Quick Introduction to Propositional Logic

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Logic and Algebra in Computer Science

Session 1

Fall 2009, Barcelona

Departament de Llenguatges i Sistemes Informàtics

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Overview of the session

- Definition of Logic
- Definition of Propositional Logic
  - Syntax
  - Semantics
- General Concepts in Logic
  - Reduction to SAT
- Parenthesis removal
- Logical equivalences
- Conversion to CNF and DNF
  - Via truth table
  - Via distributivity
  - Via Tseitin

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What is a Logic?

- A logic is a language
  - But any logic is non-ambiguous language
- Logic = syntax + semantics
- When defining a logic we have to define:
  - Syntax:
    - what is a formula F?
  - Semantics:
    - what is an interpretation I?
    - when does an interpretation satisfy a formula?
- Trade-off in every logic: expressivity vs automation
- Lots of different logics, but here only propositional logic
  - Other logics: first-order, temporal, fuzzy, ...

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Definition of Propositional Logic

SYNTAX (what is a formula?)**:**

- Vocabulary consists of a set $\mathcal{P}$ of propositional variables, usually denoted by (subscripted) $p, q, r, \ldots$
- The set of propositional formulas over $\mathcal{P}$ is defined as:
  - Every propositional variable is a formula
  - If F is a formula, $\neg F$ is also a formula
  - If F and G are formulas, $(F \land G)$ is also a formula
  - If F and G are formulas, $(F \lor G)$ is also a formula
  - Nothing else is a formula
- Formulas are usually denoted by (subscripted) $F, G, H, \ldots$
- Examples:

$$p \qquad \neg p \qquad (p \lor q) \qquad \neg(p \land q)$$

$$(p \land (\neg p \lor q)) \qquad ((p \land q) \lor (r \lor \neg q)) \qquad \ldots$$

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Definition of Propositional Logic (2)

SEMANTICS (what is an interpretation I, when $I \models F$?):

- An interpretation I over $\mathcal{P}$ is a function $I : \mathcal{P} \rightarrow \{0,1\}$.
- I satisfies F (written $I \models F$) if and only if $eval_I(F) = 1$.
- $eval_I : Formulas \rightarrow \{0,1\}$ is a function defined as follows:
  - $eval_I(p) = I(p)$
  - $eval_I(\neg F) = 1 - eval_I(F)$
  - $eval_I(\ (F \wedge G)\ ) = min\{eval_I(F), eval_I(G)\}$
  - $eval_I(\ (F \vee G)\ ) = max\{eval_I(F), eval_I(G)\}$
- If $I \models F$ we say that
  - I is a model of F or, equivalently
  - F is true in I.

Definition of Propositional Logic - Examples

EXAMPLE:

- Let F be the formula $(p \wedge (q \vee \neg r))$.
- Let I be such that $I(p) = I(r) = 1$ and $I(q) = 0$.
- Let us compute $eval_I(F)$ (use your intuition first!)

Definition of Propositional Logic - Examples

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$$eval_I((p \wedge (q \vee \neg r))) \quad =$$

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$$eval_I((p \wedge (q \vee \neg r))) \quad = \quad min\{\, eval_I(p),\, eval_I((q \vee \neg r))\, \}$$

Definition of Propositional Logic - Examples

EXAMPLE:

- Let F be the formula $(p \land (q \lor \neg r))$.
- Let I be such that $I(p) = I(r) = 1$ and $I(q) = 0$.
- Let us compute $eval_I(F)$ (use your intuition first!)

$$\begin{aligned} eval_I((p \land (q \lor \neg r))) &= min\{\, eval_I(p),\, eval_I((q \lor \neg r))\, \} \\ &= min\{\, eval_I(p),\, max\{\, eval_I(q), eval_I(\neg r)\}\, \} \end{aligned}$$

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- Is there any I such that $I \models F$?

Definition of Propositional Logic - Examples

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&= min\{\ 1,\ max\{\ 0, 1 - 1\}\ \} \\
&= 0
\end{aligned}
$$

- Is there any I such that $I \models F$?
  YES, $I(p) = I(q) = I(r) = 1$ is a possible model.

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Definition of Prop. Logic - Examples (2)

EXAMPLE

- We have 3 pigeons and 2 holes. If each hole can hold at most one pigeon, is it possible to place all pigeons in the holes?
- Vocabulary: $p_{i,j}$ means i-th pigeon is in j-th hole
- Each pigeon is placed in at least one hole:

$$(p_{1,1} \vee p_{1,2}) \wedge (p_{2,1} \vee p_{2,2}) \wedge (p_{3,1} \vee p_{3,2})$$

- Each hole can hold at most one pigeon:

$$\begin{array}{l} \neg(p_{1,1} \wedge p_{2,1}) \wedge \neg(p_{1,1} \wedge p_{3,1}) \wedge \neg(p_{2,1} \wedge p_{3,1}) \wedge \\ \neg(p_{1,2} \wedge p_{2,2}) \wedge \neg(p_{1,2} \wedge p_{3,2}) \wedge \neg(p_{2,2} \wedge p_{3,2}) \wedge \\ \neg(p_{1,3} \wedge p_{2,3}) \wedge \neg(p_{1,3} \wedge p_{3,3}) \wedge \neg(p_{2,3} \wedge p_{3,3}) \end{array}$$

- Resulting formula has no model
- Note that we have relaxed the syntax of propositional logic

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Small Syntax Extension

- We will write $(F \rightarrow G)$ as an abbreviation for $(\neg F \vee G)$
- Similarly, $(F \leftrightarrow G)$ is an abbreviation of $((F \rightarrow G) \wedge (G \rightarrow F))$

They both capture very intuitive concepts, which ones?

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Small Syntax Extension

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They both capture very intuitive concepts, which ones?

- $I \models (F \rightarrow G)$ iff $I \models F$ implies $I \models G$

  (note that if $I \not\models F$ then $(F \rightarrow G)$ is trivially satisfied by I)

- $I \models (F \leftrightarrow G)$ iff $I \models F$ and $I \models G$ or
  $I \not\models F$ and $I \not\models G$

  iff $eval_I(F) = eval_I(G)$

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General Concepts in Logic

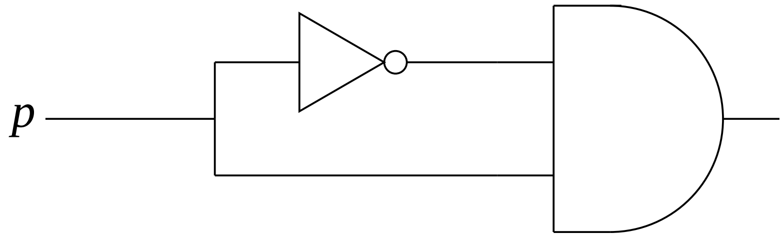
Let F and G be arbitrary formulas. Then:

- F is satisfiable if it has at least one model
- F is unsatisfiable (also a contradiction) if it has no model
- F is a tautology if every interpretation is a model of F
- G is a logical consequence of F, denoted $F \models G$, if every model of F is a model of G
- F and G are logically equivalent, denoted $F \equiv G$, if F and G have the same models

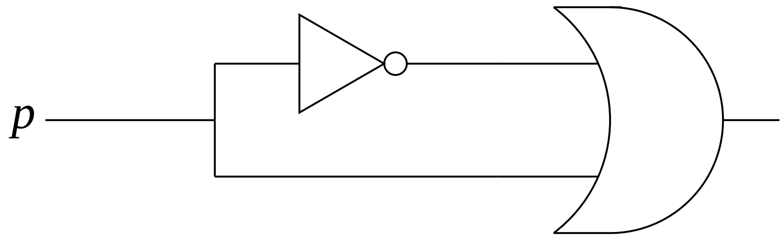
Note that:

- All these concepts are independent of the logic.
- All definitions are based on the concept of model.

