On the Logic of Expansion in Natural Language

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Abstract. We consider, for intuitionistic categorial grammar, an iteration modality with a rule of Mingle and an infinitary left rule, similar to infinitary action logic. Newly, we give Curry-Howard labelling for the iteration modality, in terms of lists, and we prove soundness and completeness of displacement calculus with additives and this modality, for phase semantics. This result has as a corollary semantic Cut-elimination. We review linguistic application of the iteration modality to unbounded addicity iterated coordination, and we present an application of a calibrated version of the iteration modality to an unbounded addicity respectively construction, this being to our knowledge the first account of respectively taking care of cases $n > 2$.

Keywords: expansion · exponentials · iterated coordination · Mingle · phase semantics · respectively construction · semantic Cut-elimination

1 Introduction

In standard logic information does not have multiplicity. Thus where + is the notion of addition of information and $\leq$ is the notion of inclusion of information we have $x{+}x \leq x$ and $x \leq x{+}x$; together these two properties amount to idempotency: $x{+}x = x$. These properties are expressed by the rules of inference of Contraction and Expansion:

(1) $\dfrac{\Delta(A, A) \Rightarrow B}{\Delta(A) \Rightarrow B}$ Contraction $\qquad \dfrac{\Delta(A) \Rightarrow B}{\Delta(A, A) \Rightarrow B}$ Expansion

In general linguistic resources do not have these properties: grammaticality is not often preserved under addition or removal of copies of expressions. However, there are some constructions manifesting something similar. In this paper we investigate categorial logic and expansion.

Iterated coordination has a kind of expansion, of unbounded addicity:

(2) John likes, Mary dislikes, . . . and Bill loves London.

Likewise an unbounded addicity *respectively* construction:

(3) Tom, Dick, ... and Harry walk, talk, ... and sing respectively.

That is, in logical grammar a *controlled* use of expansion is motivated. Girard (1987[4]) introduced exponentials for control of structural rules. For the use of nonlinearity for iterated coordination in categorial grammar see Morrill (1994[13]) and Morrill and Valentín (2015[11]).

The iteration modality is closely related to the Kleene star modality of the infinitary action logic of Buszkowski and Palka (2008[2]).[1] Our new results include the Curry-Howard annotation of the iteration modality, with (non-empty) lists, combination with the full displacement calculus, and a strong completeness result à la Okada (1999[14]), namely soundness and completeness with respect to phase semantics (Girard 1987[4]), and as a by product of this there is a semantic proof of Cut-elimination, which differs from the syntactic Cut-elimination of Palka (2007[15]). Linguistic applications include for the first time in categorial grammar syntactic and semantic analysis of an unbounded addicity *respectively* construction.[2]

In Section 2 we define a displacement calculus **DA?** with additives, and an existential exponential with a Mingle structural rule (Kamide 2002[5]) and an infinitary left rule, which entail expansion. In Section 3 we give a sound and complete phase semantics for **DA?**. The completeness has as a corollary semantic Cut-elimination. In Section 4 we present a calibrated version of the Mingle modality and present a linguistic fragment including iterated coordination and the *respectively* construction with analyses generated by a version of the categorial parser/theorem-prover CatLog2.[3]

2 The categorial logic

The multiplicative basis is the displacement calculus of Morrill et al. (2011[12]); in addition there are additives, and the existential exponential. The syntactic types of the categorial logic are sorted according to the number of points of discontinuity their expressions contain. Each type predicate letter has a sort and an arity which are naturals, and a corresponding semantic type. Assuming ordinary terms to be already given, where P is a type predicate letter of sort i and arity n and $t_1, \ldots, t_n$ are

[1] We can define the Kleene star modality $*$ in terms of our modality ? by: $A* = I \oplus ?A$.

[2] In the type logical literature iteration has been considered in Bechet et al. (2008[1]) who propose syntactic pregroup analyses but without enjoying intuitionistic Curry-Howard labelling, nor algebraic models.

[3] `https://www.cs.upc.edu/~morrill/CatLog/CatLog2/index.php`

terms, $Pt_1 \dots t_n$ is an (atomic) type of sort i of the corresponding semantic type. Compound types are formed by connectives as in Figure 1.[4]

1.	$\mathcal{F}_i ::= \mathcal{F}_{i+j}/\mathcal{F}_j$	$T(C/B) = T(B) \rightarrow T(C)$	over [9]
2.	$\mathcal{F}_j ::= \mathcal{F}_i \backslash \mathcal{F}_{i+j}$	$T(A \backslash C) = T(A) \rightarrow T(C)$	under [9]
3.	$\mathcal{F}_{i+j} ::= \mathcal{F}_i \bullet \mathcal{F}_j$	$T(A \bullet B) = T(A) \& T(B)$	continuous product [9]
4.	$\mathcal{F}_0 ::= I$	$T(I) = \top$	continuous unit [8]
5, k.	$\mathcal{F}_{i+1} ::= \mathcal{F}_{i+j} \uparrow_k \mathcal{F}_j, 1 \le k \le i+j$	$T(C \uparrow_k B) = T(B) \rightarrow T(C)$	extract [12]
6, k.	$\mathcal{F}_j ::= \mathcal{F}_{i+1} \downarrow_k \mathcal{F}_{i+j}, 1 \le k \le i+1$	$T(A \downarrow_k C) = T(A) \rightarrow T(C)$	infix [12]
7, k.	$\mathcal{F}_{i+j} ::= \mathcal{F}_{i+1} \odot_k \mathcal{F}_j, 1 \le k \le i+1$	$T(A \odot_k B) = T(A) \& T(B)$	discontinuous product [12]
8.	$\mathcal{F}_1 ::= J$	$T(J) = \top$	discontinuous unit [12]
9.	$\mathcal{F}_i ::= \mathcal{F}_i \& \mathcal{F}_i$	$T(A \& B) = T(A) \& T(B)$	additive conjunction [7, 10]
10.	$\mathcal{F}_i ::= \mathcal{F}_i \oplus \mathcal{F}_i$	$T(A \oplus B) = T(A) + T(B)$	additive disjunction [7, 10]
18.	$\mathcal{F}_0 ::= ?\mathcal{F}_0$	$T(?A) = T(A)^+$	existential exponential [13]

Fig. 1. Categorial logic types of **DA?**

For a type A, its sort $s(A)$ is the i such that $A \in \mathcal{F}_i$. Tree-based sequent calculus is as follows. Configurations are defined by:[5]

(4)
$$\begin{aligned} \mathcal{O} &::= \Lambda \\ \mathcal{O} &::= 1, \mathcal{O} \\ \mathcal{O} &::= \mathcal{F}_0, \mathcal{O} \\ \mathcal{O} &::= \mathcal{F}_{i>0}\{\underbrace{\mathcal{O} : \ldots : \mathcal{O}}_{i\,\mathcal{O}'\text{s}}\}, \mathcal{O} \end{aligned}$$

For a configuration Δ we define the *type-equivalent* $\Delta^\bullet$, which is a type which has the same algebraic meaning as Δ. Via the BNF formulation of $\mathcal{O}$ in (4) one defines recursively $\Delta^\bullet$ as follows:

(5)
$$\begin{aligned} &\Lambda^\bullet \stackrel{def}{=} I \\ &(1, \Gamma)^\bullet \stackrel{def}{=} J \bullet \Gamma^\bullet \\ &(A, \Gamma)^\bullet \stackrel{def}{=} A \bullet \Gamma^\bullet, \text{ if } s(A) = 0 \\ &(A\{\Delta_1 : \ldots : \Delta_{s(A)}\}, \Gamma)^\bullet \stackrel{def}{=} ((\cdots (A \odot_1 \Delta_1^\bullet) \cdots) \odot_{1+s(\Delta_1)+\cdots+s(\Delta_{s(A)})} \Delta_{s(A)}^\bullet) \bullet \Gamma^\bullet, \text{ if } s(A) > 0 \end{aligned}$$

[4] Observe that the iteration modality ? only applies to types of sort 0 because otherwise expansion would not preserve the equality of antecedent and succedent sorts.

[5] Note that the colons in the fourth clause of the definition punctuate the list of configurations intercalating the points of discontinuity of $\mathcal{F}_{i>0}$ of sort i; this is entirely distinct from (the standard) use of colons in type assignments made later.

For a configuration Γ, its sort $s(\Gamma)$ is $|\Gamma|_1$, i.e. the number of metalinguistic separators 1 which it contains. A sequents $\Gamma \Rightarrow A$ comprises an antecedent configuration Γ and a succedent type A such that $s(\Gamma) = s(A)$. The figure $\overrightarrow{A}$ of a type A is defined by:

$$(6)\quad \overrightarrow{A} = \begin{cases} A & \text{if } sA = 0 \\ A\{\underbrace{1 : \ldots : 1}_{sA\ 1's}\} & \text{if } sA > 0 \end{cases}$$

Where Γ is a configuration of sort i and $\Delta_1, \ldots, \Delta_i$ are configurations, the fold $\Gamma \otimes \langle \Delta_1 : \ldots : \Delta_i \rangle$ is the result of replacing the successive 1's in Γ by $\Delta_1, \ldots, \Delta_i$ respectively. Where Δ is a configuration of sort $i > 0$ and Γ is a configuration, the kth metalinguistic wrap $\Delta |_k \Gamma$, $1 \leq k \leq i$, is given by

$$(7)\quad \Delta |_k \Gamma =_{df} \Delta \otimes \langle \underbrace{1 : \ldots : 1}_{k-1\ 1\text{'s}} : \Gamma : \underbrace{1 : \ldots : 1}_{i-k\ 1\text{'s}} \rangle$$

i.e. the kth metalinguistic wrap $\Delta |_k \Gamma$ is the configuration resulting from replacing by Γ the kth separator in Δ.

