The DPLL algorithm

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Logic and Algebra in Computer Science

Session 2

Fall 2009, Barcelona

Departament de Llenguatges i Sistemes Informatics

UNIVERSITAT POLITECNICA DE CATALUNYA

Overview of the session

- Problem Solving w./ Prop. Logic
- DPLL: A Bit of History
- Abstract DPLL:
  - Rules
  - Examples
  - Theoretical Results

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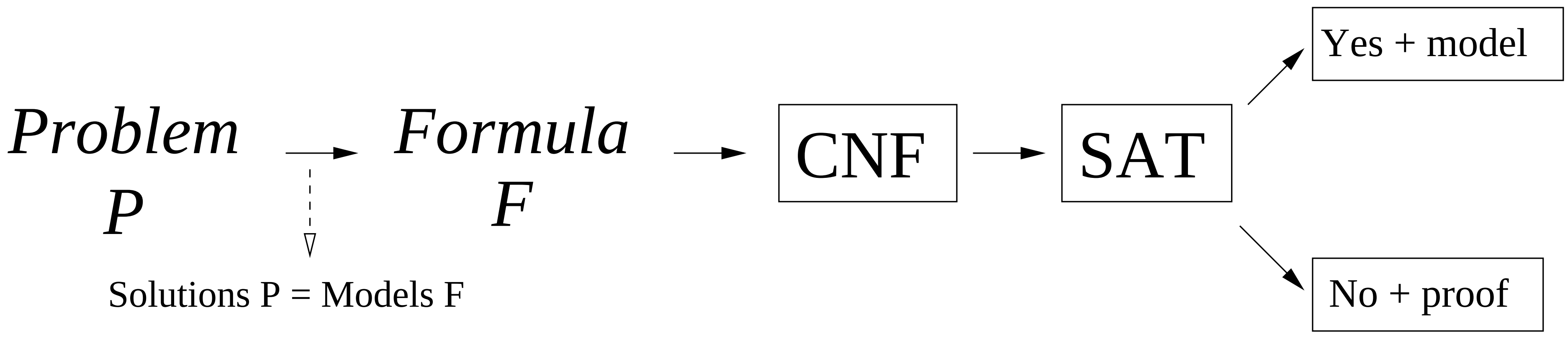
Problem Solving with Propositional Logic



- This is the standard flow used for problem solving
- Transformation P to F will be seen later in the course
  Some examples already seen (e.g. pigeon-hole problem)
- CNF conversion already explained
- Course mainly based on designing efficient SAT boxes

Designing an efficient SAT box

INPUT: formula F in CNF

OUTPUT:

- If F is SAT: YES + model
- If F is UNSAT: NO + proof

In this course we will describe two possible methods:

- DPLL-based:
  - \+ easy to obtain model
  - \- difficult to give proof
- resolution-based:
  - \- difficult to obtain model
  - \+ easy to give proof

Due to their efficiency, DPLL-based solvers are the method of choice

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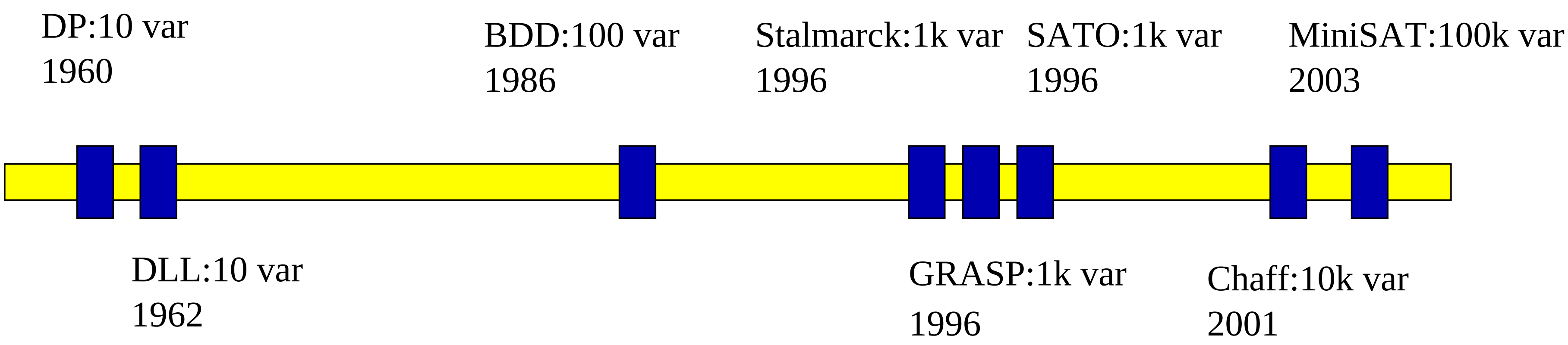
DPLL - A Bit of History

