper we address the issue of inductive proofs in this broad sense: we wish to define general methods for (semi-)automatically (dis)proving the validity of conjectures C in classes of Herbrand models of $\mathcal{E}$, where C and $\mathcal{E}$ are sets of universally quantified clauses. In order to minimize user expertise, we do not want to require the user of such a (semi-)automatic inductive theorem prover to provide explicit induction schemes. Instead, induction will be based on simple, automatically generated, well-founded orderings $\succ$ on terms, like the *recursive path ordering* [Der82]. This is what distinguishes our approach from the *explicit induction* methods that will not be addressed in this paper.

Theory imposes severe limitations on our aims. Unlike it happens when proving standard first-order consequences (i.e., validity in the class of all models), from Gödels incompleteness theorem it follows that in general there are no complete proof systems that can be used to enumerate all inductively valid formulae, and hence inductive validity is not even semi-decidable. In Section 3 we will show that this is the case even for validity of equations in the initial model of very restrictive classes of specifications $\mathcal{E}$, like the ones presented by purely equational, convergent, linear, right shallow, constructor-based term rewrite systems, or by a convergent set of length-reducing word rewrite rules. This shows why almost no decidability results for inductive validity exist in the literature: even in very restricted and simple situations the problem remains undecidable. To our knowledge, the only decidability results are for ω*-complete* theories, i.e., where (for infinite models) the inductive theory coincides with the equational theory, like the *shallow* equational theories of [CHJ94] or the *Catalog* Horn theories of [Nie96].

However, under reasonable assumptions it is possible to obtain what we will call *refutation complete* procedures for inductive validity: procedures that provide in finite time a disproof for any conjecture that is *false* in $\mathcal{M}$. Then the situation is exactly reverse—but probably less useful in practice—to what happens in standard first-order logic, where all formulae *valid* (in the class of *all* models of $\mathcal{E}$) are provable in finite time.

It is interesting to observe that refutation completeness for inductive validity amounts to refutation completeness for *ground* conjectures: a conjecture C is valid in $\mathcal{M}$ if, and only if, all its ground instances are valid (since $\mathcal{M}$ is a class of Herbrand models) and one can simply enumerate (representatives of) all ground instances until a ground counterexample is found by the ground refutation procedure. This simple idea is in essence the basis of methods like the ones of [Red90, Zha88, BR95][1]. Indeed refutation procedures or decision procedures for ground conjectures are available in many practical settings, like in the presence of a *convergent* term rewrite system or an adequate *saturated* clausal presentation for $\mathcal{E}$.

[1] Of course we do not mean that these methods were designed only with the purpose of refuting false conjectures, or that they can only detect *ground* counterexamples; but the fact that all ground instances are explored (or *covered*) is required for soundness of their validity proofs.

In fact, if $\mathcal{M}$ includes all minimal Herbrand models of $\mathcal{E}$ (and hence, in particular, if $\mathcal{M}$ is the initial model of $\mathcal{E}$), then refutation completeness for ground conjectures C, i.e., the co-semi-decidability of $\mathcal{M} \models C$, is equivalent to the decidability of $\mathcal{M} \models C$. This is true because $\mathcal{M} \models C$ is semi-decidable as well: since $\mathcal{M}$ includes the minimal models, for every ground atom A we have $\mathcal{M} \models A$ if, and only if, $\mathcal{E} \models A$ (that is, A is a first-order logical consequence of $\mathcal{E}$), which is semi-decidable[2].

Our techniques will extensively deal with refutation completeness, among other reasons because a procedure that makes progress towards a disproof when given a false conjecture, frequently also advances towards a proof when given a valid conjecture. In the class of methods described in this paper, progress is made by inference rules *reducing* (with respect to $\succ$) counter examples, i.e., false ground instances $c\sigma$ of a conjecture c. Once a minimal (irreducible) counter example appears, it is detected by other means. Here we build upon previous work on *proofs by consistency* for the purely equational case, formerly also called *inductionless induction* methods, where minimal counter examples were detected in several forms, like equations $true = false$ [Mus80], equations between constructors [HH82], or *ground irreducible* non-trivial equations [JK86, Bac88, BL90], among others. We generalize those methods for the detection of inconsistencies by introducing the notion of *I-axiomatization*. These axiomatizations will be used in a uniform framework for proof by consistency that allows us to eliminate many restrictions on syntax and semantics (saturatedness requirements on $\mathcal{E}$, arbitrary universal formulae, more general redundancy notions and classes of models). The explicit use of axiomatizations was introduced in [Fri84] for equational specifications with free constructors, but not further generalized in subsequent papers.

An I-axiomatization is a set $\mathcal{A}$ of first-order formulae such that $\mathcal{A} \cup \mathcal{E} \cup C$ is consistent if, and only if, $\mathcal{M} \models C$. Such I-axiomatizations hence allow us to reduce inductive proofs to first-order consistency proofs, which in turn makes it possible to apply general-purpose first-order theorem provers to inductive validity problems. Note that we do of course not require $\mathcal{A}$ to completely axiomatize $\mathcal{M}$, which is impossible in general.

The issue of how to compute I-axiomatizations automatically will be treated in Section 7. The other main question to be answered is of course how to design efficient[3] procedures (i) proving the inconsistency of $\mathcal{A} \cup \mathcal{E} \cup C$ whenever C is false, and (ii) proving its consistency in as many cases as possible if C is valid.

[2] Ganzinger and Stuber [GS92] state that if validity of ground conjectures is not decidable, "inductive theorem proving is hopeless anyway". But in fact refutation completeness (rather than inductive theorem proving) becomes hopeless in that case only if $\mathcal{M}$ includes all minimal Herbrand models of $\mathcal{E}$, which is not true in their setting. Their perfect model is only one of the minimal models and hence a (rather hypothetical, however) co-semidecision procedure for ground validity would suffice for obtaining refutation completeness.

[3] The meaning of the word "efficient" here is of course relative to the context in which it is used. In particular the opposition between "matching based" and "unification based" proof methods is not relevant here since the problem is the termination of saturation rather than each individual deduction step.

The remainder of this paper is devoted to this in a number of different situations, depending on the intended semantics and the syntactic properties of $\mathcal{E}$ and C. An important step forward is a completeness result for the following two-stage approach which is common to our different strategies: on the one hand, new consequences are computed from $\mathcal{E} \cup C$ in a restricted way; on the other, each new consequence is checked for inconsistency with $\mathcal{A}$. The latter can be done either by a standard first-order prover or by a dedicated —in some cases, decision—procedure (see Section 8).

In Section 4 we first concentrate on the case where $\mathcal{E}$ is Horn, $\mathcal{M}$ is the minimal Herbrand model, and C is any set of clauses. The results are hence directly applicable to the case where $\mathcal{M}$ is the class of all Herbrand models if C contains only positive clauses. Different special cases of constructor-based specifications are handled separately in Section 5. In Section 6 we generalize these ideas to non-Horn $\mathcal{E}$ and perfect model semantics.

This paper does not contain many difficult new results; its contributions are more at the conceptual level and several proofs are actually adaptations of standard proofs. We show however how inductive validity can be reduced to first-order consistency, without the usual assumptions of the proofs by consistency method. The main advantage is that any saturation-based general-purpose first-order theorem prover can be used for inductive validity. For instance, we experimented with *Saturate* [NN93, GNN95] for this purpose. On all (small-sized) examples we tried, this experiment was quite successful. We reproduce some of the *Saturate* proofs in the paper.

2 Basic Notions and Notations

We use the standard definitions of [DJ90]: $T(\mathcal{F}, \mathcal{X})$ ($T(\mathcal{F})$) is the set of (ground) terms over a set of symbols $\mathcal{F}$ and a denumerable set of variables $\mathcal{X}$ (over $\mathcal{F}$), the subterm of t at *position* p is denoted $t|_p$, the result of replacing $t|_p$ by s in t is denoted $t[s]_p$, and syntactic equality of terms is denoted by $\equiv$.

A *multiset* over a set S is a function $M: S \to \mathbb{N}$. The union of multisets is defined as usual by $M_1 \cup M_2(x) = M_1(x) + M_2(x)$, and is sometimes simply written M_1, M_2.

If $\to$ is a binary relation, then $\leftarrow$ is its inverse, $\leftrightarrow$ is its symmetric closure, $\to^+$ is its transitive closure and $\to^*$ is its reflexive-transitive closure. We write $s \to^! t$ if $s \to^* t$ and there is no t' such that $t \to t'$. Then t is called *irreducible* and a *normal form* of s (w.r.t. $\to$). The relation $\to$ is *well-founded* or *terminating* if there exists no infinite sequence $s_1 \to s_2 \to \dots$ and it is *confluent* or *Church-Rosser* if the relation $\leftarrow^* \circ \to^*$ is contained in $\to^* \circ \leftarrow^*$. It is *locally confluent* if $\leftarrow \circ \to \ \subseteq \ \to^* \circ \leftarrow^*$. By Newman's lemma, terminating locally-confluent relations are confluent. A relation $\to$ on terms is *monotonic* if $s \to t$ implies $u[s]_p \to u[t]_p$ for all terms s, t and u and positions p. A *congruence* is a reflexive, symmetric, transitive and monotonic relation on terms.

An *equation* is a multiset $\{s, t\}$, denoted $s = t$ or, equivalently, $t = s$. A first-order clause is a pair of finite multisets of equations Γ (the *antecedent*) and Δ (the *succedent*), denoted by $\Gamma \rightarrow \Delta$. It is a *Horn clause* if Δ contains at most one equation. The empty clause $\Box$ is a clause where both Γ and Δ are empty.

A *rewrite rule* is an ordered pair of terms (s, t), written $s \rightarrow t$, and a set of rewrite rules R is a *term rewrite system* (TRS). The rewrite relation with R on $T(\mathcal{F}, \mathcal{X})$, denoted $\rightarrow_R$, is the smallest monotonic relation such that $l\sigma \rightarrow_R r\sigma$ for all $l \rightarrow r \in R$ and all substitution σ, and if $s \rightarrow_R t$ then we say that *s rewrites into t* with R. R is called terminating, confluent, etc. if $\rightarrow_R$ is. A rewrite system R is *convergent* if it is confluent and terminating; then every term t has a unique normal form w.r.t. $\rightarrow_R$, denoted by $nf_R(t)$, and $s = t$ is a logical consequence of R (where R is seen as a set of equations) iff $nf_R(s) = nf_R(t)$.

Let R be a set of ground equations or rewrite rules. Then the congruence $\leftrightarrow^*_R$ defines an equality Herbrand interpretation denoted by R^*, where the only predicate $=$ is interpreted by $s = t$ iff $s \leftrightarrow^*_R t$. We write $s = t \in R^*$ if $s \leftrightarrow^*_R t$. R^* satisfies (is a model of) a ground clause $\Gamma \rightarrow \Delta$, denoted $R^* \models \Gamma \rightarrow \Delta$, if $R^* \not\supseteq \Gamma$ or $R^* \cap \Delta \neq \emptyset$. The empty clause $\Box$ is hence satisfied by no interpretation. R^* satisfies a set of clauses S, denoted by $R^* \models S$, if it satisfies every clause in S.

A (strict partial) *ordering* on $T(\mathcal{F}, \mathcal{X})$ is an irreflexive transitive relation $\succ$. It is a *reduction* ordering if it is well-founded and monotonic, and stable under substitutions: $s \succ t$ implies $s\sigma \succ t\sigma$ for all substitutions σ. It fulfils the *subterm property* if $\succ \supseteq \rhd$, where $\rhd$ denotes the strict subterm ordering.

A well-known, easily automatizable reduction ordering that is adequate for the purposes of this paper is the *recursive path ordering* (RPO), which is based on a total ordering $\succ_{\mathcal{F}}$ (the *precedence)* on $\mathcal{F}$. Furthermore let $\mathcal{F}$ be the disjoint union of two sets lex and mul, the symbols with lexicographic and multiset status, respectively, and let $=_{mul}$ denote the equality of terms up to the permutation of direct arguments of symbols with multiset status.

Then RPO (with status) on ground terms is defined as follows.