Where the notation $\Xi(\Omega)$ signifies a configuration Ξ with a distinguished subconfiguration Ω, the notation $\Delta\langle\Gamma\rangle$ abbreviates $\Delta_0(\Gamma \otimes \langle \Delta_1 : \ldots : \Delta_n \rangle)$, i.e. a configuration with a potentially discontinuous distinguished subconfiguration Γ with external context Δ_0 and internal context $\Delta_1, \ldots, \Delta_n$.

The semantically annotated identity axiom *id* and *Cut* rule are:

$$(8)\quad \frac{}{P\colon x \Rightarrow P\colon x}\ id,\ P \text{ atomic} \qquad \frac{\Gamma \Rightarrow A\colon \phi \qquad \Delta\langle \overrightarrow{A}\colon x \rangle \Rightarrow B\colon \beta}{\Delta\langle\Gamma\rangle \Rightarrow B\colon \beta\{\phi/x\}}\ Cut$$

The semantically annotated multiplicative rules of **DA?** are given in Figure 2. The semantically annotated additive and exponential rules are given in Figure 3.[6]

[6] Notice that although the sequent calculus is infinitary and has possibly infinite proofs, the proveable sequents are always finite. The system is undecidable by a result of Buszkowski and Palka (2008[2]) but a linguistically sufficient fragment, without antedent iteration modalities, is decidable.

The expansion rule with iteration modalities is derivable by the following reasoning. Given an arbitrary type A of sort 0, for every $i > 0$ and a fixed index $j_0 > 0$, by one application of $?R$ and a finite number of applications of the Mingle rule we get the infinite provable sequents indexed by i ($i > 0$) $A^i, A^{j_0} \Rightarrow ?A$. We can then apply the $?L$ rule, obtaining $?A, A^{j_0} \Rightarrow ?A$. Since j_0 is a positive natural, we have that for every $j > 0$, $?A, A^j \Rightarrow ?A$. We can apply again then the $?R$ rule, whence $?A, ?A \Rightarrow ?A$. This proves the expansion rule.

$$\frac{\Gamma \Rightarrow B\colon \psi \qquad \Delta\langle\overrightarrow{C\colon z}\rangle \Rightarrow D\colon \omega}{\Delta\langle\overrightarrow{C/B}\colon x, \Gamma\rangle \Rightarrow D\colon \omega\{(x\ \phi)/z\}}\ /L \qquad \frac{\Gamma, \overrightarrow{B}\colon y \Rightarrow C\colon \chi}{\Gamma \Rightarrow C/B\colon \lambda y\chi}\ /R$$

$$\frac{\Gamma \Rightarrow A\colon \phi \qquad \Delta\langle\overrightarrow{C\colon z}\rangle \Rightarrow D\colon \omega}{\Delta\langle\Gamma, \overrightarrow{A\backslash C}\colon y\rangle \Rightarrow D\colon \{(y\ \phi)/z\}}\ \backslash L \qquad \frac{\overrightarrow{A}\colon x, \Gamma \Rightarrow C\colon \chi}{\Gamma \Rightarrow A\backslash C\colon \lambda x\chi}\ \backslash R$$

$$\frac{\Delta\langle\overrightarrow{A}\colon x, \overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega}{\Delta\langle\overrightarrow{A\bullet B\colon z}\rangle \Rightarrow D\colon \omega\{\pi_1 z/x, \pi_2 z/y\}}\ \bullet L \qquad \frac{\Gamma_1 \Rightarrow A\colon \phi \qquad \Gamma_2 \Rightarrow B\colon \psi}{\Gamma_1, \Gamma_2 \Rightarrow A\bullet B\colon (\phi, \psi)}\ \bullet R$$

$$\frac{\Delta\langle\Lambda\rangle \Rightarrow A\colon \phi}{\Delta\langle\overrightarrow{I}\colon x\rangle \Rightarrow A\colon \phi}\ IL \qquad \frac{}{\Lambda \Rightarrow I\colon 0}\ IR$$

$$\frac{\Gamma \Rightarrow B\colon \psi \qquad \Delta\langle\overrightarrow{C\colon z}\rangle \Rightarrow D\colon \omega}{\Delta\langle\overrightarrow{C\uparrow_k B\colon x} |_k \Gamma\rangle \Rightarrow D\colon \omega\{(x\ \psi)/z\}}\ \uparrow_k L \qquad \frac{\Gamma |_k \overrightarrow{B}\colon y \Rightarrow C\colon \chi}{\Gamma \Rightarrow C\uparrow_k B\colon \lambda y\chi}\ \uparrow_k R$$

$$\frac{\Gamma \Rightarrow A\colon \phi \qquad \Delta\langle\overrightarrow{C\colon z}\rangle \Rightarrow D\colon \omega}{\Delta\langle\Gamma |_k \overrightarrow{A\downarrow_k C}\colon y\rangle \Rightarrow D\colon \omega\{(y\ \phi)/z\}}\ \downarrow_k L \qquad \frac{\overrightarrow{A}\colon x |_k \Gamma \Rightarrow C\colon \chi}{\Gamma \Rightarrow A\downarrow_k C\colon \lambda x\chi}\ \downarrow_k R$$

$$\frac{\Delta\langle\overrightarrow{A}\colon x |_k \overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega}{\Delta\langle\overrightarrow{A\odot_k B}\colon z\rangle \Rightarrow D\colon \omega\{\pi_1 z/x, \pi_2 z/y\}}\ \odot_k L \qquad \frac{\Gamma_1 \Rightarrow A\colon \phi \qquad \Gamma_2 \Rightarrow B\colon \psi}{\Gamma_1 |_k \Gamma_2 \Rightarrow A\odot_k B\colon (\phi, \psi)}\ \odot_k R$$

$$\frac{\Delta\langle 1\rangle \Rightarrow A\colon \phi}{\Delta\langle\overrightarrow{J}\colon x\rangle \Rightarrow A\colon \phi}\ JL \qquad \frac{}{1 \Rightarrow J\colon 0}\ JR$$

Fig. 2. Multiplicative rules of **DA?**

$$\frac{\Gamma\langle\overrightarrow{A}: x\rangle \Rightarrow C: \chi}{\Gamma\langle\overrightarrow{A\&B}: z\rangle \Rightarrow C: \chi\{\pi_1 z/x\}}\,\&L_1 \qquad \frac{\Gamma\langle\overrightarrow{B}: y\rangle \Rightarrow C: \chi}{\Gamma\langle\overrightarrow{A\&B}: z\rangle \Rightarrow C\chi\{\pi_2 z/y\}}\,\&L_2$$

$$\frac{\Gamma \Rightarrow A: \phi \qquad \Gamma \Rightarrow B: \psi}{\Gamma \Rightarrow A\&B: (\phi, \psi)}\,\&R$$

$$\frac{\Gamma\langle\overrightarrow{A}: x\rangle \Rightarrow C: \chi_1 \qquad \Gamma\langle\overrightarrow{B}: y\rangle \Rightarrow C: \chi_2}{\Gamma\langle\overrightarrow{A\oplus B}: z\rangle \Rightarrow C: z \to x.\chi_1; y.\chi_2}\,\oplus L$$

$$\frac{\Gamma \Rightarrow A: \phi}{\Gamma \Rightarrow A\oplus B: \iota_1\phi}\,\oplus R_1 \qquad \frac{\Gamma \Rightarrow B: \psi}{\Gamma \Rightarrow A\oplus B: \iota_2\phi}\,\oplus R_2$$

$$\frac{\Delta(A: x) \Rightarrow B: \psi([x]) \qquad \Delta(A: x, A: y) \Rightarrow B: \psi([x, y]) \qquad \cdots}{\Delta(?A: z) \Rightarrow B: \psi(z)}\,?L$$

$$\frac{\Gamma \Rightarrow A: \phi}{\Gamma \Rightarrow ?A: [\phi]}\,?R \qquad \frac{\Gamma \Rightarrow A: \phi \qquad \Delta \Rightarrow ?A: \phi'}{\Gamma, \Delta \Rightarrow [\phi|\phi']: ?A}\,?M$$

Fig. 3. Additive and exponential rules of **DA?**

3 Phase semantics

DA? incorporates the useful language-theoretic concept of *iteration*. This is done by means of an (existential) exponential modality, notated ? which licenses the structural rule of Mingle, which entails expansion.