- Originally, DPLL was incomplete method for SAT in FO logic
- First paper (Davis and Putnam) in 1960: memory problems
- Second paper (Davis, Logemann and Loveland) in 1962: Depth-first-search with backtracking
- Late 90's and early 00's improvements make DPLL efficient:
  - Break-through systems: GRASP, SATO, Chaff, MiniSAT



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Our Abstraction of DPLL

- Given formula F in CNF, DPLL tries to build a model M for F
- Each step of the algorithm modifies M (not F in this case)
- Interpretations M will be represented as sequences of literals:
  - Order in M does matter
  - No literals appear twice in M
  - No contradictory literals in M

  EXAMPLE: $M(p) = 1, M(q) = 0, M(r) = 1$ is $p\bar{q}r$
- Sequences might have decision literals, denoted l^{d}.
- We will introduce a transition system modelling DPLL
- States in the trans. system are pairs $M \parallel F$, where F is a CNF
- The rules in the transition system indicate which steps

$$M \parallel F \Longrightarrow M' \parallel F'$$

  are allowed.

Abstract DPLL - Rules

Extending the model:

UnitProp

$$M \parallel F, C \vee l \implies M\,l \parallel F, C \vee l \quad \textbf{if} \begin{cases} M \models \neg C \\ l \text{ is undefined in } M \end{cases}$$

Decide

$$M \parallel F \implies M\,l^{\mathsf{d}} \parallel F \quad \textbf{if} \begin{cases} l \text{ or } \neg l \text{ occurs in } F \\ l \text{ is undefined in } M \end{cases}$$

Abstract DPLL - Rules (2)

Repairing the model:

Fail

$$M \parallel F, C \implies \mathit{fail} \ \ \textbf{if} \begin{cases} M \models \neg C \\ M \text{ contains no decision literals} \end{cases}$$

Backtrack

$$M \, l^{\mathsf{d}} \, N \parallel F, C \implies M \, \neg l \parallel \ F, C \ \ \textbf{if} \begin{cases} M \, l^{\mathsf{d}} \, N \models \neg C \\ N \text{ contains no dec. lits} \end{cases}$$

Abstract DPLL - Example 1

$$\emptyset \parallel \overline{1} \lor 2,\ \overline{3} \lor 4,\ \overline{5} \lor \overline{6},\ 6 \lor \overline{5} \lor \overline{2} \quad \Longrightarrow$$

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Abstract DPLL - Example 2

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Abstract DPLL - Example 2

$$\emptyset \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} \implies (\text{UnitProp})$$

$$1 \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} \implies (\text{Decide})$$

$$1\ {\color{red}2^{d}} \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} \implies (\text{UnitProp})$$

$$1\ {\color{red}2^{d}}\ 3 \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} \implies$$