$s = f(s_1, \ldots, s_m) \succ_{rpo} g(t_1, \ldots, t_n) = t$ iff:

1. $s_1 \succ_{rpo} t$ or $s_1 =_{mul} t$, for some i with $1 \leq i \leq m$ or
2. $f \succ_{\mathcal{F}} g$, and $s \succ_{rpo} t_j$, for all j with $1 \leq j \leq n$ or
3. $f = g$, $f \in lex$, $\langle s_1, \ldots, s_n \rangle \succ^{lex}_{rpo} \langle t_1, \ldots, t_n \rangle$, and $s \succ_{rpo} t_j$, for all j with $1 \leq j \leq n$
4. $f = g$, $f \in mul$, $\{s_1, \ldots, s_n\} \succ^{mul}_{rpo} \{t_1, \ldots, t_n\}$

where $\langle s_1, \ldots, s_n \rangle \succ^{lex}_{rpo} \langle t_1, \ldots, t_n \rangle$ if $\exists j \leq n$ s.t. $s_j \succ_{rpo} t_j$ and $\forall i < j \;\; s_i =_{mul} t_i$. Furthermore, $\succ^{mul}_{rpo}$ is the multiset extension of $\succ_{rpo}$, defined as the smallest ordering such that $S \cup \{s\} \succ_{rpo}^{mul} S' \cup \{t_1, \ldots, t_n\}$ whenever S is equal to S' up to $=_{mul}$ and $s \succ_{rpo} t_i$ for all i in $1 \ldots n$.

The *lexicographic path ordering* is the particular case of RPO where $\mathcal{F} = lex$, and the *multiset path ordering* (or RPO without status) is the particular case where $\mathcal{F} = mul$.

3 Limitations

To our knowledge, the only (syntactically defined) classes of axiomatizations $\mathcal{E}$ with a decidable inductive validity problem are the *w-complete* ones, i.e., (for infinite I) where the inductive theory coincides with the equational theory. This is the case for classes like the *shallow* equations of [CHJ94] or the *Catalog* Horn axiomatizations of [Nie96], which are both based on presentations where variables do not occur at depth more than one.

It seems to be *folk* knowledge that it is difficult to go beyond. In this section we very briefly give some insight for the reason why: even for (syntactically) very restricted and simple $\mathcal{E}$ the problem remains undecidable. We slightly adapt two results from [KNO90] (used there for showing the undecidability of *ground confluence*) in order to obtain the following:

Theorem 1 *The following problem is undecidable:*
INSTANCE: A finite, convergent, left- and right-linear (no variable occurs more than once per side of a rule), right monadic (right hand sides have depth at most 1), constructor-based term rewrite system R and an equation $s = t$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$
PROBLEM: Is $s = t$ valid in $\mathcal{T}(\mathcal{F})/_{=R}$?

Proof: In [KNO90] for each instance $(\{u_1, \ldots, u_n\}, \{v_1, \ldots, v_n\})$ of the modified Post correspondence problem (MPCP), a rewrite system R (with the aforementioned properties) is shown to exist, such that, very roughly, all ground terms of the form $f(\ u_{i_1}(\ldots(u_{i_k}(a)\ldots),\ v_{i_1}(\ldots(v_{i_k}(a)\ldots)\ )$ for $i_j \in 1\ldots n$ rewrite into $\top$, and all others into $\perp$ (the symbols of the MPCP become unary function symbols and $\top, \perp, a$ are constants).

Applying this construction to inductive validity: it is not difficult to see that $f(x, x) = \perp$ is valid in $\mathcal{T}(\mathcal{F})/_{=R}$ if, and only if, the instance of MPCP has no solution. $\square$.

Theorem 2 *There exists a finite, length reducing, convergent (string) rewrite system R over $\mathcal{T}(\mathcal{F}, \mathcal{X})$, where $\mathcal{F}$ consists of only unary symbols (and hence there is only one variable at each side of each rule) such that, given some constant symbol a, the following inductive validity problem is undecidable:*
INSTANCE: an equation $s = t$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$
PROBLEM: Is $s = t$ valid in $\mathcal{T}(\mathcal{F} \cup \{a\})/_{=R}$?

Proof: The word TRS S given in [KNO90] has the required properties and has an undecidable *right equivalence problem*: given two words x and y, does $xw \leftrightarrow_S^* yw$ hold for all w? This amounts to an undecidable inductive validity problem. □

4 Inductive Proofs and I-Axiomatizations

In this section we assume a finite signature $\mathcal{F}$, and that $\mathcal{E}$ is a finite set of Horn clauses (the *axioms*), C is a set of clauses (the *conjectures*), $\mathcal{I}$ is the minimal Herbrand model of $\mathcal{E}$, and we address the question whether $\mathcal{I} \models C$. Finally, let $\succ$ be a total reduction ordering on $\mathcal{T}(\mathcal{F})$.

For simplicity, we can assume as well that equality is the only predicate symbol, since (positive or negative) atoms $P(t_1 \dots t_n)$ can be expressed as (positive or negative) equations $P(t_1 \dots t_n) = true$, where $true$ is a new special symbol and P is considered as a (boolean) function symbol. Indeed, if $\mathcal{I}'$ is the resulting minimal Herbrand model, then clearly for every ground atom A it holds that $\mathcal{I} \models A$ iff $\mathcal{I}' \models A = true$ (note, however that $\mathcal{I}$ and $\mathcal{I}'$ are not isomorphic since two ground atoms that are false in $\mathcal{I}$ need not be in the same congruence class of $\mathcal{I}'$). In what follows, when P is a predicate symbol, we only allow equations (resp. disequations) of the form $P(s_1, \dots, s_n) = true$ (resp. their negation) which will be sometimes abbreviated as $P(s_1, \dots, s_n)$ (resp. $\neg P(s_1, \dots, s_n)$). In other words, equations of the form $P(s_1, \dots, s_n) = Q(t_1, \dots, t_m)$ are ruled out by our syntax. This will be always consistent with ordering strategies, if we assume $true$ to be the smallest term, which we will do in the rest of the paper.

Definition 3 *A set $\mathcal{A}$ of first-order formulas is an I-*axiomatization *of $\mathcal{I}$ if*

1. *$\mathcal{A}$ contains only purely universal sentences*
2. *$\mathcal{I}$ is the only Herbrand model of $\mathcal{E} \cup \mathcal{A}$ up to isomorphism.*

In other words, an I-axiomatization must contain enough negative information to rule out all non-minimal Herbrand models of $\mathcal{E}$, or, equivalently, it must ensure the well-known concept of "no confusion". As we mentioned before, in Section 7 we will extensively explain how to automatically derive $\mathcal{I}$-axiomatizations. Let us only remark at this point that, since an ordering on ground terms $\succ$ is available, a convenient and intuitive way for defining I-axiomatizations is given by the following:

Definition 4 *A ground term t is called* normal *if it is the (unique) minimal (w.r.t. $\succ$) representative of its congruence class in $\mathcal{I}$. Similarly, a ground clause c is normal if all terms occurring in c are normal.*

Lemma 5 *Let $\mathcal{A}$ be a set of first-order clauses such that $\mathcal{I} \models \mathcal{A}$ and $\mathcal{A} \models s \neq t$ for any two distinct normal terms s and t. Then $\mathcal{A}$ is an I-axiomatization.*

Proof: If J were a non-minimal Herbrand model such that $J \models \mathcal{A} \cup \mathcal{E}$, then $J \models u = v$ for some ground u and v such that $\mathcal{I} \not\models u = v$, i.e., two different congruence classes in $\mathcal{I}$ are merged in J, which is impossible since $\mathcal{A} \models s \neq t$ for the normal representatives s and t of these classes. □

In the following, the particular kind of I-axiomatization given by this lemma will be called *normal*, and, unless stated otherwise, we assume all I-axiomatizations to be normal.

Example 6 *Consider for $\mathcal{E}$ the set of equations*

$$\begin{cases} 0 + x &= x \\ s(x) + y &= s(x + y) \end{cases}$$

Assuming that $F = \{0, s, +\}$, an $\mathcal{I}$-axiomatization could be:

$$\forall x, y. s(x) \neq 0 \wedge (s(x) = s(y) \Rightarrow x = y)$$

Note that there is no need to express differences between terms headed by $+$ because they are equivalent to terms without $+$. Indeed if terms with $+$ are bigger w.r.t. $\succ$ than terms built of only s and 0, then this I-axiomatization is normal. □

Note that not all I-axiomatizations are normal: if a, b, c, d are constants with $a \succ b \succ c \succ d$ and $\mathcal{E} = \{a = b, c = d\}$ then $\mathcal{A} = \{a \neq c\}$ is an I-axiomatization that is not normal (any normal one must entail $b \neq d$).

The following key proposition allows us to reduce the problem of proving inductive theorem to the consistency of a finite set of clauses:

Proposition 7 *Let $\mathcal{A}$ be an I-axiomatization. Then $\mathcal{A} \cup \mathcal{E} \cup C$ is consistent, if, and only if, $\mathcal{I} \models C$.*

Proof: If $\mathcal{I} \models C$, then $\mathcal{I} \models \mathcal{A} \cup \mathcal{E} \cup C$, and hence $\mathcal{A} \cup \mathcal{E} \cup C$ is consistent. Conversely, if the set $\mathcal{A} \cup \mathcal{E} \cup C$ is consistent, then it has a Herbrand model, as it is a set of purely universal formulas. Now, by the second property of I-axiomatizations, Definition 3.2, this model should be $\mathcal{I}$, since $\mathcal{I}$ is the only Herbrand model of $\mathcal{E} \cup \mathcal{A}$. □

4.1 Inductive saturation

As we mentioned before, our aim is to define an inference system that *reduces* (with respect to $\succ$) counterexamples, i.e., the smallest false ground instance $c\sigma$ of some conjecture c in C. We will do this only if $c\sigma$ is not a normal clause, since for normal clauses we have the following:

Lemma 8 *Let $\mathcal{A}$ be a normal I-axiomatization, let c be a clause, and let $c\sigma$ be a normal clause such that $\mathcal{I} \not\models c\sigma$. Then $\mathcal{A} \cup \{c\sigma\}$ is inconsistent.*

Proof: $\mathcal{A} \cup \{c\sigma\}$ is inconsistent if (i) every model of $\mathcal{A}$ satisfies the negative literals of $c\sigma$ and (ii) no model of $\mathcal{A}$ satisfies any of the positive ones. If $\mathcal{I} \not\models c\sigma$ then $\mathcal{I} \models s = t$ for all negative equations $s = t$ in $c\sigma$. Since s and t are both normal, it must be the case that $s \equiv t$, which implies (i). For (ii), let $u = v$ be a positive equation $u = v$ in $c\sigma$. Since $\mathcal{I} \not\models c\sigma$, we have $u \not\equiv v$, and hence, since $\mathcal{A}$ is normal, $\mathcal{A} \models u \neq v$ and hence no model of $\mathcal{A}$ satisfies $u = v$. This implies the inconsistency of $\mathcal{A} \cup \{c\sigma\}$. □

Now it remains to design efficient procedures that are able to reduce any non-normal counterexample. Since this is essentially a well-known problem in first-order saturation-based theorem proving, we can rely on a large amount of existing results from this field. We refer to [BG94] for more details and re-state only the main results.

4.1.1 Inference Rules

Let us recall here only the ground versions of the following inference rules for Horn clauses in sequent notation, where $s \succ (t, \Gamma)$ denotes that $s \succ t$ and $s \succ u$ for all terms u occurring in Γ:

superposition right:

$$\frac{\Gamma' \rightarrow l = r \qquad \Gamma \rightarrow s = t}{\Gamma', \Gamma \rightarrow s[r]_p = t} \qquad \text{if} \quad \begin{array}{l} s|_p \equiv l \text{ and} \\ l \succ (r, \Gamma') \text{ and } s \succ (t, \Gamma). \end{array}$$

superposition left:

$$\frac{\Gamma' \rightarrow l = r \qquad \Gamma, s = t \rightarrow \Delta}{\Gamma', \Gamma, s[r]_p = t \rightarrow \Delta} \qquad \text{if} \quad \begin{array}{l} s|_p \equiv l \text{ and} \\ l \succ (r, \Gamma'),\ s \succ t \text{ and } s \succeq (\Gamma, \Delta). \end{array}$$

equality resolution:

$$\frac{\Gamma, s = s \rightarrow \Delta}{\Gamma \rightarrow \Delta} \qquad \text{if} \quad s \succeq (\Gamma, \Delta).$$

Non-ground versions of these rules are defined as usual. For example, by equality resolution on $\Gamma, s = t \rightarrow \Delta$, the conclusion $\Gamma\sigma \rightarrow \Delta\sigma$ is obtained if σ is the most general unifier of s and t and $s\sigma\theta$ can indeed be the maximal term of $(\Gamma, s = t \rightarrow \Delta)\sigma\theta$ for some ground θ (the latter condition is decidable if $\succ$ is some recursive path ordering; otherwise, approximations are used and some more inferences than needed may be computed). Also, the usual restriction in non-ground superposition that $s|_p$ must be non-variable applies here. It is also possible to further restrict the amount of inferences by working with constrained formulae, inheriting the generated unification and ordering restrictions as constraints [NR95].