General Concepts in Logic (2)



- Circuit corresponds to formula $(\neg p \wedge p)$
- Formula unsatisfiable amounts to "circuit equals 0"



- Circuit corresponds to formula $(\neg p \vee p)$
- Formula is a tautology amounts to "circuit equals 1"

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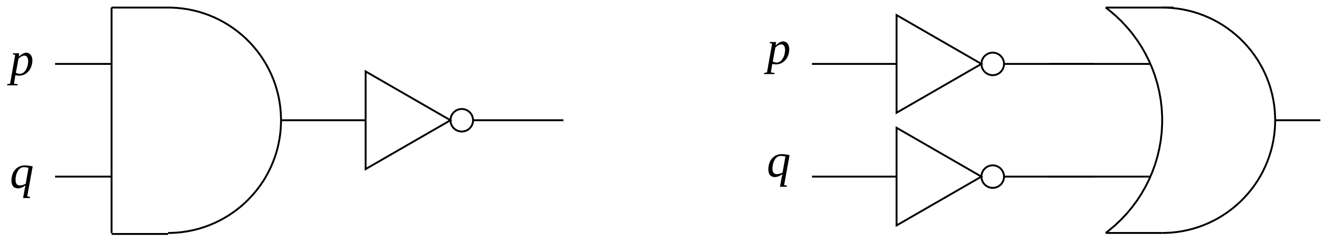
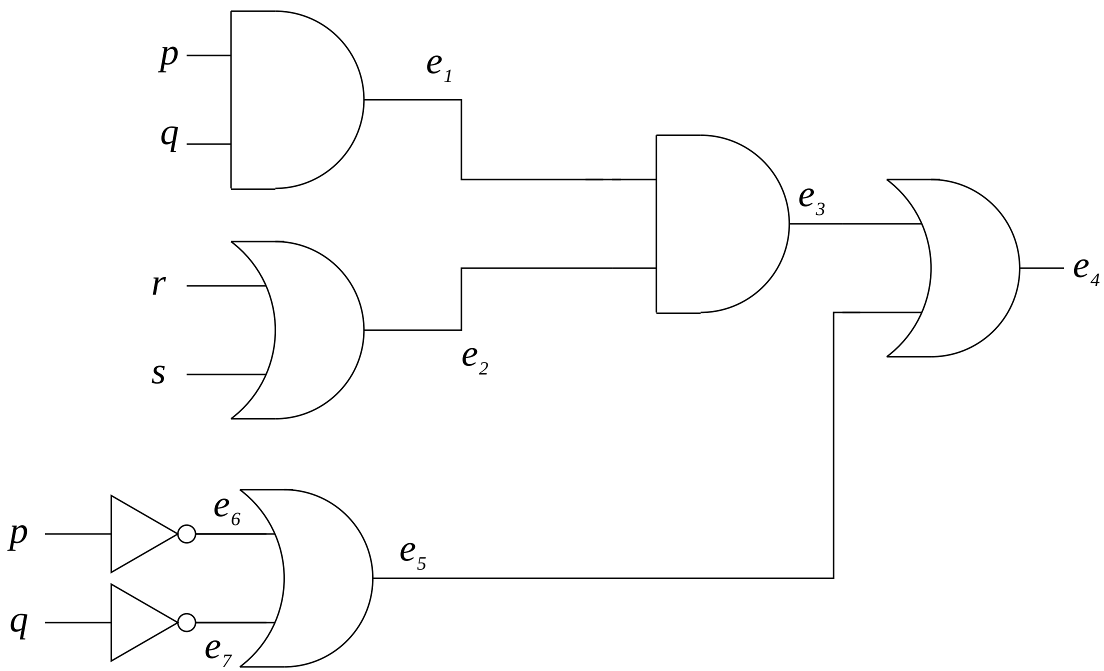
General Concepts in Logic (3)



- Circuit on the left corresponds to formula $F := \neg(p \wedge q)$
- Circuit on the right corresponds to formula $G := (\neg p \vee \neg q)$
- They are functionally equivalent, i.e. same inputs produce same output
- Cheapest, fastest, less power-consumption circuit is then chosen
- That corresponds to saying $F \equiv G$

General Concepts in Logic (4)



$$e_1 \neq e_5 \text{ in the circuit amounts to } \left.\begin{array}{lcll} e_1 & \leftrightarrow & (p \wedge q) & \wedge \\ e_2 & \leftrightarrow & (r \vee s) & \wedge \\ e_3 & \leftrightarrow & (e_1 \wedge e_2) & \wedge \\ e_4 & \leftrightarrow & (e_3 \vee e_5) & \wedge \\ e_5 & \leftrightarrow & (e_6 \wedge e_7) & \wedge \\ e_6 & \leftrightarrow & (\neg p) & \wedge \\ e_7 & \leftrightarrow & (\neg q) & \end{array}\right\} \models e_1 \leftrightarrow \neg e_5$$

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General concepts - Reduction to SAT

Assume we have a black-box **SAT** that given a formula F:

- **SAT**(F)=YES iff F is satisfiable
- **SAT**(F)=NO iff F is unsatisfiable

How to reuse SAT for detecting tautology, logical consequences, ...?

- F tautology iff **SAT**($\neg F$)=NO
- $F \models G$ iff **SAT**($F \wedge \neg G$)=NO
- $F \equiv G$ iff **SAT**($(F \wedge \neg G) \vee (\neg F \wedge G)$)=NO

Hence, a single tool suffices.

COURSE GOAL: learn how to build such a black-box SAT

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Removing parenthesis

- Formulas like $(\,(F_1 \vee F_2) \vee F_3)$ are usually written as $F_1 \vee F_2 \vee F_3$
- Given a formula $F_1 \vee F_2 \vee F_3$ we can understand it as $(\,(F_1 \vee F_2) \vee F_3)$ or $(F_1 \vee (F_2 \vee F_3)\,)$
- But this doesn't matter since $(\,(F_1 \vee F_2) \vee F_3) \equiv (F_1 \vee (F_2 \vee F_3)\,)$
- However, what if we write $F \wedge G \vee H$?
  In this case, $(\,(F \wedge G) \vee H) \not\equiv (F \wedge (G \vee H)\,)$ ...
- Similarly, take $F \longrightarrow G \longrightarrow H$.
  Again $(\,(F \longrightarrow G) \longrightarrow H) \not\equiv (F \longrightarrow (G \longrightarrow H)\,)$
- Ambiguity is fixed by assigning priorities and type of associativities

Removing parenthesis (2)

- From most to least priority: $\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$
- All connectives are left-associative

EXAMPLES:

- $\neg F_1 \wedge F_2 \vee F_3 \rightarrow \neg F_4$ is

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EXAMPLES:

- $\neg F_1 \wedge F_2 \vee F_3 \rightarrow \neg F_4$ is

$$(\,(\,(\,(\,\neg F_1) \wedge F_2) \vee F_3) \rightarrow (\neg F_4)\,)$$

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Removing parenthesis (2)

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EXAMPLES:

- $\neg F_1 \wedge F_2 \vee F_3 \longrightarrow \neg F_4$ is

$$(\, (\, (\, (\, \neg F_1) \wedge F_2) \vee F_3) \longrightarrow (\neg F_4) \,)$$

- $F_1 \wedge F_2 \wedge F_3 \longrightarrow \neg F_4 \leftrightarrow F_5$ is

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Removing parenthesis (2)

- From most to least priority: $\neg \quad \wedge \quad \vee \quad \longrightarrow \quad \longleftrightarrow$
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EXAMPLES:

- $\neg F_1 \wedge F_2 \vee F_3 \longrightarrow \neg F_4$ is

$$(\,(\,(\,(\,\neg F_1) \wedge F_2) \vee F_3) \longrightarrow (\neg F_4)\,)$$

- $F_1 \wedge F_2 \wedge F_3 \longrightarrow \neg F_4 \longleftrightarrow F_5$ is

$$(\,(\,(\,(F_1 \wedge F_2) \wedge F_3) \longrightarrow (\neg F_4)\,) \longleftrightarrow F_5)$$

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Logical Equivalences

$$
\begin{aligned}
F \wedge F &\equiv F \\
F \vee F &\equiv F \\
F \wedge G &\equiv G \wedge F \\
F \vee G &\equiv G \vee F \\
\neg\neg F &\equiv F \\
(F \wedge G) \wedge H &\equiv F \wedge (G \wedge H) \\
(F \vee G) \vee H &\equiv F \vee (G \vee H) \\
F \wedge (G \vee H) &\equiv (F \wedge G) \vee (F \wedge H) \\
F \vee (G \wedge H) &\equiv (F \vee G) \wedge (F \vee H)
\end{aligned}
$$

$$
\begin{aligned}
\neg(F \wedge G) &\equiv \neg F \vee \neg G \\
\neg(F \vee G) &\equiv \neg F \wedge \neg G
\end{aligned}
$$

If F is a tautology then

$$
\begin{aligned}
F \wedge G &\equiv G \\
F \vee G &\equiv F
\end{aligned}
$$

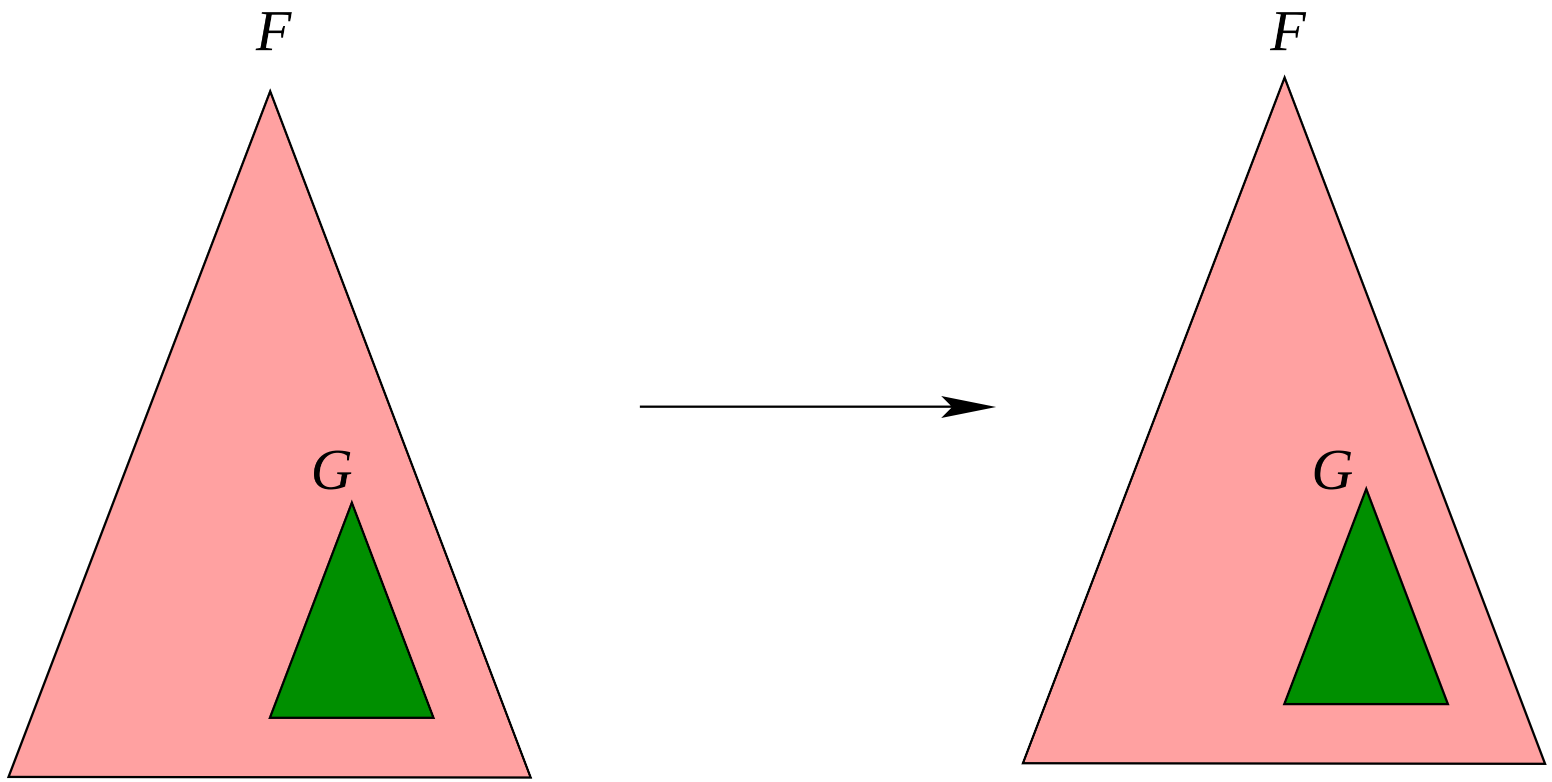
If F is unsatisfiable then

$$
\begin{aligned}
F \wedge G &\equiv F \\
F \vee G &\equiv G
\end{aligned}
$$

But we want to use them not only at top level!

Logical Equivalences - Substitution Lemma

Lemma: If we replace inside F a subformula G by G' with $G \equiv G'$, we obtain F' with $F \equiv F'$. [exercise: prove by induction]