Let i, j and k range over the set of natural numbers ω. Where A is a type of sort 0, and $i > 0$, A^i denotes $\underbrace{A, \dots, A}_{i \text{ times}}$. A^0 is the empty string Λ.

3.1 Semantic Interpretation

In the following, we describe the phase space machinery in order to give a result of strong completeness in the style of Okada (1999[14]). Phase spaces from linear logic (Girard 1987[4]) are based on (commutative) monoids. Likewise, the proper algebras for the displacement calculus **D** are the so-called *displacement algebras* (DA for short) (see Valentín 2012[17]) which can be seen as a generalisation of (non-commutative) monoids where the operations of k-th intercalation in a punctuated string are incorporated. In Valentín (2012[17]) it is proved that DAs can be axiomatised; see Figure 4). We can define the class of residuated DAs (Valentín forthcoming[18]), and therefore models.

Given a mapping $v : \mathrm{Pr} \rightarrow \mathbf{A}$ where $\mathbf{A}$ is a residuated DA, there exists a unique ω-sorted homomorphism $\widehat{v}$ which extends v as follows: $\widehat{v} : \mathbf{Tp} \rightarrow \mathbf{A}$ and $\widehat{v}(p) = v(p)$ for any primitive type. Needless to say, since we are working in an ω-sorted setting, equations, inequations and mapping and so on, are to be understood modulo sorting; in order to give a smoother reading of formulas we always avoid if possible the explicit reference to sorts.

Continuous associativity
$x + (y + z) \approx (x + y) + z$

Discontinuous associativity
$x \times_i (y \times_j z) \approx (x \times_i y) \times_{i+j-1} z$
$(x \times_i y) \times_j z \approx x \times_i (y \times_{j-i+1} z)$ if $i \leq j \leq 1 + s(y) - 1$

Mixed permutation
$(x \times_i y) \times_j z \approx (x \times_{j-s(y)+1} z) \times_i y$ if $j > i + s(y) - 1$
$(x \times_i z) \times_j y \approx (x \times_j y) \times_{i+s(y)-1} z$ if $j < i$

Mixed associativity
$(x + y) \times_i z \approx (x \times_i z) + y$ if $1 \leq i \leq s(x)$
$(x + y) \times_i z \approx x + (y \times_{i-s(x)} z)$ if $x + 1 \leq i \leq s(x) + s(y)$

Continuous unit and discontinuous unit
$0 + x \approx x \approx x + 0$ and $1 \times_1 x \approx x \approx x \times_i 1$

Fig. 4. Axiomatisation of a DA

A subset B of the carrier set A of a DA is called a *same-sort* subset iff there exists an $i \in \omega$ such that for every $a \in B$, $s(a) = i$. Notice that $\emptyset$ vacuously satisfies the *same-sort* condition. $\mathcal{P}(\mathrm{A})$ is in fact an ω-sorted subset $(\mathcal{P}(\mathrm{A})_i)_{i\in\omega}$ where for every $i, \mathcal{P}(\mathrm{A})_i = \{X : X$ is a same-sort subset of sort $i\}$.

Definition 1. *A displacement phase space* $\mathbf{P} = (\mathbf{A}, \mathbf{Closed})$ *is a structure partially ordered by the relation of subset inclusion such that:*

1. $\mathbf{A}$ *is a DA.*
2. $\mathbf{Closed} = (\mathbf{Closed}_i)_i$ *is a set of subsets such that* $\mathbf{Closed}_i \subseteq \mathcal{P}(\mathrm{A})_i$, $\mathbf{Closed}_i \cap \mathbf{Closed}_j = \{\emptyset\}$ *iff* $i \neq j$, *and:*
   a) *For every* $F \in \mathbf{Closed}_i$, *F is called a* closed *subset.*

b) **Closed** *is closed by intersections of arbitrary families of same-sort subsets. In particular, the intersection of the empty family of closed subsets of sort i is* A_i *which belongs to* $\mathbf{Closed}_i$*.*

d) *For all* $F \in \mathbf{Closed}_i$*, and for all* $x \in A_j$*:*

$$\begin{array}{ll} x\backslash F \in \mathbf{Closed}_{i-j} & F/x \in \mathbf{Closed}_{i-j} \\ F{\uparrow_k}x \in \mathbf{Closed}_{i-j+1} & x{\downarrow_k}F \in \mathbf{Closed}_{i-j+1} \end{array}$$

Closed is also called (an ω-sorted) *closure system*.

Where F, G denote subsets of A of sort i, we define the ω-sorted closure operator cl_i:

(9) $cl_i(G) \stackrel{def}{=} \bigcap\{F \in \mathbf{Closed}_i : G \subseteq F\}$

We write $\overline{G}^i$ for $cl_i(G)$. If the context is clear we omit the subscript.

Where F and G are same-sort subsets, it is readily seen that:

i) $\overline{F}$ is the least closed set of sort $s(F)$ such that $F \subseteq \overline{F}$.
ii) $cl(\cdot)$ is extensive, i.e.: $G \subseteq \overline{G}$.
iii) $cl(\cdot)$ is monotone, i.e.: if $G_1 \subseteq G_2$ then $\overline{G_1} \subseteq \overline{G_2}$.
iv) $cl(\cdot)$ is idempotent, i.e.: $cl^2(G) = cl(G)$.

We define the following operators at the level of same-sort subsets:

- $F{\circ}G \stackrel{def}{=} \{f + g : f \in F \text{ and } g \in G\}$
- $F{\circ_i}G \stackrel{def}{=} \{f \times_i g : f \in F \text{ and } g \in G\}$
- $f{\circ}G \stackrel{def}{=} \{f\}{\circ}G$ and $F{\circ}g \stackrel{def}{=} F{\circ}\{g\}$
- $f{\circ_i}G \stackrel{def}{=} \{f\}{\circ_i}G$ and $F{\circ_i}g \stackrel{def}{=} F{\circ_i}\{g\}$
- $G//F \stackrel{def}{=} \{h : \forall f \in F, h + f \in G\}$ and similarly for $F\backslash\backslash G$
- $G{\uparrow\uparrow_i}F \stackrel{def}{=} \{h : \forall f \in F, h \times_i f \in G\}$ and similarly for $F{\downarrow\downarrow_i}G$
- $G//f \stackrel{def}{=} G//\{f\}$ and similarly for $f\backslash\backslash G$
- $G{\uparrow\uparrow_i}f \stackrel{def}{=} G{\uparrow\uparrow_i}\{f\}$ and similarly for $f{\downarrow\downarrow_i}G$

The following basic properties for ω-sorted closure operators are evident:

Lemma 1.

- $F{\circ}G \subseteq H$ *iff* $F \subseteq H//G$ *iff* $G \subseteq F\backslash\backslash H$.
- $F{\circ_i}G \subseteq H$ *iff* $F \subseteq H{\uparrow\uparrow_i}G$ *iff* $G \subseteq F{\downarrow\downarrow_i}H$.
- *By construction,* $\overline{F}$ *is the least closed subset such that* $F \subseteq \overline{F}$*. Hence:*
- *If* $A \subseteq F$ *and* $\overline{F} = F$ *then* $\overline{A} \subseteq \overline{F}$*.*

Lemma 2. *If A is closed, then:*

- $A//F, F\backslash\backslash A, A\uparrow\uparrow_i F$, *and* $F\downarrow\downarrow_i A$ *are closed.*
  Proof: $A\uparrow\uparrow_i F = \bigcap_{x\in F} A\uparrow\uparrow_i x$, *whence* $A\uparrow\uparrow_i F$ *is closed.*$\square$
- *Similarly for the other implicative operations.*
- $cl(F)\circ cl(G) \subseteq cl(F\circ G)$. *Similarly,* $cl(F)\circ_i cl(G) \subseteq cl(F\circ_i G)$
- *Hence,* $\overline{\overline{F}\circ\overline{G}} \subseteq \overline{F\circ G}$, *and* $\overline{\overline{F}\circ_i\overline{G}} \subseteq \overline{F\circ_i G}$
- *It follows that* $cl(cl(F)\circ cl(G)) = cl(F\circ G)$ *and* $cl(cl(F)\circ_i cl(G)) = cl(F\circ_i G)$
  Proof: Let us see the case of $\circ_i$. $F\circ_i G \subseteq \overline{F\circ_i G}$. *By residuation,* $F \subseteq \overline{F\circ_i G}\uparrow\uparrow_i G$.
  $\overline{F\circ_i G}\uparrow\uparrow_i G$ *is a closed subset (see previous proof). Hence,* $\overline{F} \subseteq \overline{F\circ_i G}\uparrow\uparrow_i G$.
  Applying again residuation, we have $\overline{F}\circ_i G \subseteq \overline{F\circ G}$
  We repeat the process with G, obtaining $\overline{G} \subseteq \overline{F}\downarrow\downarrow_i \overline{F\circ_i G}$. *It follows that:*
  $\overline{F}\circ_i\overline{G} \subseteq \overline{F\circ_i G}$. *Hence,* $\overline{\overline{F}\circ_i\overline{G}} \subseteq \overline{F\circ_i G}$