4.1.2 Redundancy and Saturation

We now use multiset extensions for lifting the ordering $\succ$ on terms to an ordering on ground equations and clauses. Let C be a ground clause, and let $emul(s=t)$ be $\{s,t\}$ if $s=t$ is a positive equation in C, and $\{s,s,t,t\}$ if it is negative. Then we define the ordering $\succ_e$ on (occurrences of) ground equations in a clause by $e \succ_e e'$ if $emul(e) \succ_{mul} emul(e')$. Similarly, $\succ_c$ on ground clauses is defined $C \succ_c D$ if $mse(C) \; (\succ_{mul})_{mul} \; mse(D)$, where $mse(C)$ is the multiset of all $emul(e)$ for ocurrences e of equations in C. When clear from the context, we will sometimes write $\succ$ instead of $\succ_e$ and $\succ_c$. If c is a ground clause and S is a set of clauses, we denote by $S^{\prec c}$ the set of all ground instances of clauses of S that are smaller than c (w.r.t. $\succ$).

The following are well-known notions of redundancy in saturation-based first-order theorem proving (see [BG94]).

Definition 9

1. *A ground clause c is* redundant *in a set of clauses S if $S^{\prec c} \models c$. Similarly, a non-ground conjecture c is redundant if all its ground instances are.*
2. *A ground inference with rightmost premise c and conclusion c' is* redundant *in a set S if $S^{\prec c} \models c'$. Similarly, a non-ground inference is redundant if all its ground instances are.*
3. *A set of clauses S is* saturated *if all inferences with premises in C are redundant in S.*

Definition 10

1. *An* theorem proving derivation *is a sequence of sets of clauses $S_1, S_2, \ldots$ such that each S_{i+1} is obtained from S_i either by adding to S_i a logical consequence of S_i or by removing from S_i some clause that is redundant in S_i.*
2. *A clause is* persistent *in the derivation if for some j it belongs to all S_k with $k \geq j$.*
3. *A derivation is* fair *if every inference with persistent premises is redundant in S_j for some j.*

Lemma 11 ([BG94]) *Let $S_1, S_2, \ldots$ be a fair theorem proving derivation. Then the set of persistent clauses is saturated and logically equivalent to S_1.*

4.1.3 Model Generation

If $\mathcal{E}$ is consistent and saturated under superposition and equality resolution then the following construction produces a model R^* for $\mathcal{E}$ (remind that R^* demotes the congruence $\mathcal{T}(\mathcal{F})/_{=R}$) [BG94]. The construction proceeds by induction on $\succ_c$:

Definition 12 *An instance C of the form $\Gamma \rightarrow l = r$ of a clause in $\mathcal{E}$* generates *the rule $l \rightarrow r$ if*

1. $R^*_C \not\models C$,
2. $l \succ r$, *and* $(l = r) \succ_e e$ *for all equations e in Γ,*
3. *l is irreducible by R_C*

where R_C is the set of rules generated by all instances D of clauses in $\mathcal{E}$ such that $C \succ_c D$. We denote by R the set of rules generated by all ground instances of $\mathcal{E}$.

Theorem 13 ([BG94]) *The ground TRS R is convergent. Furthermore, if $\mathcal{E}$ is saturated under superposition and equality resolution then either $\Box \in \mathcal{E}$ or else $\mathcal{T}(\mathcal{F})/_{=R}$ is a model for $\mathcal{E}$, i.e., $\mathcal{E}$ is consistent. More precisely, $\mathcal{I}$ is (isomorphic to) $\mathcal{T}(\mathcal{F})/_{=R}$.*

From this theorem, clearly a term (or a clause) is normal if, and only if, it is in normal form with respect to R. Similarly, we will call a ground substitution σ *normal* (or *irreducible* by R) if $x\sigma$ is normal (or irreducible by R) for all $x \in Dom(\sigma)$.

4.1.4 Conjecture Superposition

Let us assume from now on that $\mathcal{E}$ is saturated under superposition and equality resolution. In Section 5.2 we will show how this requirement can be weakened in many cases. We now define *conjecture superposition*, a form of superposition where the leftmost premise is always a (definite) Horn clause of $\mathcal{E}$ and the rightmost premise c is a conjecture in C:

conjecture superposition:

$$\frac{D \vee l = r \qquad c}{(D \vee c[r]_p)\sigma} \quad \text{if} \quad \begin{array}{l} \sigma = mgu(c|_p, l) \text{ and } c|_p \text{ is not a variable, and} \\ \text{for some ground } \theta\text{:} \\ l\sigma\theta \succ (r\sigma\theta, D\sigma\theta), \text{ and, if } p \text{ is inside } s \text{ in a} \\ \text{negative literal } s = t \text{ of } c \text{ then } s\sigma\theta \succ t\sigma\theta \end{array}$$

Note that in this inference rule there are strong ordering restrictions on the leftmost premise, the clause of $\mathcal{E}$, but only a weak ordering restriction on the conjecture clause c has been imposed so far. Furthermore, here we have given the standard non-ground version of this inference rule, but again it is possible to apply constraint inheritance. For example, the basicness restriction can be imposed, i.e., no inferences are needed on terms introduced by unifiers of previous inferences generating ancestor conjectures (see [NR95, BGLS95] for the details).

4.1.5 Inductive Redundancy and Saturation

We now give definitions for the redundancy of conjectures and of induction superposition inferences. They roughly coincide with the ones given above, except that here they include the use, without any ordering limitations, of formulae that are known to be valid in $\mathcal{I}$. This includes *lemmas* proved by previous runs of our method or by any other means. The essence of induction is present in the fact that smaller (unproved) conjectures are applicable by what could be called the induction hypothesis. Let $\mathcal{L}$ denote a set of *lemmas*, i.e., arbitrary first-order clauses such that $\mathcal{I} \models \mathcal{L}$.

Definition 14 *A ground conjecture c is* redundant *in a set of conjectures C if $\mathcal{E} \cup \mathcal{A} \cup \mathcal{L} \cup C^{\prec c} \models c$. Similarly, a non-ground conjecture c is redundant if all its ground instances are.*

Definition 15 *A ground inference by conjecture superposition with rightmost premise c and conclusion c' is* redundant *in a set of conjectures C if $\mathcal{E} \cup \mathcal{A} \cup \mathcal{L} \cup C^{\prec c} \models c'$. Similarly, a non-ground inference is redundant if all its ground instances are.*

Definition 16 *A set of conjectures C is* saturated *if all inferences by conjecture superposition with rightmost premises in C are redundant in C.*

Definition 17

1. *An* induction derivation *is a sequence of sets of conjectures $C_1, C_2, \ldots$ such that each C_{i+1} is obtained from C_i either by adding to C_i a logical consequence of $\mathcal{E}, \mathcal{A}, \mathcal{L}, C_i$ or by removing from C_i some conjecture that is redundant in C_i.*

2. *A conjecture is* persistent *in the derivation if for some j it belongs to all C_k with $k \geq j$.*

3. *A derivation is* fair *if every conjecture superposition inference with a persistent rightmost premise is redundant in C_j for some j.*

Regarding the question of how to compute fair derivations in practice, one can proceed as in the standard first-order case; for example, the conjectures can be stored in some (priority) queue insuring that every conjecture c is eventually either proved redundant or else considered for conjecture superposition; since under our ordering restrictions adding the conclusion of an inference makes the inference redundant, this implies the fairness of the derivation.

Theorem 18 *Let $\mathcal{A}$ be a normal I-axiomatization. Let $C_0, C_1, \ldots$ be a fair induction derivation. Then $\mathcal{I} \models C_0$ if, and only if, $\mathcal{A} \cup \{c\}$ is consistent for all clauses c in $\cup_i C_i$.*

Proof: If there is some c in some C_j such that $\mathcal{A} \cup \{c\}$ is inconsistent, then $\mathcal{I} \not\models C_0$, since all such c are logical consequences from $\mathcal{E}, \mathcal{A}, \mathcal{L}, C_0$ and $\mathcal{I} \models \mathcal{E}, \mathcal{A}, \mathcal{L}$.

For the reverse implication, assume $\mathcal{A} \cup \{c\}$ is consistent for all c in $\cup_i C_i$ and that $\mathcal{I} \not\models C_0$. We will derive a contradiction from the existence of a minimal (w.r.t. $\succ$) ground instance $c\sigma$ of a clause c in $\cup_i C_i$ such that $\mathcal{I} \not\models c\sigma$. If c is redundant in some C_j then from the definition of redundancy of conjectures, it follows that there is some false instance of a conjecture in C_j that is smaller than $c\sigma$, contradicting the minimality of $c\sigma$.

Otherwise c is persistent. By Lemma 8, $c\sigma$ is not a normal clause (otherwise $\mathcal{A} \cup \{c\}$ would be inconsistent). Furthermore, we can assume that σ is normal, since otherwise σ is reducible by R into some σ' such that $\mathcal{I} \not\models c\sigma'$ and $c\sigma \succ c\sigma'$, contradicting the minimality assumption on $c\sigma$.

Hence $c\sigma$ is reducible by some rule in R at some *skeleton* position, i.e., a position p in c. Let $D \vee l = r$ be the clause in $\mathcal{E}$ that generated the rule $l\theta \rightarrow r\theta$ of R that reduces $c\sigma$. Then $(c\sigma)|_p \equiv l\theta$ and hence $c|_p$ and l are unifiable by an mgu σ'.

Furthermore, if $c\sigma$ is reducible only in some negative literal $s\sigma = t\sigma$, then, since $\mathcal{I} \models s\sigma = t\sigma$, the normal forms of $s\sigma$ and $t\sigma$ w.r.t. R coincide. Hence, if $s\sigma \succ t\sigma$, the maximal side $s\sigma$ is reducible by R. Therefore, there exists some inference by conjecture superposition

$$\frac{D \vee l = r \qquad c}{(D \vee c[r]_p)\sigma'}$$

whose conclusion has an instance $c\sigma[r\theta]_p$ such that $\mathcal{I} \not\models c\sigma[r\theta]_p$ and moreover $c\sigma \succ c\sigma[r\theta]_p$. By fairness, this conclusion is redundant in some C_j. But then from the definition of redundancy of inferences, it follows that there is some false instance of a conjecture in C_j that is smaller than $c\sigma$, contradicting the minimality of $c\sigma$. $\square$

4.2 More refined orderings

Up to now, for reasons of simplicity, we have considered only an ordering $\succ$ on ground terms and clauses. However, in some redundancy proofs it is convenient to consider more refined orderings. In particular, *subsumption* cannot be handled by the redundancy notions defined up to now; for example, the equation $f(a) = b$ is not redundant in the presence of the equation $f(x) = b$.

This can be solved by techniques that are well-known from the field of saturation-based theorem proving [BG94, NR95], and which we do not want to treat in detail here. Let us only mention one possibility: compare ground instances $t\sigma$ and $s\theta$ of terms (or clauses) t and s by an ordering $\succ_p$ on pairs defined by: $(t, \sigma) \succ_p (s, \theta)$ if either $t\sigma \succ s\theta$ or else $t\sigma \equiv s\theta$ and s subsumes t but not vice versa. The definitions of redundancy can be adapted as follows according to this idea:

If $c\sigma$ is a ground instance of a clause c and S is a set of clauses, we denote by $S^{\prec c\sigma}$ the set of all ground instances $d\theta$ of clauses of S such that $(c, \sigma) \succ_p (d, \theta)$. A ground instance $c\sigma$ of a conjecture c is then redundant in a set of conjectures C if $\mathcal{E} \cup \mathcal{A} \cup \mathcal{L} \cup C^{\prec c\sigma} \models c\sigma$, and a non-ground conjecture c is redundant if all its ground instances are.

4.3 Complete sets of positions

The only requirement for Theorem 18 to hold is that enough conjecture superposition inferences are computed in order to reduce the smallest false conjecture instance. In many cases, it is possible to determine, by analysis of a clause c, whether there is some subset P of the positions of c such that for all reducible (by R) $c\sigma$ where σ is normal, $c\sigma|_p$ is always reducible for some $p \in P$. In this case, we will call P a *complete* set of positions for c (and the given $\mathcal{E}$). In such a situation, it clearly suffices to compute conjecture superposition inferences only at the positions in P.

