Lógica en la Informática / Logic in Computer Science
June 17th, 2019. Time: 2h30min. No books or lecture notes.

Note on evaluation: eval(propositional logic) = max{ eval(Problems 1,2,3), eval(partial exam) }.
eval(first-order logic) = eval(Problems 4,5,6).

1) Let F and G be arbitrary propositional formulas. Prove your answers using only the definitions of propositional logic.

A) Is it true that if $F \models G$ and $F \models \neg G$ then F is unsatisfiable?

B) Is it true that if F is unsatisfiable then $(G \vee F) \rightarrow G$ is a tautology?

2) Using the Tseitin transformation, we can transform an arbitrary propositional formula F into a set of clauses $T(F)$ (a CNF with auxiliary variables) that is *equisatisfiable*: F is SAT iff $T(F)$ is SAT. Moreover, the size of $T(F)$ is linear in the size of F.

2A) Assuming $P \neq NP$, is there any transformation T' into an equisatisfiable linear-size DNF? If yes, which one? If not, why?

2B) Is there any similar transformation T' into a linear-size DNF, such that F is a tautology iff $T'(F)$ is a tautology? If yes, which one? If not, why?

3) A pseudo-Boolean constraint has the form $a_1x_1 + \ldots + a_nx_n \leq k$ (or the same with $\geq$), where the coefficients a_i and the k are natural numbers and the x_i are propositional variables. Which clauses are needed to encode the pseudo-Boolean constraint $2x + 3y + 4z + 6u + 8v \leq 10$ into SAT, if no auxiliary variables are used? Which clauses are needed in general, with no auxiliary variables, for a constraint $a_1x_1 + \ldots + a_nx_n \leq k$?

4) Formalize and prove by resolution that sentence D is a logical consequence of the other three. Use (among others) a binary predicate symbol $OwnsCar(x, y)$ meaning "x owns the car y".

A: Paul McCartney is rich.

B: All cars with diesel engines smell badly.

C: Rich people's cars never smell badly.

D: Paul McCartney owns no diesel car.

5A) Consider a binary function symbol s and the following first-order interpretations I and I':

I: where D_I is the set of natural numbers and where $s_I(n, m) = n + m$.

I': where $D_{I'}$ is the set of integer numbers and where $s_{I'}(n, m) = n + m$.

Write the simplest possible formula F in first-order logic with equality using only the function symbol s and the equality predicate $=$ (no other symbols), such that F is true in one of the interpretations and false in the other one. Do not give any explanations.

5B) Consider binary function symbols s and p and the first-order interpretations I and I' where D_I is the set of real numbers and I' where $D_{I'}$ is the set of complex numbers and where in both cases, s is interpreted as the sum (as before) and p is interpreted as the product. Same question as 5A: complete the formula F below, using only symbols s and p:

$F: \quad \exists y \exists z\, (\, (\forall x\, p(x, y) = \ldots) \wedge p(z, z) = s(\ldots)\,)$

6A) Let F be the formula $\forall x\, p(c, x) \wedge \exists y\, (\, q(y) \vee \neg p(y, y)\,)$. Let G be the formula $\exists z\, (\, p(z, c) \vee q(z)\,)$. Do we have $F \models G$? Prove it.

6B) Let F be the formula $\forall x\ (\ p(x, x)\ \wedge\ \neg p(x, f(x))\ \wedge\ \neg p(x, g(x))\ \wedge\ \neg p(f(x), g(x))\ )$.
Is F satisfiable? If so, give a model with the smallest possible sized domain. If not, prove unsatisfiability.