We see now operations on closed subsets which return values into the set of closed subsets. This paves the path to the definition of valuations from the set of types into phase spaces, concretely into the set of closed sets. Given F, G closed sets:

(10)
$$
\begin{array}{lll}
F\overline{\circ}G & \stackrel{def}{=} & \overline{F\circ G} \\
F\overline{\circ_i}G & \stackrel{def}{=} & \overline{F\circ_i G} \\
F\overline{\&}G & \stackrel{def}{=} & F\cap G.\ \text{In general we write } F\cap G. \\
F\overline{\cup}G & \stackrel{def}{=} & \overline{F\cup G}. \\
G\overline{\uparrow\uparrow}_i F & \stackrel{def}{=} & G\uparrow\uparrow_i F.\ \text{In general we write } \uparrow\uparrow_i \text{ avoiding the use of } \overline{\uparrow\uparrow}_i. \\
 & & \text{Similarly for the other implications.} \\
\overline{\mathbb{I}} & \stackrel{def}{=} & \overline{\{0\}}. \\
\overline{\mathbb{J}} & \stackrel{def}{=} & \overline{\{1\}}.
\end{array}
$$

Valuations in phase spaces are mappings between the set of types into the set of closed sets. More concretely, given a valuation $v : \mathrm{Pr} \to \mathbf{Closed}$, where $\mathbf{P} = (\mathbf{A}, \mathbf{Closed})$ is a phase space, we see the interpretation of v and its recursive extension $\widehat{v}$ w.r.t. any type in the set of primitive types by using the closed operation on the set of closed subsets defined in (10):[7]

- $v(p)$ is closed subset of A_i where p is primitive of sort i.
  We extend recursively v to $\widehat{v}$:

[7] The semantic interpretation of a configuration Δ (for a given valuation v) is $\widehat{v}(\Delta) \stackrel{def}{=} \widehat{v}(\Delta^\bullet)$.

- $\widehat{v}(B\uparrow_i A)\overset{def}{=}\widehat{v}(B)\uparrow\uparrow_i\widehat{v}(A)$. Similarly for the other implications.
- $\widehat{v}(A \bullet B)\overset{def}{=}\widehat{v}(A)\overline{\circ}\widehat{v}(B)$. $\qquad \widehat{v}(A \odot_i B)\overset{def}{=}\widehat{v}(A)\overline{\circ_i}\widehat{v}(B)$.
- $v(A \oplus B)\overset{def}{=} v(A)\overline{\cup}v(B)$. $\qquad v(A\& B)\overset{def}{=} v(A) \cap v(B)$.
- $\widehat{v}(I)\overset{def}{=}\overline{\mathbb{I}}$. $\qquad \widehat{v}(J)\overset{def}{=}\overline{\mathbb{J}}$.

Notice that for any type A, $v(A)$ is a closed subset.

3.2 The Semantics of the Iteration Connective

Given a phase space model $(\mathbf{P}, v)$, we define $\widehat{v}(?A)$ as:

(11) $\widehat{v}(?A) \overset{def}{=} \overline{\bigcup_{i>0} \overline{\widehat{v}(A)^i}}$

Lemma 3. *Where* $(F_i)_{i\in\omega} \subseteq \mathrm{P}$, $F, G \subseteq \mathrm{P}$, *and* A *is a type of sort* 0
We have:
$\overline{\bigcup_{i\in\omega} F_i} = \overline{\bigcup_{i\in\omega} \overline{F_i}}$

Proof. $\boxed{\subseteq}$ is obvious.
$\boxed{\supseteq}$ For every $k \in \omega$, $F_k \subset \overline{\bigcup_{i\in\omega} F_i}$. Hence, $\overline{F_k} \subset \overline{\bigcup_{i\in\omega} F_i}$ for every k. Therefore, $\bigcup_{i\in\omega} \overline{F_i} \subseteq \overline{\bigcup_{i\in\omega} F_i}$. Taking closure, we obtain $\overline{\bigcup_{i\in\omega} \overline{F_i}} \subseteq \overline{\bigcup_{i\in\omega} F_i}$. □

Let $(\mathbf{P}, v)$ be a phase space model. We know that $\Delta\langle\Gamma\rangle$ abbreviates $\Delta_0|_k(\Gamma\otimes\langle\Delta_1; \ldots; \Delta_{s(\Gamma)}\rangle)$ for a certain Δ_0, Δ_i, and $k > 0$. We recall that $\widehat{v}(\Gamma\otimes\langle\Delta_1; \ldots; \Delta_{s(\Gamma)}\rangle) \overset{def}{=} \widehat{v}(\Gamma) \times_1 \widehat{v}(\Delta_1) \ldots \times_{1+s(\Delta_1)+\ldots+s(\Delta_{s(\Gamma)})} \widehat{v}(\Delta_{s(\Gamma)})$.

(12) $\widehat{v}(\Gamma\otimes\langle\Delta_1; \ldots; \Delta_{s(\Gamma)}\rangle) \overset{def}{=} (\ldots(\widehat{v}(\Gamma) \times_1 \widehat{v}(\Delta_1))\ldots) \times_{1+s(\Delta_1)+\ldots+s(\Delta_{s(\Gamma)})} \widehat{v}(\Delta_{s(\Gamma)})$

The rhs of (12) is abbreviated overloading the symbol $\otimes$, i.e.:
$\widehat{v}(\Gamma\otimes\langle\Delta_1; \ldots; \Delta_{s(\Gamma)}\rangle)\overset{def}{=}\widehat{v}(\Gamma) \otimes \langle\widehat{v}(\Delta_1); \ldots; \widehat{v}(\Delta)\rangle$.

In order to prove soundness for phase semantics it is useful to directly compute configurations w.r.t. valuations without the use of type-equivalence. We have:

(13) $\widehat{v}(\Lambda)\overset{def}{=}\widehat{v}(I)$
$\widehat{v}(1, \Gamma)\overset{def}{=}\widehat{v}(J)\overline{\circ}\widehat{v}(\Gamma)$
$\widehat{v}(A, \Gamma)\overset{def}{=}\widehat{v}(A)\overline{\circ}\widehat{v}(\Gamma)$, if $s(A) = 0$
$\widehat{v}(A\{\Delta_1 : \ldots : \Delta_{s(A)}\}, \Gamma)\overset{def}{=}$
$((\cdots(\widehat{v}(A)\overline{\circ_1}\Delta_1)\cdots)\overline{\circ_{1+s(\Delta_1)+\cdots+s(\Delta_{s(A)})}\Delta_{s(A)}\widehat{v}(\Delta_{s(A)})}\overline{\circ}\widehat{v}(\Gamma)$, if $s(A) > 0$

But how do we interpret $\Delta\langle\Gamma\rangle$? As said before, $\Delta\langle\Gamma\rangle$ abbreviates $\Delta_0\langle\Gamma\otimes\langle\Delta_1;\ldots;\Delta_{s(\Gamma)}\rangle\rangle$. $\Gamma\otimes\langle\Delta_1;\ldots;\Delta_{s(\Gamma)}\rangle$ is a configuration. We have:

$$(14)\quad \widehat{v}(\Gamma\otimes\langle\Delta_1;\ldots;\Delta_{s(\Gamma)}\rangle)\stackrel{def}{=}(\cdots(\widehat{v}(A)\overline{\circ_1}\Delta_1)\cdots)\overline{\circ_{1+s(\Delta_1)+\cdots+s(\Delta_{s(A)}}\Delta_{s(A)}}\widehat{v}(\Delta_{s(\Gamma)})$$
$$=^{\text{by lemma 2}}=\overline{(\cdots\widehat{v}(A)\circ_1\Delta_1)\cdots)\circ_{1+s(\Delta_1)+\cdots+s(\Delta_{s(A)}\Delta_{s(A)}}\widehat{v}(\Delta_{s(A)})}$$

We abbreviate (14) as $\widehat{v}(\Gamma)\overline{\otimes}\langle\widehat{v}(\Delta_1);\ldots;\widehat{v}(\Delta_{s(\Gamma)}\rangle$ and by lemma 2 as $\overline{\widehat{v}(\Gamma)\otimes\langle\widehat{v}(\Delta_1);\ldots;\widehat{v}(\Delta_{s(\Gamma)}\rangle}$.

So $\widehat{v}(\Delta\langle\Gamma\rangle)=\overline{\widehat{v}(\Delta_0)\circ_k\overline{\widehat{v}(\Gamma)\otimes\langle\widehat{v}(\Delta_1);\ldots;\widehat{v}(\Delta_{s(\Gamma))}\rangle\rangle}}=$ $\overline{\widehat{v}(\Delta_0)\circ_k(\widehat{v}(\Gamma)\otimes\langle\widehat{v}(\Delta_1);\ldots;\widehat{v}(\Delta_{s(\Gamma))}\rangle\rangle}$, for a certain $k>0$, and where the last equality is due to lemma 2. We abbreviate $\widehat{v}(\Delta\langle\Gamma\rangle)$ as $\widehat{v}(\Delta)(\widehat{v}(\Gamma))$. By simple tonicity properties we have that if $\widehat{v}(\Gamma_1)\subseteq\widehat{v}(\Gamma_2)$ then $\widehat{v}(\Delta)(\widehat{v}(\Gamma)_1)\subseteq\widehat{v}(\Delta)(\widehat{v}(\Gamma_2))$.