This generalizes a number of notions defined in the literature for the case where $\mathcal{E}$ is purely equational, like the ones of [Fri86, Küc89]. The development of techniques for finding small complete sets of positions in our setting is related to the techniques of Section 7. In the case of a constructor discipline, described in the next section, the notion of complete set of positions will be especially useful.

4.4 Selection strategies

From, e.g., [BG94] it is known that, for first-order theorem proving, superposition remains complete with *selection:* in each clause c an arbitrary negative literal can be selected and the only inferences involving c are superpositions and equality resolution steps on this selected literal.

For the purpose of proving inductive validity, conjecture superposition remains complete with selection as well, provided we add equality resolution on selected literals in conjectures. In the following, in each conjecture clause c (the rightmost premise), a negative literal may have been selected:

conjecture superposition with selection:

$$\frac{D \vee l = r \qquad c}{(D \vee c[r]_p)\sigma} \quad \text{if} \quad \begin{array}{l} \sigma = mgu(c|_p, l) \text{ and, for some ground } \theta, \\ l\sigma\theta \succ (r\sigma\theta, D\sigma\theta), \text{ and, if } p \text{ is inside } s \text{ in a} \\ \text{negative literal } s = t \text{ of } c \text{ then } s\sigma\theta \succ t\sigma\theta, \text{ and} \\ \text{if a literal } l \text{ in } c \text{ has been selected then } p \text{ is in } l. \end{array}$$

conjecture equality resolution:

$$\frac{l \neq r \vee c}{c\sigma} \quad \text{if} \quad \begin{array}{l} \sigma = mgu(l, r) \text{ and} \\ \text{the literal } l \neq r \text{ has been selected in } l \neq r \vee c. \end{array}$$

Now, inductive derivations are defined according to these rules and using any given selection function:

Theorem 19 *Let $\mathcal{A}$ be a normal I-axiomatization. Let $C_0, C_1, \ldots$ be a fair induction derivation with selection. Then $\mathcal{I} \models C_0$ if and only if $\mathcal{A} \cup \{c\}$ is consistent for all clauses c in $\cup_i C_i$.*

Proof: We have almost the same proof as for theorem 18. Only the following is new. Suppose $I \not\models c\sigma$, where σ is normal. Then $I \models s\sigma = t\sigma$ for all negative equations $s = t$ in c. If one such a negative equation $s = t$ is selected in c, either (i) $s\sigma$ and $t\sigma$ are the same term, and then there is an inference by equality resolution on $s = t$, or (ii) $s\sigma \succ t\sigma$ and hence $s\sigma$ is not normal, and then there is an inference by conjecture superposition on s. □

Example 20 *Let us show a proof obtained by an experimental implementation of the results given so far in the* Saturate *system [NN93, GNN95].*

Let gr *be a predicate symbol denoting the strict ordering on natural numbers defined by constructors $0, s$. The conjecture is the transitivity of the ordering. Note that a user-guided negative literal selection strategy is used, and that a normal I-axiomatization could be simply the one of example 6.*

```
1    : axiom gr(s(x1),0)=true
2    : axiom gr(x1,x2)=true -> gr(s(x1),s(x2))=true
3    : conje gr(x1,x2)=true,gr(x2,x3)=true -> gr(x1,x3)=true

The total LPO precedence is [gr,s,0,true]

| ?- ind.

        Select negative literal in conjecture:
         3    : conje gr(x1,x2)=true,gr(x2,x3)=true -> gr(x1,x3)=true
        Type the number (left-to-right): 1.
```

```
Inference by conjecture superposition of 1 on 3 gives:
 4    : conje gr(0,x1)=true -> gr(s(x2),x1)=true

Inference by conjecture superposition of 2 on 3 gives:
 5    : conje gr(s(x1),x2)=true,gr(x3,x1)=true -> gr(s(x3),x2)=true

        Select negative literal in conjecture:
         4    : conje gr(0,x1)=true -> gr(s(x2),x1)=true
        Type the number (left-to-right): 1.

        Select negative literal in conjecture:
         5    : conje gr(s(x1),x2)=true,gr(x3,x1)=true -> gr(s(x3),x2)=true
        Type the number (left-to-right): 1.

Inference by conjecture superposition of 1 on 5 gives:
 6    : conje gr(x1,x2)=true -> gr(s(x1),0)=true

Inference by conjecture superposition of 2 on 5 gives:
 7    : conje true=true,gr(x1,x2)=true,gr(x2,x3)=true -> gr(s(x1),s(x3))=true

clause 6 is demodulated by rules [1] giving
 8    : conje gr(x1,x2)=true -> true=true (tautology)

Clause 7: gr(x1,x2)=true,gr(x2,x3)=true -> gr(s(x1),s(x3))=true is redundant by:
 2    : axiom gr(x1,x3)=true -> gr(s(x1),s(x3))=true
 3    : conje gr(x1,x2)=true,gr(x2,x3)=true,-->,gr(x1,x3)=true]
...

Induction derivation successfully terminated.
```

4.5 More ordering restrictions

In the conjecture superposition rule, inferences may involve any literal of the conjecture c and any side of this literal, provided it is a positive one. This contrasts with the strict superposition rules where there are more ordering restrictions. In general, we cannot do better as shown by the following simple example:

Example 21 *Assume we have only 3 constants a, b, c such that $a \succ b \succ c$ and $\mathcal{E}$ consists of the single equation $b = c$. Consider the normal axiomatization $\mathcal{A} = \{a \neq c\}$ and the (false) conjecture $a = b$. It is consistent with $\mathcal{A}$ and only superpositions on the small side are possible.* □

This is a bit annoying since in the classical proof by consistency for the equational case, only superpositions on the maximal side of the equations are considered. This is because in the classical approach the axiomatization has stronger properties, which in our setting can be generalized as follows:

Definition 22 *$\mathcal{A}$ is a* strongly normal axiomatization *if $\mathcal{I} \models \mathcal{A}$ and, moreover,* *for all ground terms s, t such that s is minimal in its congruence class, $s \succ t$ and $\mathcal{I} \not\models s = t$, we have $\mathcal{A} \models s \neq t$.*

Note that in strongly normal axiomatizations we not only require $\mathcal{A} \models s \neq t$ when s and t are distinct normal terms, like in normal axiomatizations, but also when s is normal and $s \succ t$. Actually most I-axiomatizations of section 7 are strongly normal.

Lemma 23 *Every strongly normal axiomatization is normal, and hence it also is an I-axiomatization.*

Proof: This follows from the totality of the ordering on ground terms. □

If the conjecture superposition rule is restricted to inferences on maximal sides of conjecture literals only, we obtain a new definition of induction derivation (let us call it *restricted induction derivation*) of which the classical inductive completion methods are an instance. And we still have the analog of theorem 18:

Theorem 24 *If $\mathcal{A}$ is a strongly normal axiomatization and $C_0, C_1, \ldots$ is a fair restricted induction derivation, then $\mathcal{I} \models C_0$ if, and only if, $\mathcal{A} \cup \{c\}$ is consistent for all clauses c in $\cup_i C_i$.*

The proof requires only a slight modification of the proof of theorem 18.

5 Constructors

Usually, the structure of a specification with Herbrand model semantics can be seen as a set of constructor symbols $\mathcal{F}_0$ axiomatized by a set $\mathcal{E}_0$, to which (repeatedly) the complete definition of a new symbol has been added. For example, one can specify the natural numbers with constructors 0 and s, then define $+$ in terms of 0 and s, then define $*$ in terms of the $0, s, +$, then exp in terms of $0, s, +, *$, etc.

More formally, along this section we assume the following setting. Let $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{D}$. $\mathcal{F}_0$ is a non-empty set of *constructor* symbols and $\mathcal{D}$ is a set of *defined symbols*. We assume that $\mathcal{E}_0$ is a saturated subset of $\mathcal{E}$ built over $\mathcal{T}(\mathcal{F}_0, \mathcal{X})$. Terms in $\mathcal{T}(\mathcal{F}_0, \mathcal{X})$ are called *constructor terms*. The constructors are called *free* if $\mathcal{E}_0$ is empty (or consists of only tautologies). Defined symbols will be denoted by $f_1, f_2, \ldots$.

We will assume that terms in $T(\mathcal{F}_0)$ are always smaller in the ordering than terms in $T(\mathcal{F} \setminus \mathcal{F}_0)$. This is easily achieved in gerneral-purpose orderings like LPO or RPO, for which it suffices to define the precedence on symbols in such a way that $f \succ g$ for every $f \in \mathcal{D}$ and every $g \in \mathcal{F}_0$.

In addition, we will assume that $\mathcal{E}$ is *sufficiently complete*, that is, for every ground term s there is some ground constructor term t such that $\mathcal{E} \models s = t$.

This specification method is very general and, at the same time, convenient for our proof techniques for three main reasons. First, if $\mathcal{A}$ is a normal I-axiomatization for $\mathcal{E}_0$, then it is one as well for $\mathcal{E}$, since the normal terms are constructor terms. In the particular case of free constructors, $\mathcal{A}$ simply states that constructor terms are different. Second, by enriching in this way a saturated set $\mathcal{E}_0$ with the definition of a new symbol, usually the resulting set $\mathcal{E}'$ will be saturated as well. If this is not the case, we can still apply our techniques in many cases, as shown in Subsection 5.2. The third reason is explained in the next subsection: one can restrict the conjecture superposition inferences to a smaller subset of positions.

5.1 Constructors and complete sets of positions

In this subsection, in addition to the constructor-based setting, we assume $\mathcal{E}$ to be saturated. As we mentioned before, for Theorem 18 to hold we need enough conjecture superposition inferences in order to reduce the smallest false conjecture instance. Since, by sufficient completeness, all ground terms headed by a defined symbol f must be reducible by R, we have the following result:

Lemma 25 *If p is an innermost occurrence of a defined symbol f in a conjecture c, then $P = \{p \cdot p' \mid p \cdot p' \text{ is a position of } c\}$ is a complete set of positions.*

Now suppose it is known that, for some defined symbol f, all terms of the form $f(t_1, \ldots, t_n)$ are reducible at the topmost position if the arguments t_i are constructor terms. Then the set $\{p\}$ is already a complete set of positions if $c|_p$ is such a term $f(t_1, \ldots, t_n)$. In most specifications, this is indeed the case, since, in order to ensure sufficient completeness, the axioms defining f are usually precisely written like this. A simple particular case is:

Lemma 26 *In the case of free constructors, if p is an innermost occurrence of a defined symbol f in a conjecture c, then $\{p\}$ is a complete set of positions.*

Example 27 *If constructors are not free, $\{p\}$ needs not be complete. Let g, a, b be constructors axiomatized by the convergent TRS*

$$\{g(a) \to a, \;\; g(g(x)) \to g(x)\ \}$$

Let f be a completely defined by:

$$\{f(a) \to a, \;\; f(b) \to b, \;\; f(g(b)) \to b\}$$

Then some ground instances of $f(g(x))$ (by irreducible ground substitutions) are irreducible at topmost position. The root position alone is not inductively complete alone and, indeed, only overlapping at root position with the conjecture $f(g(x)) = b$ would yield to a tautology, whereas the conjecture is false ($f(g(a)) \to f(a) \to a$). □

In practice, it is usually easy to determine for each defined symbol f whether all $f(t_1, \ldots, t_n)$ with constructor arguments are reducible at the topmost position or not, and hence, whether $\{p\}$ is complete or the weaker result of Lemma 25 has to be applied.

In practice, when computing a derivation, when and how is an innermost defined symbol of a conjecture c selected for determining a complete set of positions P? A practical implementation includes a mechanism for ensuring fairness, that eventually obliges every (apparently) persistent conjecture c to be considered for inference computation. Selection of P for a conjecture c can be done at the moment c is going to be considered for superposition or before. This can be done automatically, by some heuristic, or by user interaction.

5.2 Dropping saturatedness: reductive definitions

Sufficient completeness is an undecidable property even for finite convergent string rewrite systems [KNZ87]. For convergent TRS where $\succ$ fulfills the aforementioned requirements, decidability can be recovered: then the property of sufficient completeness is equivalent to the ground reducibility of $f(x_1, \ldots, x_n)$, where the x_i are pairwise distinct variables, for every defined symbol f [JK89].

But this result does not provide a means to effectively construct sufficiently complete specifications, and in cases when the result does not apply, like when $\mathcal{E}$ is not saturated, sufficient completeness has to be ensured in some other way.

The following is a standard method to do this, by axiomatizing each defined symbol f in such a way that all ground terms containing f are, in a certain general sense, reducible, but *without any saturatedness requirement.* Below we show that this more general notion of reducibility will suffice not only for obtaining sufficient completeness, but also for the applicability of our techniques for inductive theorem proving in non-saturated $\mathcal{E}$.

