Resolution in Propositional Logic

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Problem Solving and Constraint Programming
Session 3

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Overview of the session

- Inference rules
- Resolution
- Ordered resolution
- Practical remarks

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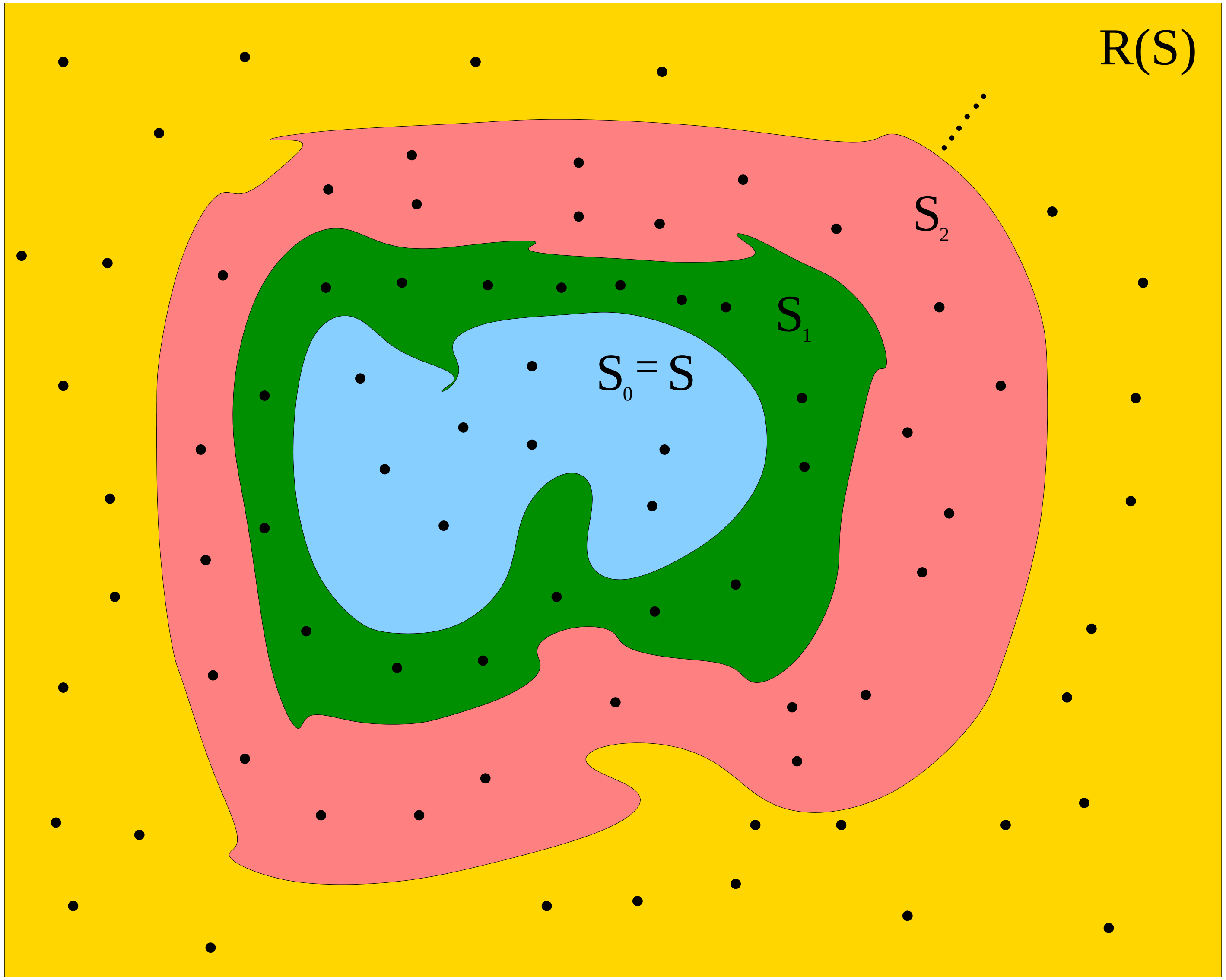
Inference rules

- They allow one to deduce new formulas from given ones
- Given an inference rule R and a set of formulas S, we define:
  - The closure of S under R, denoted $R(S)$ is the set of all formulas that can be obtained in zero or more deduction steps from S using R.
  - More formally, for $i \geq 0$

$$\begin{array}{rcl} S_0 & = & S \\ S_{i+1} & = & S_i \cup R_1(S_i) \end{array} \quad \text{and } R(S) = \bigcup_{i=0}^{\infty} S_i$$

    where $R_1(S_i)$ is the set of all formula obtained from S_i in exactly one application of R.

Inference rules - Closure



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Inference rules - properties

- R is correct iff $F \in R(S)$ implies $S \models F$
  - That is, the closure only contains logical consequences (but maybe not all of them)
- R is complete iff $S \models F$ implies $F \in R(S)$
  - That is, the closure contains all logical consequences (but maybe something more)
- Ideally, we want correct and complete inference rules
- A weaker notion of completeness is refutational completeness:

$$S \text{ unsatisfiable} \implies \Box \in R(S)$$

- If R is correct and refutationally completely, then

$$S \text{ unsatisfiable} \iff \Box \in R(S)$$

EXERCISE: prove the last property

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Resolution

- The resolution inference rule is the following:

$$\frac{p \vee C \quad \neg p \vee D}{C \vee D}$$

- We will see that:
  - Resolution is
    - Correct
    - Not complete
    - Refutationally complete
  - If S is a finite set of clauses, then $Res(S)$ is also finite
- Hence, given a set of clauses S, its satisfiability is checked by:
  1. Computing $Res(S)$
  2. **If** $\square \in Res(S)$ **Then** UNSAT ; **Else** SAT

Resolution - Properties

EXERCISE: prove that

- If S is finite, $Res(S)$ is also finite
- Resolution is not complete
- Resolution is correct
- Resolution is refutationally complete

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Ordered resolution

- The proof of refutational completeness introduces ordered resolution
- Given clauses S and a total ordering on the variables in S:

$$p_1 < p_2 < p_3 < \dots$$

we can define ordered resolution:

$$\frac{p \vee C \qquad \neg p \vee D}{C \vee D} \quad \text{if } p > q \text{ for all var. } q \in C \vee D$$

- It is easy to see that:
  - If S is finite, $ResOrd(S)$ is also finite
  - It is correct (because resolution is)
  - It is refutationally complete (same proof suffices)
- Hence, it is better from the practical point of view

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Practical Remarks

- In practice, even ordered resolution is not efficient enough
- SAT engines based on resolution not used in practice
- However, resolution plays a crucial role in DPLL