Abstract DPLL - Example 2

$$\emptyset \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} \quad \Longrightarrow \quad (\text{UnitProp})$$

$$1 \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} \quad \Longrightarrow \quad (\text{Decide})$$

$$1\ 2^{\mathsf{d}} \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} \quad \Longrightarrow \quad (\text{UnitProp})$$

$$1\ 2^{\mathsf{d}}\ 3 \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} \quad \Longrightarrow \quad (\text{Backtrack})$$

Abstract DPLL - Example 2

$$\emptyset \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow \quad (\text{UnitProp})$$

$$1 \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow \quad (\text{Decide})$$

$$1\ 2^{\mathsf{d}} \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow \quad (\text{UnitProp})$$

$$1\ 2^{\mathsf{d}}\ 3 \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow \quad (\text{Backtrack})$$

$$1\ \overline{2} \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow$$

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Abstract DPLL - Example 2

$$\emptyset \parallel \overline{1}\vee 2\vee 3,\ 1,\ \overline{2}\vee 3,\ \overline{2}\vee \overline{3},\ 2\vee 3,\ 2\vee \overline{3} \quad \Longrightarrow \quad (\text{UnitProp})$$

$$1 \parallel \overline{1}\vee 2\vee 3,\ 1,\ \overline{2}\vee 3,\ \overline{2}\vee \overline{3},\ 2\vee 3,\ 2\vee \overline{3} \quad \Longrightarrow \quad (\text{Decide})$$

$$1\ 2^{d} \parallel \overline{1}\vee 2\vee 3,\ 1,\ \overline{2}\vee 3,\ \overline{2}\vee \overline{3},\ 2\vee 3,\ 2\vee \overline{3} \quad \Longrightarrow \quad (\text{UnitProp})$$

$$1\ 2^{d}\ 3 \parallel \overline{1}\vee 2\vee 3,\ 1,\ \overline{2}\vee 3,\ \overline{2}\vee \overline{3},\ 2\vee 3,\ 2\vee \overline{3} \quad \Longrightarrow \quad (\text{Backtrack})$$

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Abstract DPLL - Example 2

$$\emptyset \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow \quad (\text{UnitProp})$$

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$$1\ \overline{2}\ 3 \parallel \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} \quad \Longrightarrow$$

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Abstract DPLL - Example 2

$$\emptyset \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} \quad \Longrightarrow \quad (\text{UnitProp})$$

$$1 \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} \quad \Longrightarrow \quad (\text{Decide})$$

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$$1\ \overline{2}\ 3 \parallel \overline{1} \lor 2 \lor 3,\ 1,\ \overline{2} \lor 3,\ \overline{2} \lor \overline{3},\ 2 \lor 3,\ 2 \lor \overline{3} \quad \Longrightarrow \quad (\text{Fail})$$

Abstract DPLL - Example 2

$$
\begin{array}{rcll}
\emptyset \parallel & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\text{UnitProp}) \\
1 \parallel & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\text{Decide}) \\
1\ 2^{\text{d}} \parallel & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\text{UnitProp}) \\
1\ 2^{\text{d}}\ 3 \parallel & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\text{Backtrack}) \\
1\ \overline{2} \parallel & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\text{UnitProp}) \\
1\ \overline{2}\ 3 \parallel & \overline{1} \vee 2 \vee 3,\ 1,\ \overline{2} \vee 3,\ \overline{2} \vee \overline{3},\ 2 \vee 3,\ 2 \vee \overline{3} & \Longrightarrow & (\text{Fail}) \\
\textit{fail} & & &
\end{array}
$$

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Abstract DPLL - Theoretical Results

- There are no infinite sequences of the form $\emptyset \parallel F \implies \ldots$
- If $\emptyset \parallel F \implies^* M \parallel F$ with state $M \parallel F$ final, then
  - F is satisfiable
  - M is a model of F
- If $\emptyset \parallel F \implies^*$ *fail* then F is unsatisfiable

Hence the transition system gives a decision procedure for SAT

EXERCISE: prove the three properties

Bibliography - Some further reading

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