Definition 28 *Let $\mathcal{E}$ be a (possibly non saturated) constructor-based specification where $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_1$ is a set of Horn clauses.*

Furthermore, assume that for every ground term s of the form $f(t_1, \ldots, t_n)$ where $t_j \in \mathcal{T}(\mathcal{F}_0)$ for $j \in 1 \ldots n$ and $f \in \mathcal{D}$, there is some clause in $\mathcal{E}_1$ with an instance $\Gamma \Rightarrow l = r$ such that $\mathcal{E} \models \Gamma, l = s$, l is headed by f, and $s \succ (\Gamma, r)$.

Then $\mathcal{E}$ is called a reductive definition*.*

Lemma 29 *Every reductive definition $\mathcal{E}$ is sufficiently complete.*

Proof: By contradiction. Let u be the smallest term (w.r.t. $\succ$) in $\mathcal{T}(\mathcal{F})$ such that $\mathcal{E} \models u = v$ for no v in $\mathcal{T}(\mathcal{F}_0)$. Let s be an innermost non constructor subterm of u. Then s is of the form $f(t_1, \ldots, t_n)$, where the t_j are constructor terms and $f \in \mathcal{D}$. Hence there is some clause in $\mathcal{E}_1$ with an instance $\Gamma \Rightarrow l = r$ such that $\mathcal{E} \models \Gamma, l = s$,

l is headed with f and $s \succ r$. But then $\mathcal{E} \models s = r$, and hence $\mathcal{E} \models u[s]_p = u[r]_p$, with $u[s]_p \succ u[r]_p$, contradicting the minimality of u. □

We believe that most practical cases of sufficiently complete non saturated specifications can be covered by this notion of reductive definition. For instance, consider the following example from [KZ95]. It is interesting because for the authors it is supposed to illustrate the weakness of the proof by consistency approach.

Example 30

$$\mathcal{E} = \left\{ \begin{array}{rcl} x+0 & = & x \\ s(x)+y & = & s(x+y) \\ gcd(0,x) & = & x \\ gcd(x,0) & = & x \\ gcd(x,x+y) & = & gcd(x,y) \\ gcd(x+y,y) & = & gcd(x,y) \end{array} \right.$$

This $\mathcal{E}$ is not saturated and cannot be turned into a finite saturated (or convergent) set of equations (even modulo the commutativity of $+$). Furthermore, note that the definition of gcd is clearly reductive: for every term s of the form $gcd(s^n(0), s^m(0))$, if $n = 0$ or $m = 0$ then s is equivalent to the smaller $s^n(0)$ or to $s^m(0)$; otherwise wlog. let $m = n + n'$; then $\mathcal{E} \models gcd(s^n(0), s^{n+n'}(0)) = gcd(s^n(0), s^n(0) + s^{n'}(0))$, which can be reduced by the fifth rule into $gcd(s^n(0), s^{n'}(0))$ which is smaller w.r.t. $\succ$ than s. The definition of gcd is hence complete by the previous lemma. □

Indeed, since reductive definitions $\mathcal{E}$ are not saturated in general, our techniques as explained in Section 4 are not refutation complete anymore. For instance, in the previous example, we cannot disprove a false conjecture like $gcd(x,x) = 0$, since the only conjecture superposition inferences produce $0 = 0$, and $gcd(x,x) = 0$ can be consistent with a normal I-axiomatization $\mathcal{A}$.

The cause of this problem is that Theorem 18 requires every counterexample to be reducible, which is not the case for $gcd(x,x) = 0$, whose minimal false instance $gcd(s(0), s(0)) = 0$ is not reducible by $\mathcal{E}$. However, a more careful analysis reveals that this problem does not appear if the false conjecture has some subterm of the form $gcd(x,y)$ where x and y are distinct variables. Then, for every instance $gcd(s,t)$ of it there exists an equivalent instance $gcd(s',t')$ that is reducible to some r such that $gcd(s,t) \succ r$, which suffices for Theorem 18. These ideas lead us to the following.

Definition 31 *A* definition pattern *is a term of the form $f(x_1, \ldots, x_n)$ where f is a defined symbol and x_i and x_j are distinct variables for $1 \leq i, j \leq n$ where $i \neq j$.*

Lemma 32 *Let $\mathcal{E}$ be a reductive definition and let c be a conjecture such that $c|_p$ is a definition pattern.*

Then for every ground instance $c\sigma$ where σ is normal, there exists some inference by conjecture superposition at position p with a conclusion c', and a normal substitution σ' such that $\mathcal{I} \not\models c\sigma$ implies $\mathcal{I} \not\models c'\sigma'$, and furthermore, $c\sigma \succ c'\sigma'$.

Proof: Let $c|_p$ be $f(x_1...x_n)$ and let s be $f(x_1...x_n)\sigma$. By Definition 28, there exists an instance $c_2 \vee l = r$ of a clause c_1 in $\mathcal{E}_1$ such that $\mathcal{E} \models l = s$, l is headed with f, and $s \succ (c_2, r)$. Furthermore, since σ is normal, $l \succeq s \succ r$. Hence there exists an inference of c_1 on c, whose conclusion has an instance of the form $c_2 \vee c\sigma[r]_p$ with the desired properties. □

From Lemma 32, it is clear that if $c|_p$ is a definition pattern, then $\{p\}$ is something similar to what we called before a complete set of positions of c. Indeed, below we will see that it plays the same role. As a consequence, in the remainder of this section we assume that every conjecture c has a selected position $p(c)$ (or simply p) such that the symbol at $c|_p$ is an innermost defined symbol.

Definition 33 *An induction derivation $C_1, C_2, \ldots$ is* reductively fair *if for every persistent conjecture c the term $c|_p$ is a definition pattern and every conjecture superposition on c at p is redundant in C_j for some j.*

Theorem 34 *Let $\mathcal{A}$ be a normal I-axiomatization. Let $C_0, C_1, \ldots$ be a reductively fair induction derivation. Then $\mathcal{I} \models C_0$ if, and only if, $\mathcal{A} \cup \{c\}$ is consistent for all clauses c in $\cup_i C_i$.*

Proof: We again replicate the proof of theorem 18, but now without using the saturatedness property, which was used for showing that if $c\sigma$ is not normal then $c\sigma$ is reducible by R. The purpose of this was to show that, for non-normal $c\sigma$ such that $\mathcal{I} \not\models c\sigma$, there is a conjecture superposition yielding a smaller false conjecture. However, this part can be replaced with the result of lemma 32 and the rest of the proof is the same. □

The previous theorem leaves us with the problem of how to achieve reductive fairness when a certain conjecture c appears such that c has no definition pattern and c cannot be proved redundant either. Hence this kind of derivation may *fail*.

In order to completely avoid failure, instead of requiring reductive fairness, we now slightly generalize the inference rule of conjecture superposition that handles persistent conjectures c without definition pattern in a different way. This is done by *abstracting out* some subterms of c, creating a logically equivalent conjecture c' that does have a definition pattern subterm:

Definition 35 *Let c be a clause and let $c|_q$ be a term t. Then the (logically equivalent) clause $x \neq t \vee c[x]_q$ is called a* variable abstraction *of c.*

Definition 36 *Let c be a conjecture and let p the selected innermost defined symbol in c. Let c' be the clause obtained from c by the smallest number of variable abstraction steps such that $c'|_p$ is a definition pattern. Furthermore, let d be a clause obtained by conjecture superposition on the position p in c'.*

Then we say that d can be obtained from c by an inference of conjecture superposition with abstraction.

Now we can state the corresponding results of Lemma 32 and Theorem 34 for conjecture superposition with abstraction.

Lemma 37 *Let $\mathcal{E}$ be a reductive definition and let c be a conjecture such that $c|_p$ is headed by a defined symbol.*

Then for every ground instance $c\sigma$, there exists some inference by conjecture superposition with abstraction at position p, whose conclusion d has an instance $d\sigma'$ such that $\mathcal{I} \not\models c\sigma$ implies $\mathcal{I} \not\models d\sigma'$, and moreover $c\sigma \succ d\sigma'$.

Proof: The proof is similar to the one of Lemma 32. Let $c\sigma|_p$ be the term t of the form $f(t_1...t_n)\sigma$ with $f \in \mathcal{D}$. Let s be the term $f(s_1...s_n)$, where each s_i is a minimal constructor representative of $t_i\sigma$ and hence $t \succeq s$. By Definition 28, there exists an instance $c_2 \vee l = r$ of a clause c_1 in D_i such that $\mathcal{E} \models l = s$, l is headed with f and $s \succ (c_2, r)$. Furthermore, since the s_i are minimal, $l \succeq s \succ r$.

The clause c by abstraction becomes a clause c' of the form

$c[f(x_1...x_n)]_p \vee x_1 \neq t_1 \vee \ldots \vee x_n \neq t_n$

Hence as in Lemma 32 there exists an inference of c_1 on c', whose conclusion has an instance of the form $c_2 \vee c\sigma[r]_p \vee s_1 = t_1 \vee \ldots \vee s_n \neq t_n$ with the desired properties: since all additional literals are smaller than one containing the original $f(t_1...t_n)$, the final result is smaller. Note that since t is replaced with something smaller, even if a finite number of equations between proper subterms of t is added, this still yields a smaller clause wrt. $\succ_c$. □

Theorem 38 *Let $\mathcal{A}$ be a normal I-axiomatization. Let $C_0, C_1, \ldots$ be a fair induction derivation with respect to conjecture superposition with abstraction. Then $\mathcal{I} \models C_0$ if, and only if, $\mathcal{A} \cup \{c\}$ is consistent for all clauses c in $\cup_i C_i$.*

The proof is the same as for theorem 34, replacing lemma 32 with lemma 37.

Example 39 *Continuing with example 30, here we show the proof of commutativity of gcd by the* Saturate *system. Note that the commutativity of $+$ is used as a lemma, and that indeed for all conjectures a definition pattern is selected.*

```
1    : axiom   0+x1=x1
2    : axiom   s(x1)+x2=s(x1+x2)
3    : axiom   gcd(x1,0)=x1
4    : axiom   gcd(0,x1)=x1
5    : axiom   gcd(x1,x1+x2)=gcd(x1,x2)
6    : axiom   gcd(x1+x2,x2)=gcd(x1,x2)
7    : lemma   x1+x2=x2+x1
8    : conje   gcd(x1,x2)=gcd(x2,x1)

The total LPO precedence is [gcd,+,s,0]
```

```
| ?- ind.

        Select innermost defined symbol in conjecture:
         8   : conje   gcd(x1,x2)=gcd(x2,x1)
        Type the number (left-to-right) of symbol: 1.

Inference by conjecture superposition of 3 on 8 gives:
 9    : conje   x1=gcd(0,x1)

Inference by conjecture superposition of 4 on 8 gives:
 10   : conje   x1=gcd(x1,0)

Inference by conjecture superposition of 5 on 8 gives:
 11   : conje   gcd(x1,x2)=gcd(x1+x2,x1)

Inference by conjecture superposition of 6 on 8 gives:
 12   : conje   gcd(x1,x2)=gcd(x2,x1+x2)

Clause 9: x1=gcd(0,x1) is redundant by instances:
         4    : axiom   gcd(0,x1)=x1
     clausal rewrite proof:
        by 4 we get x1=x1

Clause 10: x1=gcd(x1,0) is redundant by instances:
         3    : axiom   gcd(x1,0)=x1
     clausal rewrite proof:
        by 3 we get x1=x1

Clause 11: gcd(x1,x2)=gcd(x1+x2,x1) is redundant by instances:
 7    : lemma   x1+x2=x2+x1
 6    : axiom   gcd(x1+x2,x2)=gcd(x1,x2)
 8    : conje   gcd(x1,x2)=gcd(x2,x1)
     clausal rewrite proof:
        by 7 we get gcd(x1,x2)=gcd(x2+x1,x1)
        by 6 we get gcd(x2,x1)=gcd(x1,x2)
        by 8 we get gcd(x2,x1)=gcd(x2,x1)

Clause 12: gcd(x1,x2)=gcd(x2,x1+x2) is redundant by instances:
 7    : lemma   x1+x2=x2+x1
 5    : axiom   gcd(x1,x1+x2)=gcd(x1,x2)
 8    : conje   gcd(x1,x2)=gcd(x2,x1)
     clausal rewrite proof:
        by 7 we get gcd(x1,x2)=gcd(x2,x2+x1)
        by 5 we get gcd(x2,x1)=gcd(x1,x2)
        by 8 we get gcd(x2,x1)=gcd(x2,x1)

Induction derivation successfully terminated.
```