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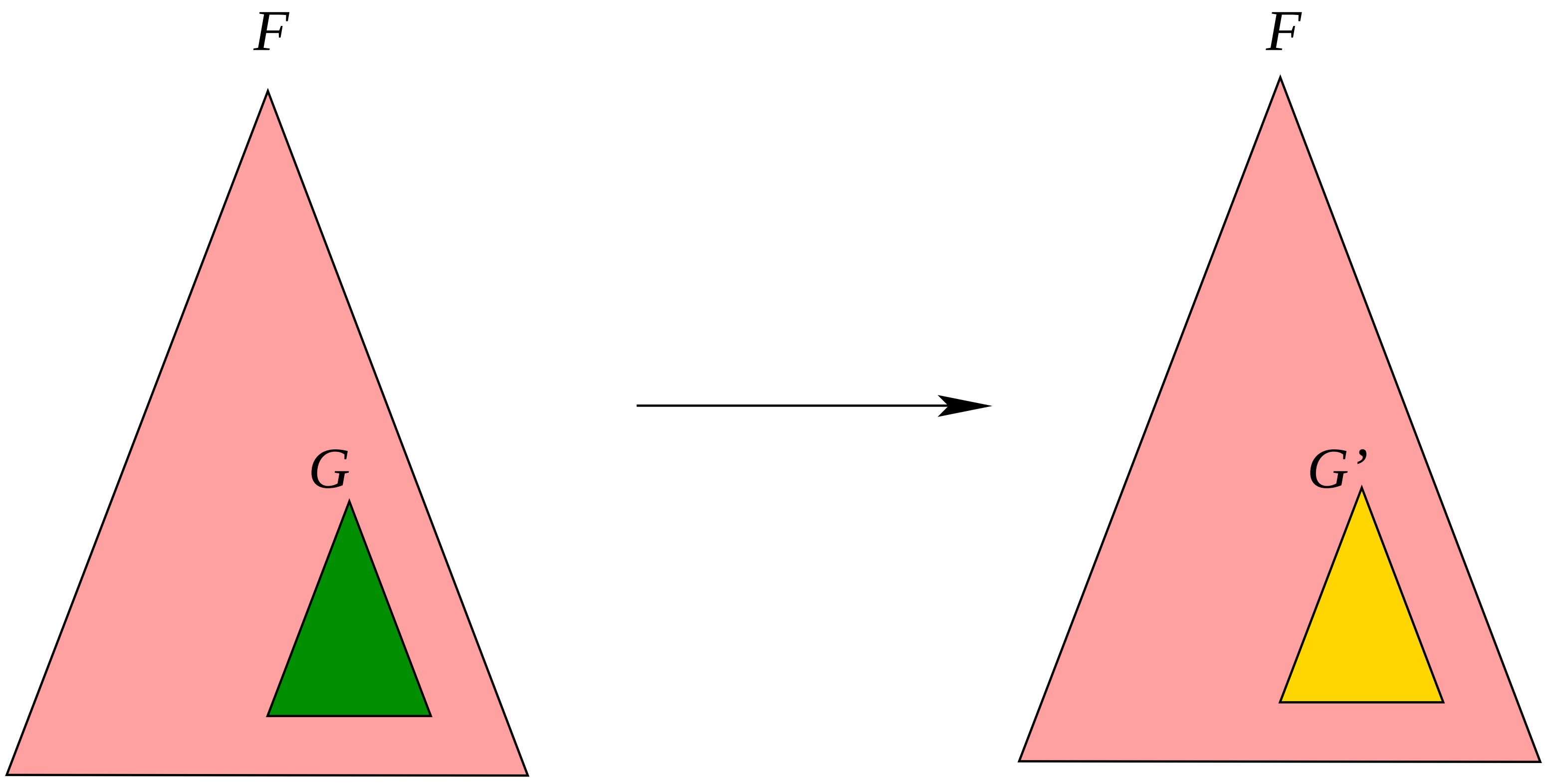

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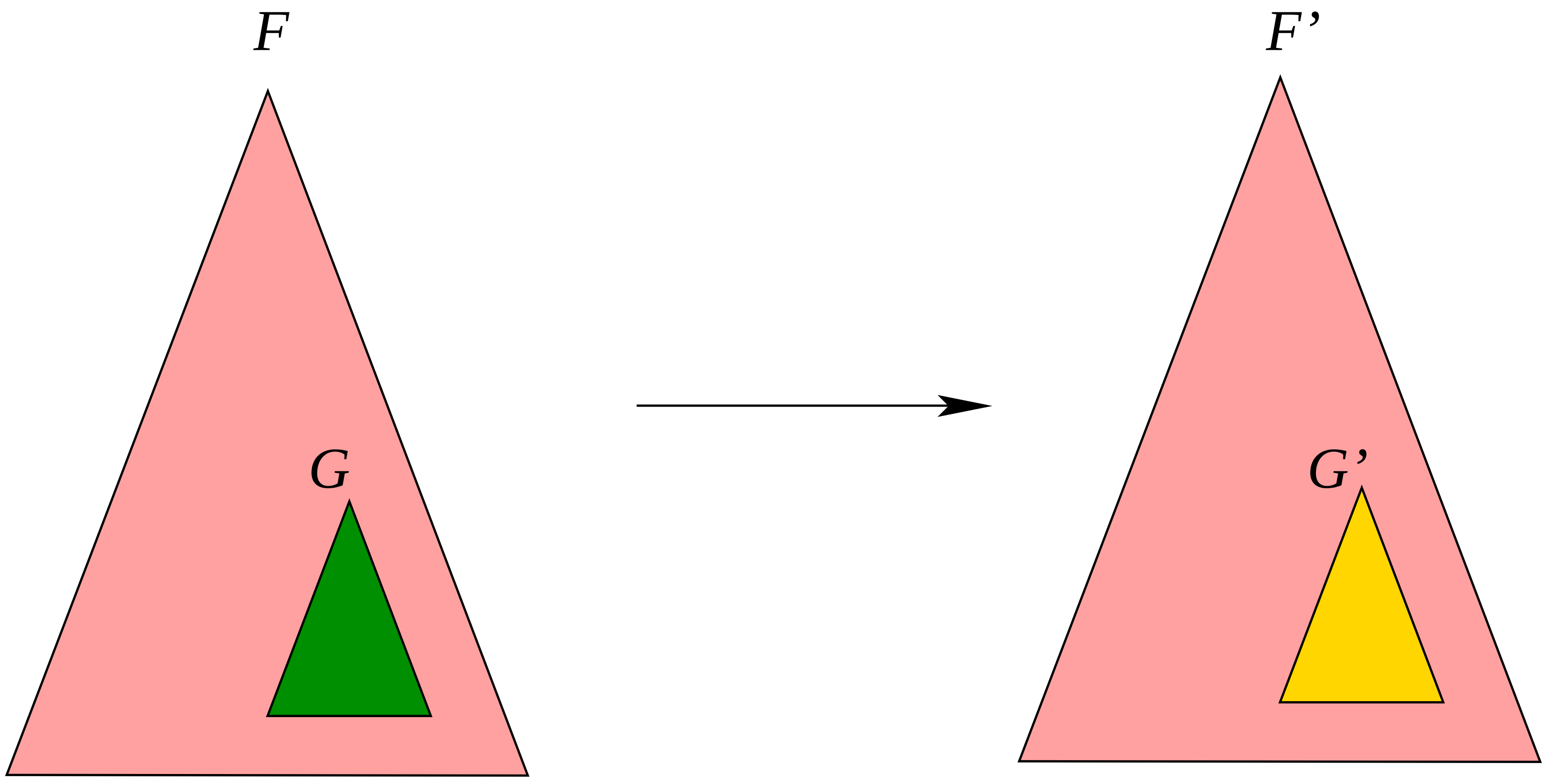

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CNFs and DNFs

In order to construct our **SAT** black-box it would simplify our job to assume that the formula F has a given format.

- A literal is a prop. variable (p) or a negation of one ($\neg p$)
- A clause is a disjunction of zero or more literals ($l_1 \vee \ldots l_n$)
- The empty clause (zero lits.) is denoted $\Box$ and is unsatisfiable
- A formula is in Conjunctive Normal Form (CNF) if it is a conjunction of zero of more clauses
- A formula is in Disjunctive Normal Form (DNF) if it is a disjunction of conjunctions of literals

Examples:

$p \wedge (q \vee \neg r) \wedge (q \vee p \vee \neg r)$ is in CNF

$p \vee (q \wedge \neg r) \vee (q \wedge p \wedge \neg r)$ is in DNF

CNFs and DNFs (2)

- Given a formula F there exist formulas
  - G in CNF with $F \equiv G$ and
  - H in DNF with $F \equiv H$

- Which is the complexity of checking whether F is satisfiable
  - if F is an arbitrary formula?
  - if F is in CNF?
  - if F is in DNF?

- Then, why not choosing always F in DNF?