Theorem 1. **DA**? *is sound w.r.t. phase semantics.*

Proof. By induction on the derivation of **DA**? sequents. For reasons of space we omit the proof cases of the remaining multiplicative and additive connectives, and units, and we only prove a representative case of the discontinuous implicative extract connective, and the case of the iteration connective.

Case of $\uparrow_k L$ $k>0$ (similar for the $\downarrow_k$ connective) we have:

$$(15)\quad \frac{\Gamma\Rightarrow A\qquad \Delta\langle\overrightarrow{B}\rangle\Rightarrow C}{\Delta\langle\overrightarrow{C\uparrow_k B}|_k\Gamma\rangle\Rightarrow C}\uparrow_k L$$

By induction hypothesis (i.h.), $\widehat{v}(\Gamma)\subseteq\widehat{v}(A)$. We have $\widehat{v}(\overrightarrow{B\uparrow_k A}|_k\Gamma)=\overline{\widehat{v}(\overrightarrow{B\uparrow_k A})\circ_k\widehat{v}(\Gamma)}\subseteq\widehat{v}(B)$. Hence $\widehat{v}(\Delta)(\widehat{v}(\overrightarrow{B\uparrow_k A}|_k\Gamma))\subseteq\widehat{v}(\Delta)(\widehat{v}(\overrightarrow{B})\subseteq\widehat{v}(C)$, where the last equality follows from the i.h.

Let us see rule $?L$. By i.h. for every $i>0$ $\widehat{v}(\Delta\langle A^i\rangle)\subseteq\widehat{v}(B)$. $\widehat{v}(\Delta\langle A^i\rangle)=\overline{\widehat{v}(\Delta)\circ_k\widehat{v}(A)^i}$, for a certain $k>0$. $\widehat{v}(\Delta)\circ_k\widehat{v}(A)^i\subseteq\overline{\widehat{v}(\Delta)\circ_k\widehat{v}(A)^i}$. Hence $\bigcup_{i>0}\widehat{v}(\Delta)\circ_k\widehat{v}(A)^i\subseteq\widehat{v}(B)$. But $\bigcup_{i>0}\widehat{v}(\Delta)\circ_k\widehat{v}(A)^i=\widehat{v}(\Delta)\circ_k\bigcup_{i>0}\widehat{v}(A)^i$. Taking closure $\overline{\widehat{v}(\Delta)\circ_k\bigcup_{i>0}\widehat{v}(A)^i}=^{\text{lemma 3}}=$ $\overline{\widehat{v}(\Delta)\circ_k\overline{\bigcup_{i>0}\widehat{v}(A)^i}}=\overline{\widehat{v}(\Delta)\circ_k\widehat{v}(?A)}=\widehat{v}(\Delta(?A))\subseteq\widehat{v}(B)$.

Rule $?R$ soundness is due to the fact that by i.h. $\widehat{v}(\Delta)=\widehat{v}(A)\subseteq\bigcup_{i>0}\widehat{v}(A)^i=\widehat{v}(?A)$.

Finally, let us see the Mingle rule $?M$:

$$(16)\quad \frac{\Gamma_1\Rightarrow A\qquad \Gamma_2\Rightarrow ?A}{\Gamma_1,\Gamma_2\Rightarrow ?A}?M$$

By i.h $\widehat{v}(\Gamma_1) \subseteq A$ and $\widehat{v}(\Gamma_2) \subseteq \widehat{v}(?A)$. $\widehat{v}(\Gamma_1) \circ \widehat{v}(\Gamma_2) \subseteq \widehat{v}(A) \circ \bigcup_{i>0} v(A)^i \subseteq \bigcup_{i>0} v(A)^i$. Taking closure we obtain $\overline{\widehat{v}(\Gamma_1) \circ \widehat{v}(\Gamma_2)} \subseteq \overline{\bigcup_{i>0} v(A)^i} = \overline{\bigcup_{i>0} v(A^i)} = \widehat{v}(?A)$. □

Let us use the following notation:

(17) For any type A, $[A] \stackrel{def}{=} \{\Delta \in \mathcal{O} : \Delta \Rightarrow^- A\}$
where $\Rightarrow^-$ means provability without Cut

The strategy of the proof of strong completeness is to construct a canonical model which we call the syntactic phase space. Its underlying DA is the DA of configurations $\mathcal{O}$ with its operations of concatenation and intercalation, so that we define the phase space $(\mathbf{M}, \mathbf{cl})$ where $\mathbf{M} = (\mathcal{O}, conc, (interc_i)_{i>0}, \Lambda, 1)$. $\mathbf{cl}$ is the least ω-sorted closure system such that it is generated by the family $([D])_{D \in \mathbf{Tp}}$. The condition 2.d) from definition 1 is satisfied (by way of example we prove it only for one discontinuous implication): Let F be a closed set and Γ be a configuration. Let us see that $F \uparrow\uparrow_i \Gamma$ is a closed set. By definition there exists a same-sort family of types $\mathcal{G}$ such that $F = \bigcap_{D \in \mathcal{G}} [D]$. We have $\Delta \in F \uparrow\uparrow_i \Gamma$ iff $\Delta |_i \Gamma \in F$ iff for any $D \in \mathcal{G}$ $\Delta |_i \Gamma \in [D]$ iff $D \in \mathcal{G}$ $\Delta |_i \Gamma^\bullet \in [D]$ iff for any $D \in \mathcal{G}$ $\Delta \in [D \uparrow_i \Gamma^\bullet]$. Therefore since $F \uparrow\uparrow_i \Gamma$ is the intersection of a same-sort family of sets, it is a closed set.

Lemma 4. *Let v be the valuation $v : \mathbf{Pr} \to \mathbf{cl}$ such that $v(p) \stackrel{def}{=} [p]$ for any primitive type p. There holds:*

(18) $\overrightarrow{A} \in \widehat{v}(A) \subseteq [A]$ *for any type A*

Proof. By induction on the structure of type A:

- If $A = p$ where p is a primitive type, we have by definition $v(A) = [A]$. Hence, $\overrightarrow{A} \in v(A) \subseteq [A]$.

- Case $A = J$ (the discontinuous unit). By the JR rule, $1 \in [J]$, i.e. $\{1\} \subseteq [J]$. Applying closure $\widehat{v}(J) = \overline{\{1\}} \subseteq [J]$.
On the other hand $\widehat{v}(J) = \bigcap_{D \in \mathcal{G}}$ for a certain family $\mathcal{G}$. $1 \in \widehat{v}(J)$, i.e., for every $D \in \mathcal{G}$, $1 \in [D]$. By JL rule, $\overrightarrow{J} \in [D]$. Therefore $\overrightarrow{J} \in \widehat{v}(J)$.

- Suppose $A = B \odot_i C$. $v(B) \circ_i v(C) = \{\Gamma_B |_i \Gamma_C : \Gamma_B \in \widehat{v}(B),$ and $\Gamma_C \in \widehat{v}(C)\}$. By i.h. $v(B) \subseteq [B]$ and $\widehat{v}(C) \subseteq [C]$. Hence, by application of $\odot_i L$ $\widehat{v}(B) \circ_i \widehat{v}(C) \subseteq [B \odot_i C]$. Hence, $\overline{\widehat{v}(B) \circ_i \widehat{v}(C)} \subseteq [B \odot_i C]$. This proves $\widehat{v}(B \odot_i C) \subseteq [B \odot_i C]$.

On the other hand, $\widehat{v}(B \odot_i C) = \bigcap_{D\in\mathcal{G}}[D]$ for a certain $\mathcal{G}$. By i.h. $\overrightarrow{B} \in \widehat{v}(B)$ and $\overrightarrow{C} \in \widehat{v}(C)$. Hence $\overrightarrow{B}|_i\overrightarrow{C} \in \widehat{v}(B)\circ_i\widehat{v}(C) \subseteq \widehat{v}(B \odot_i C)$. Then, for every $D \in \mathcal{G}$ $\overrightarrow{B}|_i\overrightarrow{C} \in [D]$. By application of $\odot_i L$, $\overrightarrow{B \odot_i C} \in [D]$. Hence, $\overrightarrow{B \odot_i C} \in \widehat{v}(B \odot_i C)$.

- Suppose $A = C\uparrow_i B$. The case for the other implicative connectives is completely similar. Let $\Gamma \in v(C)\uparrow\uparrow_i v(B)$. By i.h., $\overrightarrow{B} \in v(B)$. We have $\Gamma|_i\overrightarrow{B} \Rightarrow v(C)$ and $v(C) \subseteq [C]$ by i.h. Hence, $\Gamma|_i\overrightarrow{B} \subseteq [C]$, and by application of $\uparrow_i R$, $\Gamma \in [C\uparrow_i B]$.