Example 40 *Again continuing with example 30, here we show the refutation of* $gcd(x,x) = 0$ *by* Saturate*. The input is the same as in the previous example, except that the conjecture, after abstraction for creating a definition pattern, is now:*

```
8 :  conje x1=x2 -> gcd(x1,x2)=0
```

```
...

inconsistency detected:

 24   : conje  -> x1=0

Disproof:
input: 1    : axiom   0+x1=x1
input: 4    : axiom   gcd(0,x1)=x1
input: 6    : axiom   gcd(x1+x2,x2)=gcd(x1,x2)
input: 8    : conje   x1=x2 -> gcd(x1,x2)=0

conjecture superp. of 6 on 8
 12   : conje   x1+x2=x2 -> gcd(x1,x2)=0

conjecture superp. of 4 on 12
 21   : conje   0+x1=x1 -> x1=0

demodulation of 21 by 1
 24   : conje   x1=0
```

Example 41 *Without any lemmas, we now prove $gcd(x, x) = x$, which by abstraction becomes:* `7 :  conje x1=x2 -> gcd(x1,x2)=x1`

```
 1    : axiom 0+x1=x1
 2    : axiom s(x1)+x2=s(x1+x2)
 3    : axiom gcd(x1,0)=x1
 4    : axiom gcd(0,x1)=x1
 5    : axiom gcd(x1,x1+x2)=gcd(x1,x2)
 6    : axiom gcd(x1+x2,x2)=gcd(x1,x2)
 7    : conje x1=x2 -> gcd(x1,x2)=x1

...

Inference by conjecture superposition of 3 on 7 gives:
 8    : conje x1=0 -> x1=x1 (tautology)

Inference by conjecture superposition of 4 on 7 gives:
 9    : conje 0=x1 -> x1=0 (tautology)

Inference by conjecture superposition of 5 on 7 gives:
 10   : conje x1=x1+x2 -> gcd(x1,x2)=x1

Inference by conjecture superposition of 6 on 7 gives:
 11   : conje x1+x2=x2 -> gcd(x1,x2)=x1+x2

Inference by conjecture superposition of 1 on 10 gives:
 12   : conje 0=x1 -> gcd(0,x1)=0

Inference by conjecture superposition of 2 on 10 gives:
 13   : conje s(x1)=s(x1+x2) -> gcd(s(x1),x2)=s(x1)
```

```
clause 12 is demodulated by rules [4] giving
 14   : conje 0=x1 -> x1=0 (tautology)

Clause 13: s(x1)=s(x1+x2) -> gcd(s(x1),x2)=s(x1) is redundant by instances:
...

Inference by conjecture superposition of 1 on 11 gives:
 15   : conje x1=x1 -> gcd(0,x1)=0+x1

Inference by conjecture superposition of 2 on 11 gives:
 16   : conje s(x1+x2)=x2 -> gcd(s(x1),x2)=s(x1)+x2

clause 15 is demodulated by rules [1,4] giving
 17   : conje x1=x1 (tautology)

clause 16 is demodulated by rules [2] giving
 18   : conje s(x1+x2)=x2 -> gcd(s(x1),x2)=s(x1+x2)

Clause 18: s(x1+x2)=x2 -> gcd(s(x1),x2)=s(x1+x2) is redundant by instances:
...

Induction derivation successfully terminated.
```

The redundancy proofs of 13 and 18 are left out here, because they are quite long, although easy: for 13, instances $s(x1) + x2 = s(x1 + x2)$ *of 2 are used to enable the instance* $gcd(s(x1), s(x1) + x2) = gcd(s(x1), x2)$ *of 5 and then create* $s(x1) = s(x1 + x2) \rightarrow gcd(s(x1), s(x1 + x2)) = s(x1)$ *which is subsumed by 7. The redundancy proof of 18 is similar, using 6 instead of 5.*

6 Non-Horn Axioms

It is well-known that if $\mathcal{E}$ contains some non-Horn axiom, then in general no longer a unique minimal Herbrand model exists. For example, if $\mathcal{E} \equiv \{p \vee q\}$ then both the models $\{p\}$ and $\{q\}$ are minimal.

A total reduction ordering $\succ$ on ground literals provides a way to single out one of the minimal models, the so-called *perfect model* (of $\mathcal{E}$ and $\succ$). The perfect model is the minimal one with respect to the set extension $\succ_{set}^{-1}$ of $\succ^{-1}$. If $\mathcal{E} \equiv \{p \vee q\}$ and $q \succ p$ then $\{p\} \succ_{set}^{-1} \{q\}$ and hence $\{q\}$ is the perfect model (see [BG91] for details).

In logic programming, the ordering $\succ$ is usually induced from the way non-Horn clauses are written: one positive atom is written in the head of the clause, and the other ones are written negatively in the tail. For instance, $p \vee q$ can be written $p :- \neg q$ or $q :- \neg p$. Heads are made big in the ordering. If the resulting ordering is not contradictory, then the program has a perfect model: roughly, a logic program with negation $\mathcal{E}$ is called (locally) *stratified* if there is some ordering $\succ$ such that for all ground instances of clauses $A :- A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ it holds that $A \succeq A_i$ and $A \succ B_j$ for $i \in 1 \ldots n$ and $j \in 1 \ldots m$ [Prz88].

Local stratification is too strong a condition for the existence of a perfect model, and it has been relaxed into *weak stratification*, where, roughly, only ground instances contributing to the model need to fulfill the requirements [PP90]. These ideas are generalized and extended to arbitrary clausal specifications with equality in [BG91]. There it is shown that the perfect model of $\mathcal{E}$ is precisely $\mathcal{T}(\mathcal{F})/_{=R}$, where R is the TRS *generated* by the saturation of $\mathcal{E}$ with respect to $\succ$. Apart from the aforementioned inference rules of superposition and equality factoring, for non-Horn clauses rules for factoring and *merging paramodulation* are needed as well (see again [BG91] for details).

If we assume that $\mathcal{E}$ is saturated under this inference system, our techniques for inductive theorem proving for Horn $\mathcal{E}$ given in the previous sections smoothly extend to the non-Horn case and perfect model semantics.

We consider *normal* terms, clauses, substitutions, and I-axiomatizations with respect to the perfect model, and normality in this sense coincides again with irreducibility with respect to R, the ground TRS *generated* by the saturated set of axioms $\mathcal{E}$. The ordering restrictions for the inference rule for conjecture superposition are now slightly more complicated, since the left premise $D \vee l = r$ can now be non-Horn. In this case (for ground clauses; at the non-ground level this has to be expressed as explained in Section 4). one cannot require l to be the strictly maximal term anymore. Instead one can impose that $l \succ r$ and $l \succ u$ for all terms u occurring in negative equations in D, and $l \succeq l'$ for all other positive equations $l' = r'$, and if $l \equiv l'$ then $r \succ r'$.

Theorem 42 *Let $\mathcal{I}$ be the perfect model of a saturated set $\mathcal{E}$, let $\mathcal{A}$ be a normal I-axiomatization, and let C_0 be any set of universal conjectures. Let $C_0, C_1, \ldots$ be a fair induction derivation with respect to non-Horn conjecture superposition.*

Then $\mathcal{I} \models C_0$ if, and only if, $\mathcal{A} \cup \{c\}$ is consistent for all clauses c in $\cup_i C_i$.

The results of [GS92] are closely related to our previous theorem. In both approaches, validity is proved by finite saturation, in our case under conjecture superposition, in their case, after an encoding by `gnd` predicates, by a larger set of inference rules with selection. They refer to perfect model semantics, as we do here, and also were the first to state the close relationship between redundancy (defined very similarly to the way it is done here) and inductive theorem proving.

But the methods are different in essence, however. We derive minimal counterexamples with respect to normal I-axiomatizations, whereas in [GS92] counterexamples are required to be ground: it is assumed that validity of ground clauses is decidable, and essentially the saturation with their encoding by `gnd` predicates amounts to an enumeration of the ground instances of the conjectures. Hence their method is more similar to the methods of [Red90, Zha88, BR95] which we mentioned in Section 1, than to proof by consistency.

7 Computing I-axiomatizations

In the previous sections, we designed inductive saturation methods, leaving two unanswered questions:

1. how is it possible to compute a (strongly) normal I-axiomatization ?

2. how do we derive (in)consistency ?

When the normal I-axiomatization is finite, it is easy to answer the second question: any saturation-based theorem prover can be used. We will see however that it is sometimes useful to consider infinite axiomatizations $\mathcal{A}$, in which case we have to address the second question. That is what we do in section 8.

Concerning the first question, we will see below how the work which has been done on proofs by consistency in equational theories can be reformulated in our framework: for instance, each of the procedures given in [Mus80, HH82, Bac88, JK89], which were designed in a pure equational setting, is an inductive saturation procedure corresponding to some (implicit) I-axiomatization.

We give in sections 7.1 and 7.2 the axiomatizations corresponding respectively to [Mus80] and [HH82]. Then we sketch how I-axiomatizations can be computed when $\mathcal{E}$ is a set of Horn clauses without equality in section 7.3. Finally, we give a general I-axiomatization for an arbitrary saturated set of axioms (or an extension of it with sufficiently complete definitions) in section 7.4. However, such an I-axiomatization, though recursive, will be infinite; we delay the problem of (in)consistency proofs for such axiomatizations until section 8.

7.1 Musser's approach

D. Musser in [Mus80] assumes that $\mathcal{E}$ is a convergent rewrite system such that there is a particular function symbol eq satisfying, for all terms s and t not containing $true, false, eq$, that $eq(s,t) =_{\mathcal{E}} true \Leftrightarrow s =_{\mathcal{E}} t$ and $eq(s,t) =_{\mathcal{E}} false \Leftrightarrow s \neq_{\mathcal{E}} t$. Moreover, $true$ and $false$ are assumed to be irreducible.

This corresponds to a very simple I-axiomatization:

Lemma 43 *With Musser's assumptions* $\{true \neq false\}$ *is an I-axiomatization.*

Proof: $\mathcal{I} \models \mathcal{A}$ as $true \neq_{\mathcal{E}} false$ as soon as the initial algebra is not trivial. If $\mathcal{M} \models \mathcal{E} \cup \mathcal{A}$, then, for any two ground terms s, t, if $\mathcal{I} \not\models s = t$, then the normal forms of s and t are different. It follows that $\mathcal{E} \models eq(s,t) = false$, hence $\mathcal{M} \models eq(s,t) = false$. Since $\mathcal{M} \models true \neq false$, $\mathcal{M} \not\models eq(s,t) = true$, hence $\mathcal{M} \not\models s = t$.

For any two ground terms, we have $\mathcal{M} \models s = t$ iff $\mathcal{I} \models s = t$. If $\mathcal{M}$ is an Herbrand model, it is isomorphic to $\mathcal{I}$. $\square$

This axiomatization is not normal. However, a strongly normal I-axiomatization can be obtained in an easy way. Consider

$$\mathcal{A} = \mathcal{D}_{eq} \cup \{s \neq t \Leftrightarrow eq(s,t) = false\}$$

where $\mathcal{D}_{eq}$ is the set of equations which defines eq.

Lemma 44 *With Musser's assumptions $\mathcal{A}$ is a (strongly normal) finite I-axiomatization.*

Proof: $\mathcal{I} \models \mathcal{A}$ follows from the assumptions. Ground terms that are minimal in their equivalence class are irreducible by convergence of $\mathcal{E}$. Now, if $s \succ t$ and s is an irreducible ground term, then $s \neq_{\mathcal{E}} t$, hence $eq(s,t) =_{\mathcal{D}} false$ by hypothesis. Then $\mathcal{A} \models eq(s,t) = false$, hence $\mathcal{A} \models s \neq t$. □

Note that the definition of eq needs not to be put inside $\mathcal{E}$ any more and that we do not need the property $eq(s,t) =_{\mathcal{E}} true \Leftrightarrow s =_{\mathcal{E}} t$. Actually, Musser's conditions imply that, roughly, there is an *equational* I-axiomatization.

7.2 Huet and Hullot's approach

G. Huet and J.-M. Hullot introduced [HH82] the constructor disciplines. This corresponds to the following axiomatization, as noticed by L. Fribourg [Fri84]:

$$\forall x_1,\ldots,x_n,y_1,\ldots,y_n.\ c(x_1,\ldots,x_n) = c(y_1,\ldots,y_n) \Rightarrow x_1 = y_1 \wedge \ldots \wedge x_n = y_n$$
For every constructor symbol c
$$\forall x_1,\ldots,x_n,y_1,\ldots,y_m.\ c(x_1,\ldots,x_n) \neq c'(y_1,\ldots,y_m)$$
For every pair of distinct constructors c, c'

Huet and Hullot assume moreover that $\mathcal{E}$ is given by a convergent rewrite system and that pure constructors ground terms are smaller than terms which contain at least a defined symbol. (Besides, every definition of a symbol is supposed to be sufficiently complete with respect to the constructors).