- For all our purposes, we will assume F in CNF

Transformation to CNF via truth table

Let us take the formula $F := (p \wedge q) \vee \neg(\neg p \wedge (q \vee \neg r))$
Its truth table is:

p	q	r	
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

- It is easy to compute a DNF for F:

$$(\neg p \wedge \neg q \wedge r) \quad \vee \quad (p \wedge \neg q \wedge \neg r) \quad \vee$$
$$(p \wedge \neg q \wedge r) \quad \vee \quad (p \wedge q \wedge \neg r) \quad \vee$$
$$(p \wedge q \wedge r)$$

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- It is easy to compute a DNF for F:

  $(\neg p \wedge \neg q \wedge r) \quad \vee \quad (p \wedge \neg q \wedge \neg r) \quad \vee$
  $(p \wedge \neg q \wedge r) \quad \vee \quad (p \wedge q \wedge \neg r) \quad \vee$
  $(p \wedge q \wedge r)$

- Similarly a DNF for $\neg F$:

  $(\neg p \wedge \neg q \wedge \neg r) \quad \vee \quad (\neg p \wedge q \wedge \neg r) \quad \vee$
  $(\neg p \wedge q \wedge r)$

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Transformation to CNF via truth table

Let us take the formula $F := (p \wedge q) \vee \neg(\, \neg p \wedge (q \vee \neg r)\,)$
Its truth table is:

p	q	r	
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

- It is easy to compute a DNF for F:

  $(\neg p \wedge \neg q \wedge r) \quad \vee \quad (p \wedge \neg q \wedge \neg r) \quad \vee$
  $(p \wedge \neg q \wedge r) \quad \vee \quad (p \wedge q \wedge \neg r) \quad \vee$
  $(p \wedge q \wedge r)$

- Similarly a DNF for $\neg F$:

  $(\neg p \wedge \neg q \wedge \neg r) \quad \vee \quad (\neg p \wedge q \wedge \neg r) \quad \vee$
  $(\neg p \wedge q \wedge r)$

- Now, using deMorgan, a CNF for F is:

  $(\, p \vee q \vee r) \quad \wedge \quad (p \vee \neg q \vee r) \quad \wedge$
  $(p \vee \neg q \vee \neg r)$

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Tranformation to CNF via distributivity

1. Apply the three transformation rules up to completion:
   - $\neg\neg F \Rightarrow F$
   - $\neg(F \wedge G) \Rightarrow \neg F \vee \neg G$
   - $\neg(F \vee G) \Rightarrow \neg F \wedge \neg G$

   After that, the formula is in Negation Normal Form (NNF)
2. Now apply the distributivity rule up to completion:
   - $F \vee (G \wedge H) \Rightarrow (F \vee G) \wedge (F \vee H)$

EXAMPLE: let F be $(p \wedge q) \vee \neg(\neg p \wedge (q \vee \neg r))$

1. $(p \wedge q) \vee \neg(\neg p \wedge (q \vee \neg r)) \Rightarrow$

Tranformation to CNF via distributivity

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EXAMPLE: let F be $(p \wedge q) \vee \neg(\, \neg p \wedge (q \vee \neg r)\,)$

1. $(p \wedge q) \vee \neg(\, \neg p \wedge (q \vee \neg r)\,) \Rightarrow (p \wedge q) \vee (\neg\neg p \vee \neg(q \vee \neg r)) \Rightarrow$

Tranformation to CNF via distributivity

1. Apply the three transformation rules up to completion:
   - $\neg\neg F \Rightarrow F$
   - $\neg(F \land G) \Rightarrow \neg F \lor \neg G$
   - $\neg(F \lor G) \Rightarrow \neg F \land \neg G$

   After that, the formula is in Negation Normal Form (NNF)
2. Now apply the distributivity rule up to completion:
   - $F \lor (G \land H) \Rightarrow (F \lor G) \land (F \lor H)$

EXAMPLE: let F be $(p \land q) \lor \neg(\, \neg p \land (q \lor \neg r)\,)$

1. $(p \land q) \lor \neg(\, \neg p \land (q \lor \neg r)\,) \Rightarrow (p \land q) \lor (\neg\neg p \lor \neg(q \lor \neg r)) \Rightarrow$
   $(p \land q) \lor (p \lor (\neg q \land \neg\neg r)) \Rightarrow$

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EXAMPLE: let F be $(p \wedge q) \vee \neg(\ \neg p \wedge (q \vee \neg r)\ )$

1. $(p \wedge q) \vee \neg(\ \neg p \wedge (q \vee \neg r)\ ) \Rightarrow (p \wedge q) \vee (\neg\neg p \vee \neg(q \vee \neg r)) \Rightarrow$
   $(p \wedge q) \vee (p \vee (\neg q \wedge \neg\neg r)) \Rightarrow (p \wedge q) \vee (p \vee (\neg q \wedge r))$

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EXAMPLE: let F be $(p \wedge q) \vee \neg(\neg p \wedge (q \vee \neg r))$

1. $(p \wedge q) \vee \neg(\neg p \wedge (q \vee \neg r)) \Rightarrow (p \wedge q) \vee (\neg\neg p \vee \neg(q \vee \neg r)) \Rightarrow$
   $(p \wedge q) \vee (p \vee (\neg q \wedge \neg\neg r)) \Rightarrow (p \wedge q) \vee (p \vee (\neg q \wedge r))$
2. $(p \wedge q) \vee (p \vee (\neg q \wedge r)) \Rightarrow$

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   $(p \wedge q) \vee (p \vee (\neg q \wedge \neg\neg r)) \Rightarrow (p \wedge q) \vee (p \vee (\neg q \wedge r))$
2. $(p \wedge q) \vee (p \vee (\neg q \wedge r)) \Rightarrow (p \vee p \vee (\neg q \wedge r)) \wedge (q \vee p \vee (\neg q \wedge r)) \Rightarrow$

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EXAMPLE: let F be $(p \wedge q) \vee \neg(\neg p \wedge (q \vee \neg r))$

1. $(p \wedge q) \vee \neg(\neg p \wedge (q \vee \neg r)) \Rightarrow (p \wedge q) \vee (\neg\neg p \vee \neg(q \vee \neg r)) \Rightarrow$
   $(p \wedge q) \vee (p \vee (\neg q \wedge \neg\neg r)) \Rightarrow (p \wedge q) \vee (p \vee (\neg q \wedge r))$
2. $(p \wedge q) \vee (p \vee (\neg q \wedge r)) \Rightarrow (p \vee p \vee (\neg q \wedge r)) \wedge (q \vee p \vee (\neg q \wedge r)) \Rightarrow$
   $(p \vee p \vee \neg q) \wedge (p \vee p \vee r) \wedge (q \vee p \vee \neg q) \wedge (q \vee p \vee r) \Rightarrow$

Tranformation to CNF via distributivity

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   - $\neg(F \wedge G) \Rightarrow \neg F \vee \neg G$
   - $\neg(F \vee G) \Rightarrow \neg F \wedge \neg G$