- $v(C) = \bigcap_{D\in\mathcal{G}}[D]$ for some $\mathcal{G}$. By i.h., $\overrightarrow{C} \in v(C)$. Applying $\uparrow_i L$, we get $\overrightarrow{C\uparrow_i B}|_i\Gamma_B \in [D]$ for all $\Gamma_B \in \widehat{v}(B)$ (by i.h. $\widehat{v}(B)[B]$). We have then that $\overrightarrow{C\uparrow_i B}\circ_i\widehat{v}(B) \subseteq [D]$ for all $D \in \mathcal{G}$, whence $\overrightarrow{C\uparrow_i B}\circ_i\widehat{v}(B) \subseteq \widehat{v}(C)$. By applying residuation, $\overrightarrow{C\uparrow_i B} \in \widehat{v}(C)\uparrow\uparrow_i\widehat{v}(B) = \widehat{v}(C\uparrow_i B)$.

- Case $A = B \oplus C$. By i.h. $v(B) \subseteq [B]$ and $v(C) \subseteq [C]$. Hence, $v(B) \cup v(C) \subseteq cl([B] \cup [C]) \subseteq [B \oplus C]$. The first inclusion is due to the monotony property and properties of cl. In fact, we have $[B] \cup [C] \subseteq [B \oplus C]$. For, $[B] \subseteq [B \oplus C]$ and $[C] \subseteq [B \oplus C]$ by $\oplus iR$ $(i = 1, 2)$. It follows that $cl(v(B) \cup v(C)) \subseteq [B \oplus C]$.

- On the other hand, $v(B \oplus C) = \bigcap_{D\in\mathcal{G}}[D]$ for a certain $\mathcal{G}$. By i.h $\overrightarrow{B} \in v(B)$. Hence, $\overrightarrow{B} \subseteq cl(v(B) \cup v(C))$. Similarly, $\overrightarrow{C} \subseteq cl(v(B) \cup v(C))$. Therefore, for any $D \in \mathcal{G}$, $\overrightarrow{B} \in [D]$ and $\overrightarrow{C} \in [D]$. By $\oplus L$ we get $\overrightarrow{B \oplus C} \in [D]$. It follows that $\overrightarrow{B \oplus C} \subseteq v(B \oplus C)$.

- Case $C = ?A$

$$(19)\qquad \dfrac{\Gamma_1 \Rightarrow A \qquad \dfrac{\dfrac{\Gamma_{i-1} \Rightarrow A \qquad \dfrac{\Gamma_i \Rightarrow A}{\Gamma_i \Rightarrow ?A}\,?R}{\vdots}\,?M}{\Gamma_2, \ldots, \Gamma_i \Rightarrow ?A}\,?M}{\Gamma_1, \ldots, \Gamma_i \Rightarrow ?A}\,?M$$

The proof above shows that for every $i > 0$ $\widehat{v}(A)^i \subseteq [?A]$. We have then $\bigcup_{i>0}\widehat{v}(A)^i \subseteq [?A]$. Applying the closure map we get $\overline{\bigcup_{i>0}\widehat{v}(A)^i} \subseteq [?A]$, whence $\widehat{v}(?A) \subseteq [?A]$.
We prove now $?A \in \widehat{v}(?A)$. We know that $\widehat{v}(?A) = \bigcap_{D\in\mathcal{G}}[D]$, for a certain family of closed sets $\mathcal{G}$. By i.h. $A \in \widehat{v}(A)$. It follows that for every $i > 0$

$A^i \in \widehat{v}\,(A^i)$, whence $A^i \in \bigcup_{k>0} \widehat{v}\,(A^i) \subseteq \widehat{v}\,(?A)$. We have therefore:

$$\begin{array}{lll} \text{For every } i > 0\ A^i \in \widehat{v}\,(?A) & \text{iff} & \text{For every } i > 0, \text{ and for every } \mathrm{D} \in \mathcal{G}, A^i \in [D] \\ & \text{iff} & \text{For every } \mathrm{D} \in \mathcal{G}, ?A \in [D], \text{ by application of } ?R \\ & \text{iff} & ?A \in \widehat{v}\,(?A) \end{array}$$

□

Theorem 2 (Strong Completeness à la Okada).
Let $\Delta \Rightarrow A$ be such that for every $(\mathbf{P}, v)$, $(\mathbf{P}, v) \models \Delta \Rightarrow B$. It follows that $\Delta \Rightarrow^{-} B$.

Proof. In particular, this sequent holds in the syntactic phase displacement model. By the previous lemma, for any A, $\overrightarrow{A} \in \widehat{v}\,(A)$. Hence $\Delta \in \widehat{v}\,(\Delta)$. By soundness, for every $(\mathbf{P}, w)$ $\widehat{w}\,(\Delta) \subseteq \widehat{w}\,(B)$. Therefore we have that $\widehat{v}\,(\Delta) \subseteq \widehat{v}\,(B)$. Since $\Delta \in \widehat{v}\,(\Delta)$, $\Delta \in \widehat{v}\,(A)$, which entails (by the truth lemma) that $\Delta \in [A]$, i.e. $\Delta \Rightarrow^{-} A$. □

By the previous theorem $\Delta \Rightarrow A$ is provable without Cut, whence:

Corollary 1 (Cut admissibility). *The Cut rule is admissible.*

□

4 CatLog2 analyses

In Figure 5 we give a mini-lexicon for a fragment. The heart of the analysis of iterated coordination is the assignment to a coordinator of types of the form $(?A\backslash A)/A$. For a *respectively* construction we employ in conjunction with displacement connectives a calibrated version $?_n$ of the Mingle exponential as follows, with list Curry-Howard labelling:

$$\frac{\Delta(A_1\colon x_1, \ldots, A_n\colon x_n) \Rightarrow B\colon \psi([x_1, \ldots, x_n])}{\Delta(?_n A\colon z) \Rightarrow B\colon \psi(z)}\ ?_n L$$

$$\frac{\Gamma \Rightarrow A\colon \phi}{\Gamma \Rightarrow ?_1 A\colon [\phi]}\ ?_n R \qquad \frac{\Gamma \Rightarrow A\colon \phi \qquad \Delta \Rightarrow ?_n A\colon \phi'}{\Gamma, \Delta \Rightarrow [\phi|\phi']\colon ?_{n+1} A}\ ?_n M$$

A crucial aspect of what makes the respectively construction work here is the information sharing between two $?_A$ connectives in the type assignment to *respectively* — an implicit quantification over the natural A in the type: i.e. a kind of dependent type.

The output of a version of CatLog2 for some examples is as follows:

and : $(?Sf\backslash Sf)/Sf$: $(\Phi^{n+}\ \mathit{0}\ and)$
and : $(?(Sf/NA)\backslash(Sf/NA))/(Sf/NA)$: $(\Phi^{n+}\ (s\ 0)\ and)$
and : $(?(Sf/!NA)\backslash(Sf/!NA))/(Sf/!NA)$: $(\Phi^{n+}\ (s\ 0)\ and)$
and : $(?(NA\backslash Sf)\backslash(NA\backslash Sf))/(NA\backslash Sf)$: $(\Phi^{n+}\ (s\ 0)\ and)$
and : $(?((NA\backslash Sf)/NB)\backslash((NA\backslash Sf)/NB))/((NA\backslash Sf)/NB)$: $(\Phi^{n+}\ (s\ (s\ 0))\ and)$
and : $(?((NA\backslash Sf)/!NB)\backslash((NA\backslash Sf)/!NB))/((NA\backslash Sf)/!NB)$: $(\Phi^{n+}\ (s\ (s\ 0))\ and)$
and+1+and+1+respectively :$?_A NB\backslash((SC{\uparrow}(ND\backslash SC)){\uparrow}(NE\bullet ?_A(NF\backslash SC)))$:
$\lambda G\lambda H\lambda I(((\Phi^{n+}\ 0\ and)\ (I\ \pi_1 H))\ (\beta^+\ \pi_2 H\ G))$
Bill : $Nt(s(m))$: b
danced : $NA\backslash Sf$: $\lambda B(Past\ (dance\ B))$
John : $Nt(s(m))$: j
Mary : $Nt(s(f))$: m
laughed : $NA\backslash Sf$: $\lambda B(Past\ (laugh\ B))$
likes : $(Nt(s(A))\backslash Sf)/NB$: $like$
London : $\blacksquare Nt(s(n))$: l
love : $(NA\backslash Sb)/NB$: $love$
praised : $(NA\backslash Sf)/NB$: $\lambda C\lambda D(Past\ ((praise\ C)\ D))$
sang : $NA\backslash Sf$: $\lambda B(Past\ (sing\ B))$
sings : $Nt(s(A))\backslash Sf$: $sing$
talks : $Nt(s(A))\backslash Sf$: $talk$
walks : $Nt(s(A))\backslash Sf$: $walk$
will : $(NA\backslash Sf)/(NA\backslash Sb)$: $\lambda B\lambda C(Fut\ (B\ C))$

Fig. 5. Lexicon

4.1 Iterated coordination

To express the lexical semantics of (iterated) coordination, including iterated coordination and various arities (zeroary e.g. sentence, unary e.g. verb phrase, binary e.g. transtive verb, ...), we use combinators: a non-empty list map apply α^+, a non-empty list list apply β^+, and a non-empty list map $\mathbf{\Phi^n}$ combinator $\mathbf{\Phi^{n+}}$.[8]

The non-empty list map apply combinator α^+ is as follows:

(20) $$(\alpha^+\ [x]\ y) = [(x\ y)]$$ $$(\alpha^+\ [x, y|z]\ w) = [(x\ w)|(\alpha^+\ [y|z]\ w)]$$

The non-empty list list apply combinator α^+ is as follows:

(21) $$(\alpha^+\ [x]\ [y]) = [(x\ y)]$$ $$(\alpha^+\ [x|y]\ [z|w]) = [(x\ z)|(\alpha^+\ y\ w)]$$

The non-empty list map $\mathbf{\Phi^n}$ combinator $\mathbf{\Phi^{n+}}$ is thus:

[8] For the list map apply cf. Schiehlen (2005[16]. The combinator $\mathbf{\Phi}$ is such that $\mathbf{\Phi}\ x\ y\ z\ w = x\ (y\ w)\ (z\ w)$ (Curry and Feys 1958[3]).