Lemma 45 *Under these assumptions, the above axioms form a (strongly) normal finite I-axiomatization.*

Proof: if s,t are ground terms such that $s \succ t$ and s is minimal in its equivalence class. By convergence, s is irreducible, hence s is a constructor term. Moreover, t being smaller is also a constructor term. Since s,t are distinct, it follows that $\mathcal{A} \models s \neq t$. □

This extends in a straightforward way to reductive specifications:

Lemma 46 *If $\mathcal{E}$ is a reductive definition, then Huet and Hullot's axiomatization is strongly normal.*

Proof: This follows from lemma 29. □

7.3 Domain closure and axiomatization computation

Now we want to design a general procedure for the computation of (strongly) (normal) I-axiomatizations. For this purpose, we start with a known procedure for Horn-clauses without equality which we borrow from the logic programming area. The idea is to use a domain closure axiom to compute a set of Horn clauses which defines the negation of the predicates. Such a computation is valid in the least fixed point of the set of Horn clauses and is used in the "explicit negation as failure" [Stu91].

The following is known as *Clark's completion* [Cla78]: write every set of clauses whose head has a top predicate P as a single implication

$$P(x_1, \ldots, x_n) \Leftarrow \phi$$

where $x_1, \ldots, x_n$ are variables and ϕ is a disjunction of conjunctions of the form

$$\exists \vec{y}. x_1 \equiv t_1 \wedge \ldots \wedge x_n \equiv t_n \wedge B$$

B being the body of the clause whose head is $P(t_1, \ldots, t_n)$. In the least fixed point of the set of clauses, the converse implication holds, hence, in this model, we have

$$\neg P(x_1, \ldots, x_n) \Leftarrow \neg \phi$$

Now, we may use a quantifier elimination procedure for the theory of finite trees and get a definition of $\neg P$. (See e.g. [Com91] for more details on the quantifier elimination procedures). The only weakness is that, if some clauses contain variables in the body which do not appear in the head, then universal quantifiers cannot always be eliminated. However, if this is not the case, then the result is an I-axiomatization.

Lemma 47 *If we assume that all variables occurring in the body of the clauses also occur in the head of the clause, then the above procedure computes a strongly normal I-axiomatization for Horn clauses without equality.*

Proof: (sketch) $\mathcal{I} \models \mathcal{A}$ and $\mathcal{A}$ satisfies $\mathcal{E} \not\models P(t_1, \ldots, t_n) = true$ if, and only if, $\mathcal{A} \models P(t_1, \ldots, t_n) \neq true$. Since $true$ is the smallest term, this guarantees the strong normality property. □

Example 48 *Let us consider the most simple example of Horn clauses.*

$$\begin{array}{rcl} & E(0) & \\ E(s(s(x))) & \Leftarrow & E(x) \end{array}$$

Then we write

$$E(x) \Leftarrow (x \equiv 0) \vee (\exists y. x \equiv s(s(y)) \wedge E(y))$$

by negating both members we get

$$\neg E(x) \Leftarrow (x \not\equiv 0) \wedge (\forall y. x \not\equiv s(s(y)) \vee \neg E(y))$$

which gives after quantifier elimination:

$$\neg E(x) \Leftarrow (x \equiv s(0) \vee (\exists z.x \equiv s(s(z)) \wedge \neg E(z)))$$

and hence the I-axiomatization:

$$\mathcal{A} = \begin{cases} \neg E(s(0)) & \\ \neg E(s(s(x))) & \Leftarrow \quad \neg E(x) \end{cases}$$

Another simple example is the definition of the strict ordering on natural numbers:

Example 49

$$\begin{array}{rcl} s(x) > 0 & & \\ s(x) > s(y) & \Leftarrow & x > y \end{array}$$

whose axiomatization can be computed, yielding:

$$\mathcal{A} = \begin{cases} 0 \not> x & \\ s(x) \not> s(y) & \Leftarrow \quad x \not> y \end{cases}$$

Note that in the case of Horn clauses without equality, proof by consistency is actually a negation as failure rule.

7.4 A strongly normal I-axiomatization for any saturated $\mathcal{E}$

J.-P. Jouannaud and E. Kounalis introduced in [JK86] the notion of *ground reducibility* in a pure equational setting: if $\mathcal{E}$ is a ground convergent rewrite system (hence saturated), then a term s is ground reducible when all its ground instances are reducible. The equation $s = t$ is inconsistent when $s \succ t$ and s is not ground reducible. This corresponds to the following axiomatization:

$$\mathcal{A} = \{s \neq t \mid s, t \in T(\mathcal{F}), \neg Red(s), s \succ t\}$$

where Red is the predicate which holds true on ground terms which are reducible by the convergent ground rewrite system R of theorem 13.

Lemma 50 *When $\mathcal{E}$ is finite and saturated, the above set $\mathcal{A}$ is a strongly normal I-axiomatization.*

Proof: $\mathcal{I} \models \mathcal{A}$ since, by theorem 13 and saturatedness of $\mathcal{E}$, $\mathcal{I}$ is isomorphic to $T(\mathcal{F})/_{=_R}$ and, by ground convergence of R, for every ground terms s, t, $R \models s = t$ and $s \succ t$ implies that the normal forms of s and t are identical, hence s is reducible.

Let s be a minimal ground term in its equivalence class and t be a ground term such that $s \succ t$ and $\mathcal{I} \not\models s = t$. s is irreducible by minimality, hence $s \neq t \in \mathcal{A}$, by definition. □

Of course, this result can be extended to finite sets of clauses $\mathcal{E}$ which are sufficiently complete w.r.t. a saturated subset $\mathcal{E}_0$.

8 (In)consistency proofs for $\mathcal{A} \cup \{c\}$

Our deduction procedure reduced the consistency of $\mathcal{E} \cup \mathcal{A} \cup \mathcal{C}$ to the consistency of each single conjecture c with $\mathcal{A}$. It remains to explain how we (dis)prove such simple consistency results.

We have seen that there are simple finite normal I-axiomatizations for constructor-based specifications. For such axiomatizations $\mathcal{A}$, the consistency of $\mathcal{A} \cup \{c\}$ is easily decided.

If we do not assume a constructor-based specification, we have still the normal axiomatization $\mathcal{A}$ of section 7.4. However, such a $\mathcal{A}$ is infinite, hence it cannot be computed in extension. Our purpose here is to show how such a set of axioms can be effectively handled, by lifting the ground disequalities at a non-ground level and using again a general purpose consistency prover.

Our first concern is to show that existing methods for the pure equational case (e.g. [JK86] and [Bac88]) can be abstracted and actually consist in a way of lifting the infinite set $\mathcal{A}$ to a finite one (sections 8.1 and 8.2). Once these (in)consistency proofs have been abstracted, we can use them in other (more general) settings. This is illustrated in section 8.3: we drop the restrictions of [JK86, Bac88] at a possible price of non-termination, while keeping the refutation completeness and avoiding to enumerate the Herbrand base.

Yet another advantage of our approach is to separate clearly the deduction phase (computing overlaps of $\mathcal{E}$ into the conjectures), the computation of axiomatizations, and the (in)consistency proofs (i.e. lifting of the axiomatization and, roughly, computing overlaps of $\mathcal{A}$ into c). Then, it is possible to replace one of the pieces with another (e.g. use another lifting), while keeping the other pieces (e.g. deduction rules and axiomatization).

First, let us observe that we do not really need to consider the (in)consistency of $\mathcal{A} \cup \{c\}$ for an arbitrary non-ground clause c. It follows from lemma 8 and theorems 18, 19, 24, 34, 38 that:

Theorem 51 *Let $\mathcal{A}$ be a normal (resp. normal, resp. strongly normal,....) I-axiomatization. Let $C_0, C_1, \ldots, C_n, \ldots$ be a fair induction derivation (resp. fair induction derivation with selection, resp. fair restricted induction derivation, resp. reductively fair induction derivation, resp. fair induction derivation with respect to conjecture superposition with abstraction).*

Then $\mathcal{I} \models C_0$ if and only if, for every clause $c \in \cup_i C_i$ and for every ground substitution σ, $\mathcal{A} \cup \{c\sigma\}$ is consistent.

Indeed, in the proofs of the theorems, we only use the above weaker assumption.

8.1 Jouannaud and Kounalis approach

In this section we assume (as Jouannaud and Kounalis do [JK86]) that $\mathcal{E}$ is a finite ground convergent rewrite system.

The idea is to lift $\mathcal{A}$ from the ground to the non-ground level.

First, the predicate *Red* can be lifted to non-ground terms, by means of the following finite set of Horn clauses:

$$\mathcal{S}_{Red} = \begin{cases} Red(l) & \text{For every left hand side } l \text{ of a rule in } \mathcal{E} \\ Red(f(x_1, \ldots, x_n)) \Leftarrow Red(x_i) & \text{For every function symbol } f \text{ and index } i \end{cases}$$

The minimal Herbrand model of this set of clauses is exactly the set of reducible ground terms.

The negation of *Red* is computed as explained in section 7.3, yielding a finite set of Horn clauses $\mathcal{S}_{\neg Red}$ such that, $\neg Red(s)$ holds in the minimal Herbrand model of $\mathcal{S}_{Red}$ if and only if $\mathcal{S}_{\neg Red} \models \neg Red(s)$. It corresponds actually to a finite tree automaton with disequality constraints (see [CJ97]).

Example 52 *Consider the following very simple example of integers with addition: function symbols are* $0, s, p, +$ *and* $\mathcal{E} = \{0 + x = x;\ \ s(x) + y = s(x+y),\ \ p(x) + y = p(x+y),\ \ s(p(x)) = x,\ \ p(s(x)) = x\}$. $\neg Red$ *is defined by the following set of Horn clauses:*

$$\mathcal{S}_{\neg Red} = \begin{cases} \neg Red(x) \Leftrightarrow P(x) \vee N(x) \\ P(0) \\ N(0) \\ P(s(x)) \Leftarrow P(x) \\ N(p(x)) \Leftarrow N(x) \end{cases}$$

Next, we replace the equality symbol in c with a new binary predicate symbol eq, yielding a clause $\widetilde{c}$. Finally, $\mathcal{A}$ can be lifted into the set $\widetilde{\mathcal{A}}$:

$$\widetilde{\mathcal{A}} = \{\forall \vec{x}.\ eq(s,t) \Rightarrow Red(s) \mid s, t \in T(\mathcal{F}, \mathcal{X}), s \succ t\} \quad \cup \quad \{\forall x. eq(x,x)\} \quad \cup \quad \mathcal{S}_{\neg Red}$$

where $\vec{x}$ is the set of variables occurring in s, t and $\succ$ is the lifting of the ordering to the non-ground level.

Lemma 53 *Let* $\mathcal{A}$ *be the normal axiomatization of section 7.4.* $\mathcal{A} \cup \{c\sigma\}$ *is consistent for all ground substitutions* σ *if and only if* $\widetilde{\mathcal{A}} \cup \widetilde{c}$ *is consistent.*

Proof: $\mathcal{A} \cup \{c\sigma\}$ is inconsistent if and only if

1. for every positive literal $s = t$ in c with $s\sigma \succ t\sigma$, $s\sigma$ is irreducible.

2. for every negative literal $s = t$ in c, $s\sigma$ and $t\sigma$ are identical.

By definition, this implies that $\mathcal{S}_{\neg Red} \models \neg Red(s\sigma)$, hence $\mathcal{A}' \models \neg eq(s\sigma, t\sigma)$, which shows that $\widetilde{\mathcal{A}} \cup \widetilde{c}$ is inconsistent.