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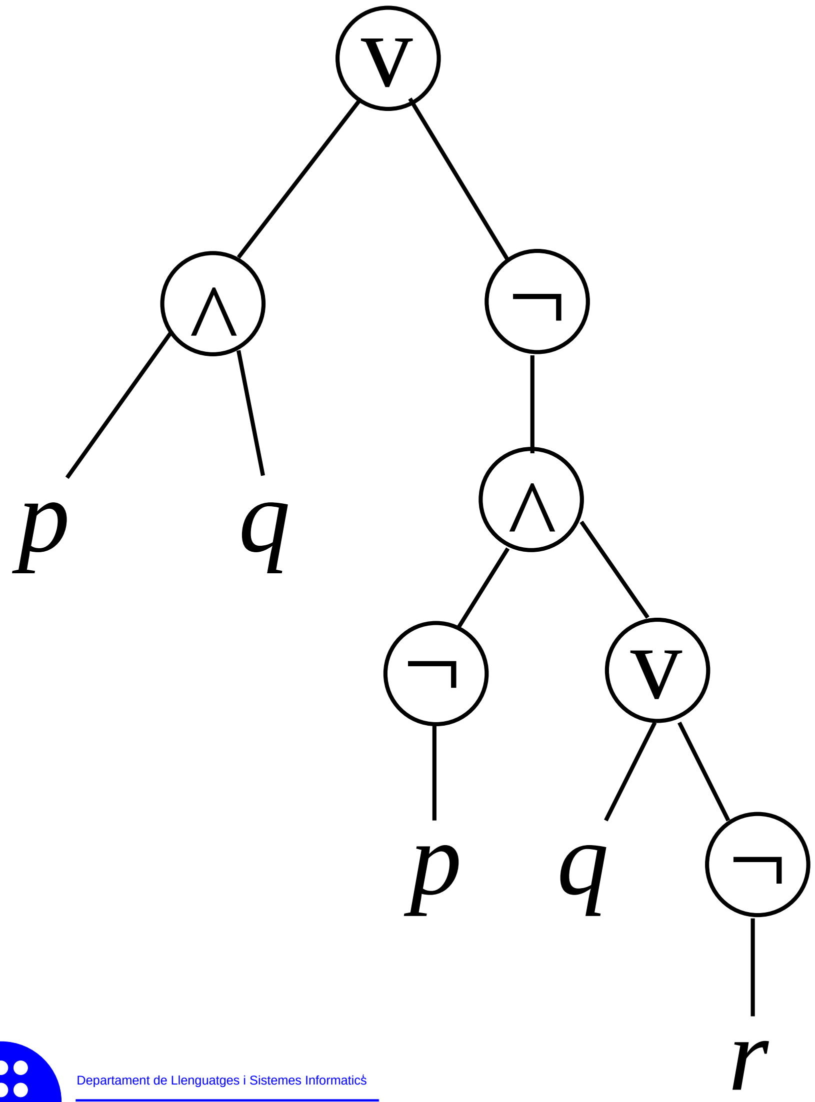
EXAMPLE: let F be $(p \wedge q) \vee \neg(\neg p \wedge (q \vee \neg r))$

1. $(p \wedge q) \vee \neg(\neg p \wedge (q \vee \neg r)) \Rightarrow (p \wedge q) \vee (\neg\neg p \vee \neg(q \vee \neg r)) \Rightarrow$
   $(p \wedge q) \vee (p \vee (\neg q \wedge \neg\neg r)) \Rightarrow (p \wedge q) \vee (p \vee (\neg q \wedge r))$
2. $(p \wedge q) \vee (p \vee (\neg q \wedge r)) \Rightarrow (p \vee p \vee (\neg q \wedge r)) \wedge (q \vee p \vee (\neg q \wedge r)) \Rightarrow$
   $(p \vee p \vee \neg q) \wedge (p \vee p \vee r) \wedge (q \vee p \vee \neg q) \wedge (q \vee p \vee r) \Rightarrow$
   $(p \vee \neg q) \wedge (p \vee r) \wedge (q \vee p \vee r)$

Tranformation to CNF via Tseitin

Let F be $(p \wedge q) \vee \neg(\ \neg p \wedge (q \vee \neg r)\ )$



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Tranformation to CNF via Tseitin

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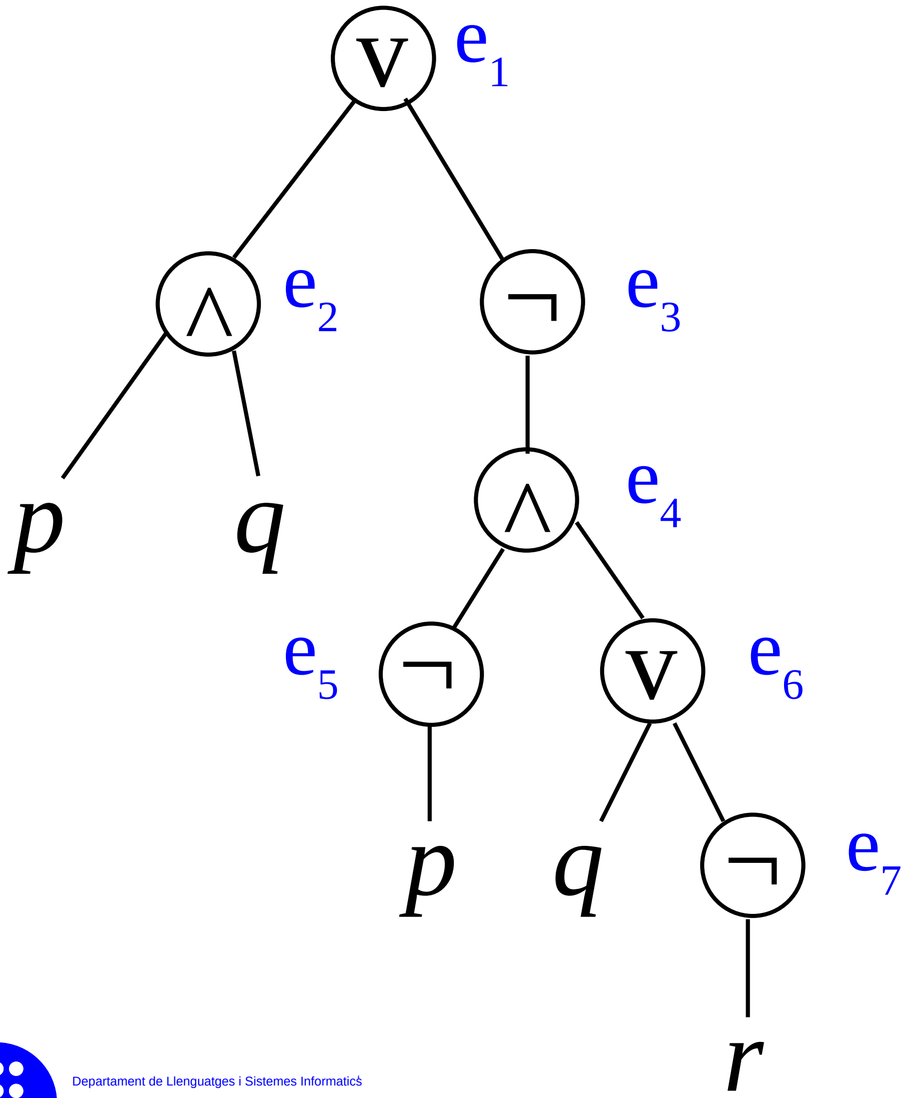