(22) $(((\mathbf{\Phi^{n+}}\ 0\ and)\ x)\ [y]) = [y \wedge x]$

$$(((\mathbf{\Phi^{n+}}\ 0\ and)\ x)\ [y, z|w]) = [y \wedge (((\mathbf{\Phi^{n+}}\ 0\ and)\ x)\ [z|w])]$$

$$((((\mathbf{\Phi^{n+}}\ (s\ n)\ c)\ x)\ y)\ z) = (((\mathbf{\Phi^{n+}}\ n\ c)\ (x\ z))\ (\alpha^{+}\ y\ z))$$

These equations mean that in semantic evaluation any subterm of the form on the left is to be replaced by that on the right, successively.

The first example is of iterated sentence coordination:

(23) **John**+**walks**+**Mary**+**talks**+**and**+**Bill**+**sings** : Sf

Lexical lookup yields the following annotated sequent:

$Nt(s(m)) : j, Nt(s(A))\backslash Sf : walk, Nt(s(f)) : m, Nt(s(B))\backslash Sf : talk, (?Sf\backslash Sf)/Sf : (\Phi^{n+}\ 0\ and), Nt(s(m)) : b, Nt(s(C))\backslash Sf : sing \Rightarrow Sf$

The derivation is given in Figure 6. This delivers semantics:

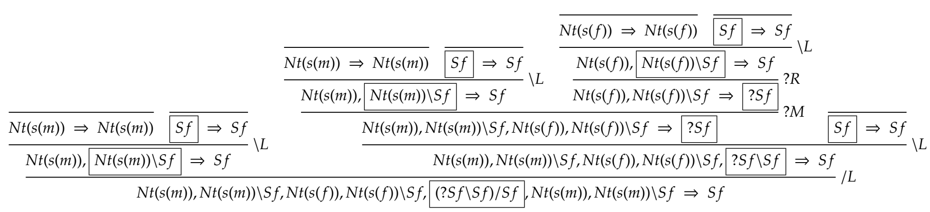

Fig. 6. Derivation of *John walks, Mary talks, and Bill sings*

$[(walk\ j) \wedge [(talk\ m) \wedge (sing\ b)]]$

The second example is of iterated verb phrase coordination:

(24) **John**+**walks**+**talks**+**and**+**sings** : Sf

Lexical lookup yields:

$Nt(s(m)) : j, Nt(s(A))\backslash Sf : walk, Nt(s(B))\backslash Sf : talk,$
$(?(NC\backslash Sf)\backslash(NC\backslash Sf))/(NC\backslash Sf) : (\Phi^{n+}\ (s\ 0)\ and), Nt(s(D))\backslash Sf : sing \Rightarrow Sf$

The derivation is given in Figure 7. This delivers semantics:

$[(walk\ j) \wedge [(talk\ j) \wedge (sing\ j)]]$

The next example is of iterated transitive verb coordination, with a non-standard constituent in the right hand conjunct:

(25) **John**+**praised**+**likes**+**and**+**will**+**love**+**London** : Sf

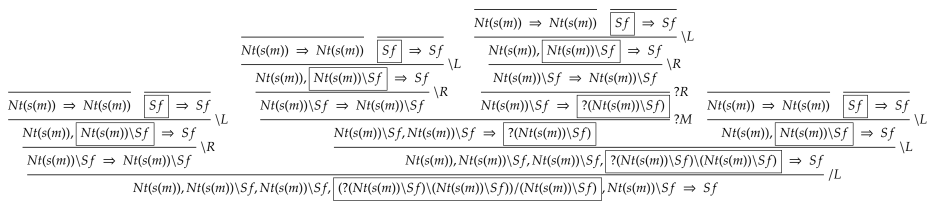

Fig. 7. Derivation of *John walks, talks, and sings*

Lexical lookup yields:

$Nt(s(m)) : j, (NA\backslash Sf)/NB : \lambda C\lambda D(Past\ ((praise\ C)\ D)), (Nt(s(E))\backslash Sf)/NF : like, (?((NG\backslash Sf)/NH)\backslash((NG\backslash Sf)/NH))/((NG\backslash Sf)/NH) : (\Phi^{n+}\ (s\ (s\ 0))\ and), (NI\backslash Sf)/(NI\backslash Sb) : \lambda J\lambda K(Fut\ (J\ K)), (NL\backslash Sb)/NM : love, \blacksquare Nt(s(n)) : l \Rightarrow Sf$

The derivation is given in Figure 8. This delivers semantics:

$[(Past\ ((praise\ l)\ j)) \wedge [((like\ l)\ j) \wedge (Fut\ ((love\ l)\ j))]]$

Finally we have an example of iterated coordination with right node raising:

(26) **John+praised+Bill+likes+and+Mary+will+love+London** : Sf

Lexical lookup yields:

$Nt(s(m)) : j, (NA\backslash Sf)/NB : \lambda C\lambda D(Past\ ((praise\ C)\ D)), Nt(s(m)) : b, (Nt(s(E))\backslash Sf)/NF : like, (?(Sf/NG)\backslash(Sf/NG))/(Sf/NG) : (\Phi^{n+}\ (s\ 0)\ and), Nt(s(f)) : m, (NH\backslash Sf)/(NH\backslash Sb) : \lambda I\lambda J(Fut\ (I\ J)), (NK\backslash Sb)/NL : love, \blacksquare Nt(s(n)) : l \Rightarrow Sf$

There is the derivation in Figure 9. This delivers semantics:

$[(Past\ ((praise\ l)\ j)) \wedge [((like\ l)\ b) \wedge (Fut\ ((love\ l)\ m))]]$

4.2 The *respectively* construction

Kubota and Levine (2016[6]) provide a type logical account of binary *respectively* constructions using empty operators. By contrast we account here for unbounded addicity *respectively* constructions, without empty operators.

Our first example synchronises parallel pairs of items:

(27) **Bill+and+Mary+danced+and+sang+respectively** : Sf

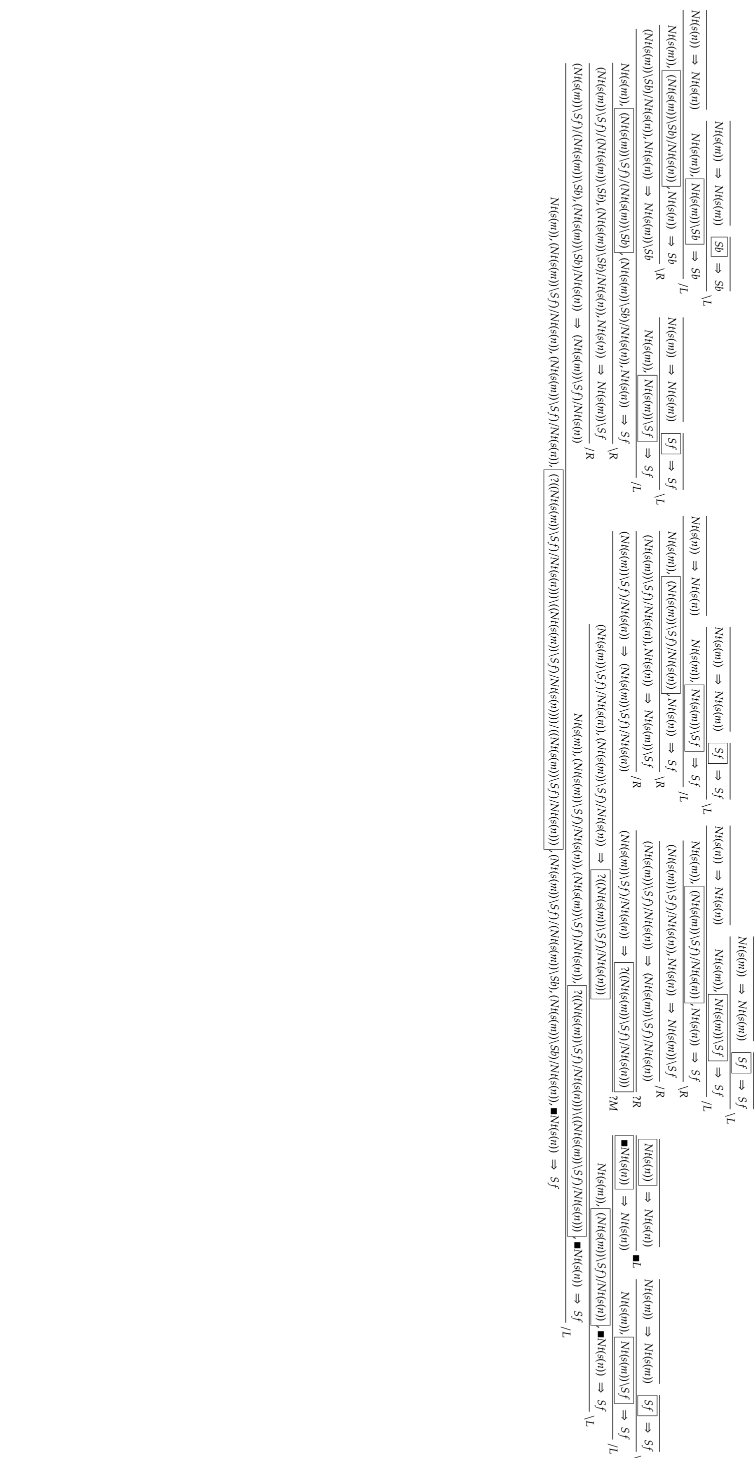

Fig. 8. Derivation of *John praised, likes, and will love, London*

Fig. 9. Derivation for *John praised, Bill likes, and Mary will love, London*

Lexical lookup yields:

$Nt(s(m)) : b, ?_A NB\backslash((SC{\uparrow}(ND\backslash SC)){\uparrow}(NE{\bullet}?_A(NF\backslash SC)))\{Nt(s(f)) : m, NJ\backslash Sf :$
$\lambda K(Past\ (dance\ K)) : NL\backslash Sf : \lambda M(Past\ (sing\ M))\} :$
$\lambda G\lambda H\lambda I(((\Phi^{n+}\ 0\ and)\ (I\ \pi_1 H))\ (\beta^{+}\ \pi_2 H\ G))\ \Rightarrow\ Sf$

There is the derivation given in Figure 10. This delivers semantics:

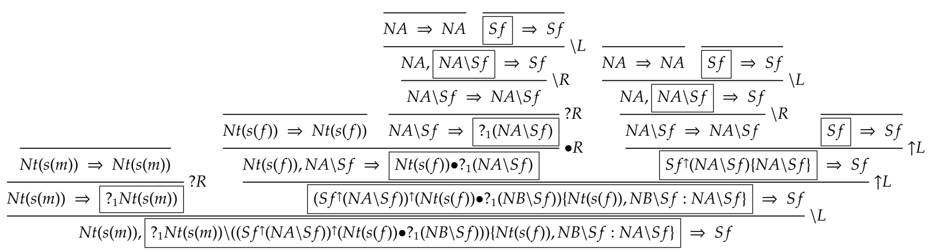

Fig. 10. Derivation for *Bill and Mary danced and sang respectively*

$[(Past\ (dance\ b)) \wedge (Past\ (sing\ m))]$

Our other example of the *respectively* construction synchronises parellel triples of items:

(28) **John+Bill+and+Mary+laughed+danced+and+sang+respectively** : Sf

Lexical lookup yields the following:

$Nt(s(m)) : j, Nt(s(m)) : b, ?_A NB\backslash((SC{\uparrow}(ND\backslash SC)){\uparrow}(NE{\bullet}?_A(NF\backslash SC)))\{Nt(s(f)) :$
$m, NJ\backslash Sf : \lambda K(Past\ (laugh\ K)), NL\backslash Sf : \lambda M(Past\ (dance\ M)) : NN\backslash Sf :$
$\lambda O(Past\ (sing\ O))\} : \lambda G\lambda H\lambda I(((\Phi^{n+}\ 0\ and)\ (I\ \pi_1 H))\ (\beta^{+}\ \pi_2 H\ G))\ \Rightarrow\ Sf$

There is the derivation given in Figure 11. This delivers semantics:

$[(Past\ (laugh\ j)) \wedge [(Past\ (dance\ b)) \wedge (Past\ (sing\ m))]]$

Interestingly, our account syntactically blocks examples of the kind *John and Peter walk, talk, and sing, respectively* since the calibrated numbers of occurrences are not equal. A variation of our account with uncalibrated modalities would need to appeal to a semantic anomaly in relation to the combinators.

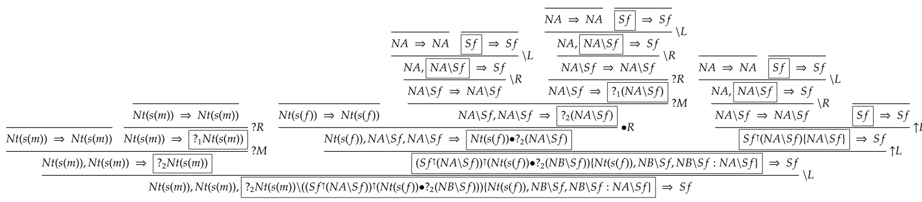

Fig. 11. Derivation for *John, Bill, and Mary laughed, danced, and sang, respectively*

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Bibliography

[1] D. Bechet, A. Dikowsky, A. Foret, and E. Garel. Optional and iterated types for pregroup grammars. In C. Martin-Vide, F. Otto, and H. Fernau, editors, *Language and Automata Theory and Applications*, number 5196 in LNCS, pages 88–100. Springer, 2008.

[2] Wojciech Buszkowski and Ewa Palka. Infinitary action logic: Complexity, models and grammars. *Studia Logica*, 89(1):1–18, 2008.

[3] Haskell B. Curry and Robert Feys. *Combinatory Logic*, volume I. North-Holland, Amsterdam, 1958.

[4] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.

[5] Norihiro Kamide. Substructural logics with mingle. *Journal of Logic, Language and Information*, 11(2):227–249, 2002.

[6] Yusuke Kubota and Bob Levine. The syntax-semantics interface of respective predication: a unified analysis in hybrid type-logical categorial grammar. *Natural Language and Linguistic Theory*, 34(3):911–973, August 2016.

[7] J. Lambek. On the Calculus of Syntactic Types. In Roman Jakobson, editor, *Structure of Language and its Mathematical Aspects, Proceedings of the Symposia in Applied Mathematics XII*, pages 166–178. American Mathematical Society, Providence, Rhode Island, 1961.

[8] J. Lambek. Categorial and Categorical Grammars. In R.T. Oehrle, E. Bach, and D. Wheeler, editors, *Categorial Grammars and Natural Language Structures*, volume 32 of *Studies in Linguistics and Philosophy*, pages 297–317. D. Reidel, Dordrecht, 1988.

[9] Joachim Lambek. The mathematics of sentence structure. *American Mathematical Monthly*, 65:154–170, 1958.

[10] Glyn Morrill. Grammar and Logical Types. In M. Stockhof and L. Torenvliet, editors, *Proceedings of the Seventh Amsterdam Colloquium*, pages 429–450, Amsterdam, 1990. University of Amsterdam.

[11] Glyn Morrill and Oriol Valentín. Computational Coverage of TLG: Nonlinearity. In M. Kanazawa, L.S. Moss, and V. de Paiva, editors, *Proceedings of NLCS'15. Third Workshop on Natural Language and Computer Science*, volume 32 of *EPiC*, pages 51–63, Kyoto, 2015. Workshop affiliated with Automata, Languages and Programming (ICALP) and Logic in Computer Science (LICS).

[12] Glyn Morrill, Oriol Valentín, and Mario Fadda. The Displacement Calculus. *Journal of Logic, Language and Information*, 20(1):1–48, 2011.

[13] Glyn V. Morrill. *Type Logical Grammar: Categorial Logic of Signs*. Kluwer Academic Publishers, Dordrecht, 1994.

[14] Mitsuhiro Okada. Phase semantic cut-elimination and normalization proofs of first- and higher-order linear logic. *Theoretical Computer Science*, 227(1–2):333–396, September 1999.

[15] Ewa Palka. An infinitary sequent system for the equational theory of *-continuous action lattices. *Fundam. Inf.*, 78(2):295–309, April 2007.

[16] Michael Schiehlen. The role of lists in a categorial analysis of coordination. In P. Dekker and M Franke, editors, *Proceedings of the 15th Amsterdam Colloquium*, pages 221–226, 2005.

[17] Oriol Valentín. *Theory of Discontinuous Lambek Calculus*. PhD thesis, Universitat Autònoma de Barcelona, Barcelona, 2012.

[18] Oriol Valentín. Models for the displacement calculus. In A. Foret, G. Morrill, R. Muskens, R. Osswald, and S. Pogodalla, editors, *Formal Grammar 2015: Revised Selected Papers. Formal Grammar 2016: Proceedings*, volume 9804 of *LNCS*, pages 147–163, Berlin, 2016. Springer.