Conversely, if $\widetilde{\mathcal{A}} \cup \widetilde{c}$ is inconsistent, then there is a ground substitution σ which falsifies $\widetilde{c}$. This implies that, for every negative literal $\neg s = t \in c$, $s\sigma$ and $t\sigma$ are identical and that, for every positive literal $s = t$ in c, $S_{\neg Red} \cup \{eq(s\sigma, t\sigma)\}$ is inconsistent. Then, assuming $s\sigma \succ t\sigma$, $s\sigma$ must be irreducible. It follows that $\mathcal{A} \cup \{c\}$ is inconsistent. □

There is still a problem since $\widetilde{\mathcal{A}}$ is still infinite. However, if, for every positive equation $s = t$ in c, s and t are comparable in the ordering $\succ$, then (dis)proving the consistency of $\widetilde{\mathcal{A}} \cup \widetilde{c}$ amounts to (dis)proving the consistency of the finite set of clauses:

$$\{Red(s) \mid s \succ t, s = t \in c\} \cup \mathcal{S}_{\neg Red}$$

On the other hand, if c is an equation $s = t$ where s, t are incomparable, $\widetilde{\mathcal{A}}$ is infinite, which corresponds to a *failure case* in Jouannaud and Kounalis method. This is not fully satisfactory and has been answered by L. Bachmair, as explained in the next section.

However, if a failure does not occur, lemma 53 together with theorem 51 allows to use a general-purpose (in)consistency prover, encompassing Jouannaud and Kounalis method: when Jouannaud and Kounalis's method does not fail, we have to check the consistency of a finite set of clauses without equality and, when their method fails, we have to check the consistency of an infinite set of clauses without equality.

Finally, let us point out that, under Jouannaud and Kounalis hypotheses, *Red* is a recursive predicate on non-ground terms (i.e. it is possible to decide whether $\mathcal{S}_{\neg Red} \models \exists \vec{x}. \neg Red(s))$. There are several proofs in the literature (from [Pla85] to [CJ97]). This implies the decidability of the consistency of an equation with $\mathcal{A}$.

The dedicated proofs of ground reducibility are certainly "more efficient" than the general method (inconsistency proof) we are currently designing. But, of course, we have more freedom, both in the design of $\widetilde{\mathcal{A}}$ and in the axioms $\mathcal{E}$ we can handle.

8.2 Bachmair's approach

L. Bachmair implicitly uses in [Bac91] the same axiomatization as in Jouannaud and Kounalis approach (with the same hypotheses on $\mathcal{E}$), but with a different (more adequate) lifting at the non-ground level. Now, instead of $\widetilde{A}$, we consider:

$$\mathcal{B} = \begin{cases} eq(x, y) \Rightarrow Red(x) \vee Red(y) \vee x \equiv y & \\ eq(s, t) \Rightarrow Red(s) & \text{For all terms } s, t \text{ such that } s \succ t \\ eq(x, x) & \\ \mathcal{S}_{\neg Red} \cup \mathcal{S}_{\neg \equiv} & \end{cases}$$

where $\equiv$ is the syntactic equality on ground terms (whose negation can easily be defined by a finite set of Horn clauses $\mathcal{S}_{\neg \equiv}$).

Note that the first axiom is not necessarily a consequence of the second set of formulas when s, t are not comparable at the non ground level. Similarly, the second set of formulas is not a consequence of the first axiom since s may be both irreducible and strictly larger than a reducible term t.

Lemma 54 *$\mathcal{B} \cup \{\widetilde{c}\}$ is consistent iff $\mathcal{A} \cup \{c\}$ is consistent.*

Actually, the same result holds if we remove either of the two first lines in the definition of $\mathcal{B}$.

This improves over $\widetilde{\mathcal{A}}$ since, now, for every clause c, we can compute a finite subset $\mathcal{B}_c$ of $\mathcal{B}$ such that $\mathcal{B}_c \cup \{\widetilde{c}\}$ is consistent iff $\mathcal{B} \cup \{\widetilde{c}\}$ is consistent: it suffices to restrict the second line in the definition of $\mathcal{B}$ to the pairs of terms s, t such that $s = t$ occurs in c. (We may also simply not consider the second line in the definition of $\mathcal{B}$, but we keep it because, in case this clause applies, it will result in more efficient inconsistency proofs).

Another slight extension would consist in merging the two cases, considering the more elegant axiomatization:

$$\mathcal{B}' = \{(eq(x, y) \wedge x > y) \Rightarrow Red(x)\} \quad \cup \mathcal{S}_{\neg Red} \cup \mathcal{S}_{\neg\equiv} \cup \mathcal{S}_{>}$$

(where $\mathcal{S}_{>}$ is a definition of $\succ$) of which all axioms in Bachmair's axiomatization would be a consequence. Again, the consistency of $\mathcal{A} \cup \{c\}$ would be equivalent to that of $\mathcal{B}' \cup \{\widetilde{c}\}$. However, we know that the consistency of $\mathcal{B} \cup \{\widetilde{c}\}$ is decidable, but the decidability of the consistency of $\mathcal{B}' \cup \{\widetilde{c}\}$ is an open problem, even if the interpretation $\succ$ of $>$ is an RPO.

8.3 The general case

For arbitrary Horn clauses with equality, the reducibility predicate can be defined as in section 8.1. We need however the equality:

$$\mathcal{S}_{Red} = \left\{ \begin{array}{rcl} Red(s) & \Leftarrow & s_1 = t_1 \wedge \ldots \wedge s_n = t_n \wedge s > t \\ & & \text{for every clause } s_1 \neq t_1 \vee \ldots \vee s_n \neq t_n \vee s = t \\ Red(f(x_1, \ldots, x_n)) & \Leftarrow & Red(x_i) \end{array} \right.$$

Using the method described in section 7.3, we can compute an axiomatization (assuming that there is no variable occurring in the conditions which does not occur in the corresponding body). We get a finite set of clauses $\mathcal{S}_r$ of the form :

$$\begin{array}{l} \neg Red(s) \Leftarrow s'_1 \neq t'_1 \wedge \ldots s'_m \neq t'_m \wedge \neg Red(u_1) \wedge \ldots \wedge \neg Red(u_k) \\ \qquad\qquad\qquad \wedge v_1 > w_1 \wedge \ldots \wedge v_p > w_p \wedge x_1 \not\equiv t_1 \wedge \ldots \wedge x_q \not\equiv t_q \end{array}$$

using the totality of the ordering on ground terms. Now, we get an automaton with semantic constraints. In the pure equational case, we get an automaton with disequality and inequality constraints. If, moreover, every equation can be oriented we are back to an automaton with disequality constraints. Finally, if $t \equiv t_1 \equiv \ldots \equiv t_n \equiv true$ we are back in the scope of the previous section.

Unfortunately, nothing guarantees in general that the semantic constraints can be eliminated. This is a problem: even if the theory in which the disequality constraints are interpreted is restricted to AC (associativity and commutativity), Red may be no longer recursive [KNRZ91]. Even if there are no disequality constraints $s'_i \neq t'_i$, Red is no longer recursive, as shown as a side result in [CNNR98]).

Let $\mathcal{B}^r$ be the set $\mathcal{B}$ of axioms, replacing $\mathcal{S}_{\neg Red}$ with $\mathcal{S}_r$.

Lemma 55 *$\mathcal{B}^r \cup \mathcal{E} \cup \{\widetilde{c}\}$ is consistent if and only if for every ground substitution σ, $\mathcal{A} \cup \{c\sigma\}$ is consistent.*

As before, for every clause c, we only need a finite subset (known in advance) $\mathcal{B}^r_c$ of $\mathcal{B}^r$. Hence, together with theorem 51, we have the desired set of axioms.

However, this lemma is not fully satisfactory in general as, for the inconsistency proof, $\mathcal{E}$ has to be considered again. There are however several restricted classes for which the disequalities $s'_i \neq t'_i$ are not present in $\mathcal{S}_r$. In such a case, $\mathcal{B}' \cup \{\widetilde{c}\}$ is consistent if and only if for every ground σ, $\mathcal{A} \cup \{c\sigma\}$ is consistent. We give below two such examples, illustrating the benefits of our general setting.

First when the negative literals in the clauses of $\mathcal{E}$ are always of the form $s = true$ or $s = false$ (i.e. Boolean equations), there is no disequalities in $\mathcal{S}_r$ (hence no need for $\mathcal{E}$ in the previous lemma). This is illustrated in the next example.

Example 56 *This is a natural definition of finite sets (of, say, natural numbers, but it does not matter here). Function symbols include ins (binary), $\emptyset$ (constant) and $\in$ (binary). $\in$ and ins are defined by:*

$$\left\{\begin{array}{lcl} ins(x,l) = l & \Leftarrow & x \in l \\ x \in ins(y,l) & \Leftarrow & x \in l \\ x \in ins(x,l) & & \end{array}\right.$$

(the $= true$ have been removed for the predicates definitions). Note that there is no finite set of free constructors and that there is no purely equational specification.

The first computation step gives a definition of the membership predicate and of the reducibility predicate for lists:

$$\left\{\begin{array}{rcl} x \in l & \Leftarrow & \exists y, l'.x \in l' \wedge l \equiv ins(y,l') \vee \exists l'.l \equiv ins(x,l') \\ Red(l) & \Leftarrow & \exists x\, \exists l'.l \equiv ins(x,l') \wedge x \in l' \end{array}\right.$$

The second step negates both sides:

$$\left\{\begin{array}{rcl} x \notin l & \Leftarrow & (\forall y, l'.x \notin l' \vee l \not\equiv ins(y,l')) \wedge \forall l'.l \not\equiv ins(x,l') \\ \neg Red(l) & \Leftarrow & \forall x\, \forall l'.l \not\equiv ins(x,l') \vee x \notin l' \end{array}\right.$$

The last step is quantifier elimination, the patterns being inserted back in the body of the clause:

$$\mathcal{S}_r = \begin{cases} x \notin \emptyset & \\ x \notin ins(y, l') & \Leftarrow \quad y \not\equiv x \wedge x \notin l' \\ \neg Red(\emptyset) & \\ \neg Red(ins(x, l)) & \Leftarrow \quad \neg Red(l) \wedge x \notin l \end{cases}$$

And $\mathcal{E}$ is not needed in $\mathcal{A}$ as there is no (semantic) disequality constraint in $\mathcal{S}_r$.

Another example which illustrates this method is the way AC symbols can be handled. Assume that the ordering on terms is a lexicographic path ordering and that $+$ is associative and commutative. Let t be any ground term which is minimal in its equivalence class. A subterm of t which is headed by $+$ has the following restricted syntactic form: $+(t_1, +(t_2, \ldots, +(t_n, t_{n+1}) \ldots))$ (i.e. it is a right comb) where $t_1, \ldots, t_n$ are not headed with $+$ and $t_1 \leq \ldots \leq t_{n+1}$. $\mathcal{S}_r$, which specifies the minimal terms in each equivalence class, is defined accordingly. In general this yields a definition of $\mathcal{S}_r$ which does not contain disequalities, but may contain ordering constraints. Let us give an example (in which even ordering constraints are not necessary):

Example 57 *Assume $\mathcal{F} = \{0, 1, +, *\}$ and*

$$\mathcal{E} = \begin{cases} x + y & = \quad y + x \\ x + (y + z) & = \quad (x + y) + z \\ x + (y + z) & = \quad y + (x + z) \\ 0 + x & = \quad x \\ 0 * x & = \quad 0 \\ 1 * x & = \quad x \\ (1 + x) * y & = \quad y + (x * y) \end{cases}$$

The four first equations $\mathcal{E}_0$ form a saturated set (see [MN90]) and $\mathcal{E}$ is a reductive definition w.r.t. $\mathcal{E}_0$.

$\mathcal{S}_r$ specifies the irreducible ground terms w.r.t. $\mathcal{E}_0$:

$$\mathcal{S}_r = \begin{cases} \neg Red(0) & \\ \neg Red(x) & \Leftarrow \quad NZ(x) \\ NZ(1) & \\ NZ(1 + x) & \Leftarrow \quad NZ(x) \end{cases}$$

9 Conclusion and Future Work

The method of proofs by consistency has lost part of its popularity since it was developed in the early 80s. The main reasons are its requirements: pure equational specifications (this is not true of some extensions such as in [BL90] for Horn clauses), ground convergent presentations,... and its inefficiency, due to the divergence of completion.

Here we have generalized the technique, taking advantage of the powerful recent developments on saturation methods for first-order theorem proving. This was possible because of the generalization of ideas already present in [Fri84, KM87] for pure constructor systems: give explicitly a first-order axiomatization which reduces inductive proofs to proofs by consistency.

Besides this generalization (the introduction of I-axiomatizations) we have shown how to generalize the proofs by consistency methods to Horn clauses and arbitrary clauses, taking advantage of ordered strategies, and we have also shown how to drop the ground convergence requirement for constructor specifications.

Regarding future work, apart from studying further possible applications to other semantics like behavioural ones, we are seeking for more significant examples that could be automatized using the *Saturate* system, in order to investigate up to which degree the methods are automatizable, and how they compare with the other existing methods.

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