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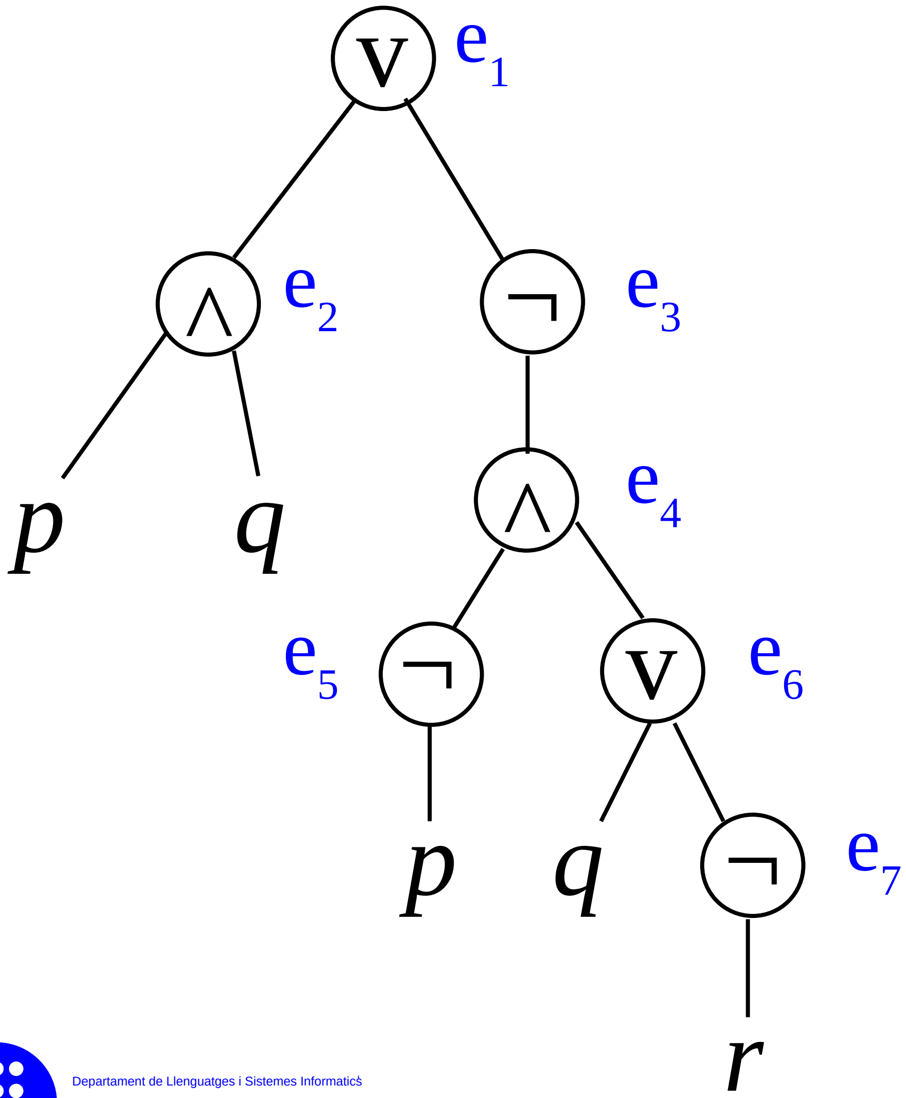

- e_1
- $e_1 \leftrightarrow e_2 \vee e_3$
- $e_2 \leftrightarrow p \wedge q$
- $e_3 \leftrightarrow \neg e_4$
- $e_4 \leftrightarrow e_5 \wedge e_6$
- $e_5 \leftrightarrow \neg p$
- $e_6 \leftrightarrow q \vee \neg e_7$
- $e_7 \leftrightarrow \neg r$

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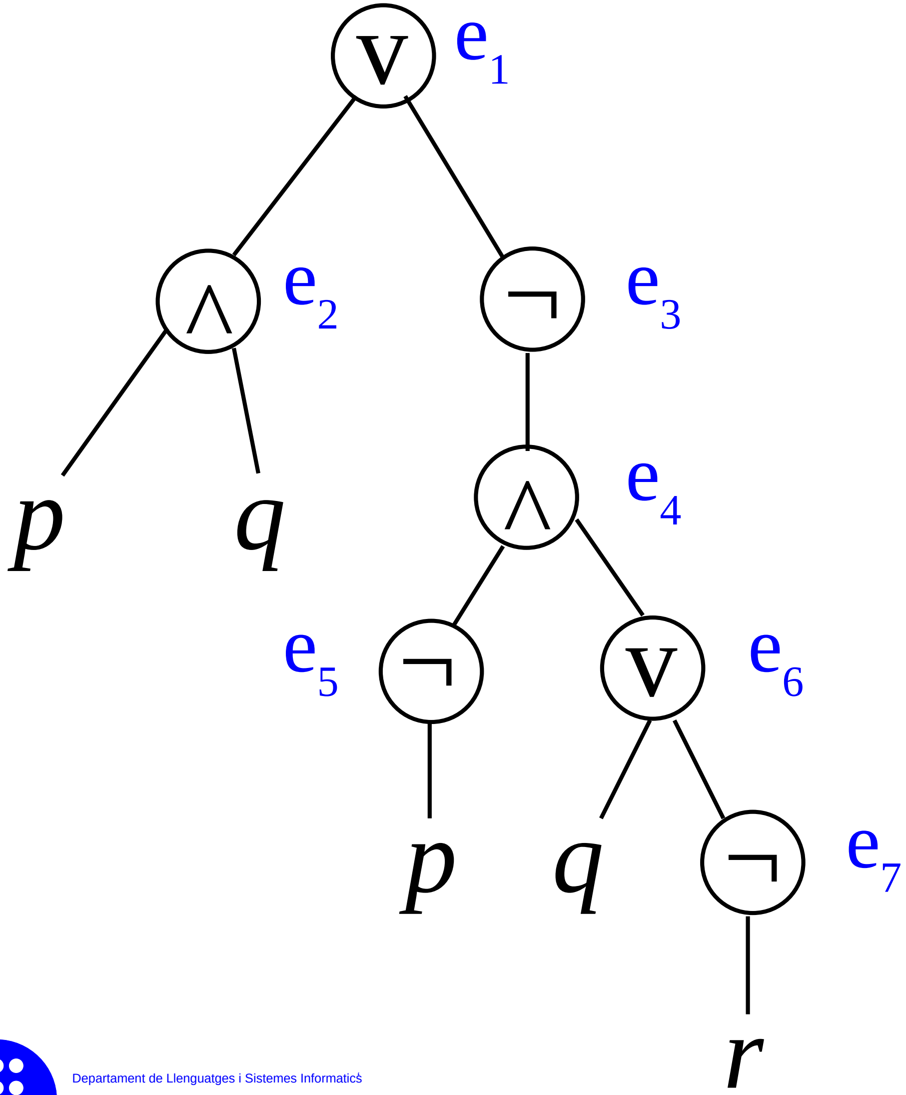
Tranformation to CNF via Tseitin

Let F be $(p \wedge q) \vee \neg(\neg p \wedge (q \vee \neg r))$



- e_1
- $e_1 \leftrightarrow e_2 \vee e_3$

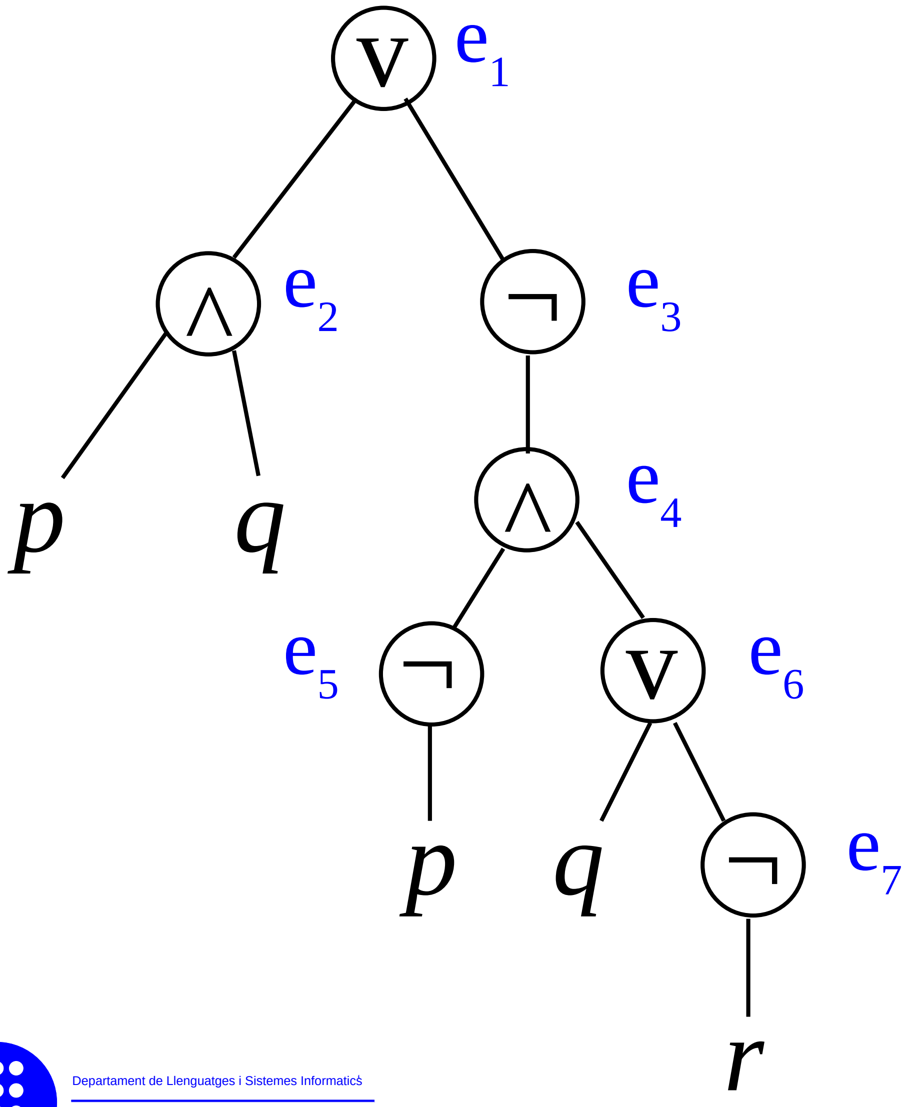
  $\neg e_1 \vee e_2 \vee e_3$

  $\neg e_2 \vee e_1$

  $\neg e_3 \vee e_1$
- $e_2 \leftrightarrow p \wedge q$
- $e_3 \leftrightarrow \neg e_4$
- $e_4 \leftrightarrow e_5 \wedge e_6$
- $e_5 \leftrightarrow \neg p$
- $e_6 \leftrightarrow q \vee \neg e_7$
- $e_7 \leftrightarrow \neg r$

Tranformation to CNF via Tseitin

Let F be $(p \wedge q) \vee \neg(\ \neg p \wedge (q \vee \neg r)\ )$



- e_1
- $e_1 \leftrightarrow e_2 \vee e_3$

  $\neg e_1 \vee e_2 \vee e_3$

  $\neg e_2 \vee e_1$

  $\neg e_3 \vee e_1$

- $e_2 \leftrightarrow p \wedge q$

  $\neg p \vee \neg q \vee e_2$

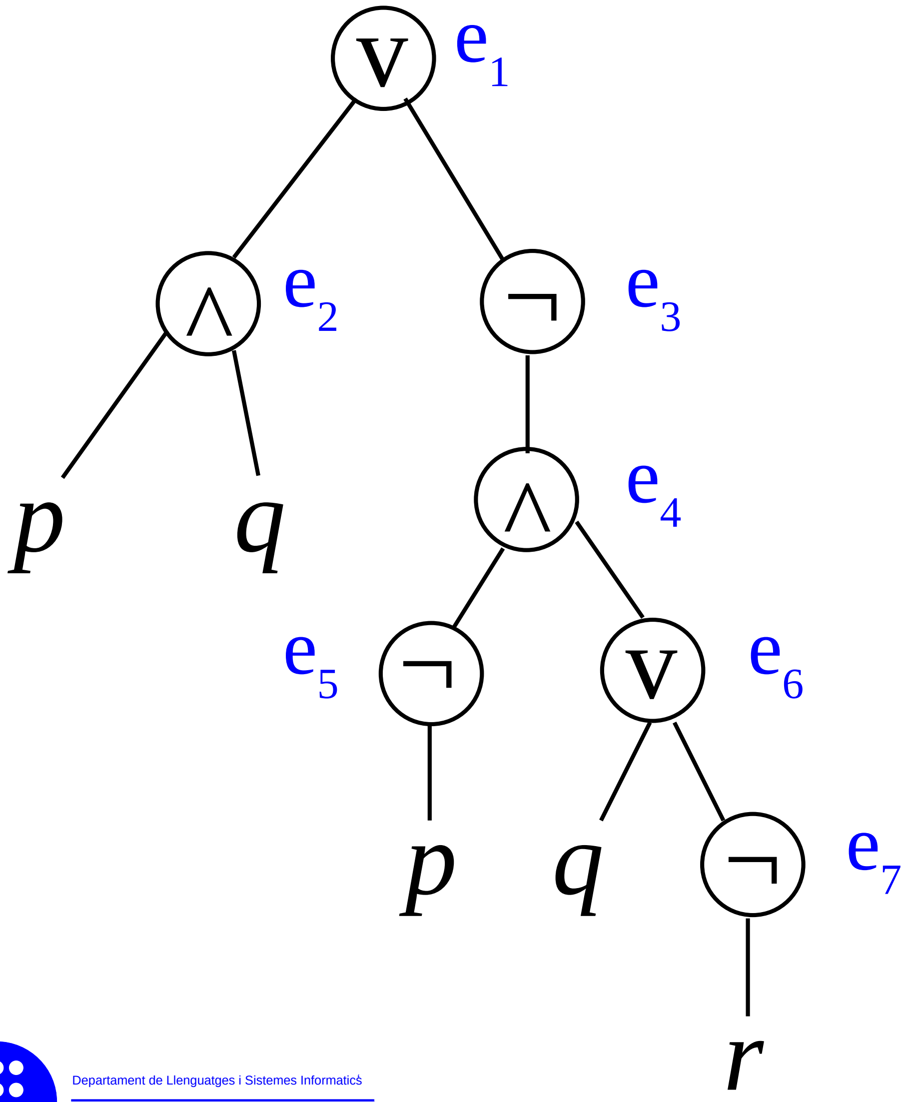
  $\neg e_2 \vee p$

  $\neg e_2 \vee q$

- $e_3 \leftrightarrow \neg e_4$
- $e_4 \leftrightarrow e_5 \wedge e_6$
- $e_5 \leftrightarrow \neg p$
- $e_6 \leftrightarrow q \vee \neg e_7$
- $e_7 \leftrightarrow \neg r$

Tranformation to CNF via Tseitin

Let F be $(p \wedge q) \vee \neg(\ \neg p \wedge (q \vee \neg r)\ )$



- e_1
- $e_1 \leftrightarrow e_2 \vee e_3$

$$\begin{array}{lllll} \neg e_1 & \vee & e_2 & \vee & e_3 \\ \neg e_2 & \vee & e_1 & & \\ \neg e_3 & \vee & e_1 & & \end{array}$$

- $e_2 \leftrightarrow p \wedge q$

$$\begin{array}{lllll} \neg p & \vee & \neg q & \vee & e_2 \\ \neg e_2 & \vee & p & & \\ \neg e_2 & \vee & q & & \end{array}$$

- $e_3 \leftrightarrow \neg e_4$

$$\begin{array}{lll} \neg e_3 & \vee & \neg e_4 \\ e_3 & \vee & e_4 \end{array}$$

- $e_4 \leftrightarrow e_5 \wedge e_6$
- $e_5 \leftrightarrow \neg p$
- $e_6 \leftrightarrow q \vee \neg e_7$
- $e_7 \leftrightarrow \neg r$

Tranformation to CNF via Tseitin (2)

- Variations of Tseitin are the ones used in practice
- Tseitin does not produce an equivalent CNF
- Given F, the CNF obtained has three important properties:
  - It is equisatisfiable to F
  - Any model of CNF can be projected to the variables in F giving a model of F
  - Any model of F can be completed to a model of the CNF
- Hence no model is lost nor added in the conversion