Relational interpretation and geometrical form*

Glyn Morrill
Departament de Llenguatges i Sistemes Informàtics,
Universitat Politècnica de Catalunya,
Mòdul C 5 - Campus Nord,
Jordi Girona Salgado 1-3,
E-08034 Barcelona.

Abstract

We describe the possibility of a systematic correspondence between combined algebraic and relational interpretation of categorial logic, and the form of proof structures and paths in proof nets, illustrating with reference to medial extraction, in situ binding and discontinuity.

Type logic for linguistic description (e.g. Moortgat 1988, 1997; Morrill 1994; Carpenter 1997) is based on what we may refer to as a Lambek-van Benthem correspondence: (logical) formulas as (linguistic) categories. Lexical signs are classified by category formulas, and the language model projected by a lexicon is determined by the consequence relation induced on category formulas by their interpretation.

In this logical model of language, (logical) proofs correspond to (linguistic) derivations, but such syntax serves just to calculate what is generated, not to define it. Although syntax plays no definitional role linguistically, from a computational linguistic point of view we are interested in the processing of language, and we can reinstate syntactic structure as the trace of such processing. Addressing the question 'What is the essential structure of the relevant kinds of proofs?' yields an answer to the question 'What is syntactic structure?' under the application of proof nets as syntactic structures. We suggest, with reference to type logic for medial extraction, in situ binding and discontinuity, the possibility of a systematic correspondence between relational interpretation and paths in proof nets.

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1 Introduction

Whereas phrase structure grammar models language as a formal system, i.e. a set of strings, categorial grammar models language as a communicative system, i.e. a set of signs (form-meaning associations). Parse trees for CFG are concrete structures defining the equivalence classes of string rewriting derivations. Corresponding structures for categorial grammar must be deeper, since they incorporate also semantics. Here we pursue the idea that proof nets (Girard 1987, Danos and Regnier 1990[1]) are those structures (see e.g. Moortgat 1990b, 1992; Hendriks and Roorda 1991; Lecomte 1992, 1993; Lecomte and Retoré 1995; Oehrle 1994, 1995; Morrill 1996, 1999; Merenciano and Morrill 1997; de Groote and Retoré 1996), that proof nets are for categorial grammar what parse trees are for CFG. This provides a particularly vivid realisation of the notion of categorial syntactic connection of Ajdukiewicz (1935) as a harmonic mutual connectivity of the valencies of the words making up a sentence.

The syntactic calculus **L** of Lambek (1958) provides a logical model of language which presents formulas as categories and proofs as derivations. Proof nets for the calculus, recognizable as a multiplicative fragment of non-commutative intuitionistic linear logic (Girard 1989; Abrusci 1990), were developed in (Roorda 1991). The question arises as to how to characterise proof nets for phenomena which go beyond the expressivity of **L**. A line of approach will be described here.

1.1 Associative Lambek calculus

In the (associative) Lambek calculus **L** the *category formulas* $\mathcal{F}$ are constructed from atomic category formulas $\mathcal{A}$ (*atoms*) by a product operator $\bullet$ and two directional divisors, \ ("under"), and / ("over"), as follows:

$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{F}_1{\bullet}\mathcal{F}_2 \mid \mathcal{F}_1\backslash\mathcal{F}_2 \mid \mathcal{F}_1/\mathcal{F}_2 \qquad (1)$$

Lambek (1958, 1988) gives an *algebraic* interpretation in a semigroup $(L, +)$, a set L closed under an associative binary operation $+$ (we may think of the set of strings over some vocabulary, and the operation of concatenation). Formulas are interpreted as subsets of L. Given an interpretation $[\![P]\!]$ for each atom P, each category formula A receives an interpretation $[\![A]\!]$ thus:

$$\begin{array}{lcl} [\![A\backslash B]\!] & = & \{s \mid \forall s' \in [\![A]\!], s'{+}s \in [\![B]\!]\} \\ [\![B/A]\!] & = & \{s \mid \forall s' \in [\![A]\!], s{+}s' \in [\![B]\!]\} \\ [\![A{\bullet}B]\!] & = & \{s_1{+}s_2 \mid s_1 \in [\![A]\!] \ \& \ s_2 \in [\![B]\!]\} \end{array} \qquad (2)$$

Van Benthem (1991) gives a *relational* interpretation in a set V (we may think of the starting and ending moments of utterances). Formulas are interpreted as binary relations, i.e. as subsets of $V \times V$. Given an interpretation $[\![P]\!]$ for each

[1] Cf. also Gallier 1992.

atom P, each category formula A receives an interpretation $[\![A]\!]$ thus:

$$
\begin{array}{rcl}
[\![A\backslash B]\!] & = & \{\langle v_2, v_3\rangle \mid \forall v_1, \langle v_1, v_2\rangle \in [\![A]\!] \rightarrow \langle v_1, v_3\rangle \in [\![B]\!]\} \\
[\![B/A]\!] & = & \{\langle v_1, v_2\rangle \mid \forall v_3, \langle v_2, v_3\rangle \in [\![A]\!] \rightarrow \langle v_1, v_3\rangle \in [\![B]\!]\} \\
[\![A\bullet B]\!] & = & \{\langle v_1, v_3\rangle \mid \exists v_2, \langle v_1, v_2\rangle \in [\![A]\!] \;\&\; \langle v_2, v_3\rangle \in [\![B]\!]\}
\end{array} \tag{3}
$$

A sequent $\Gamma \Rightarrow A$ comprises a succedent category formula A and an antecedent configuration Γ which is a finite sequence of category formulas.[2] A sequent $A_1, \ldots, A_n \Rightarrow A$ asserts that for all algebraic interpretations, for all $s_1, \ldots, s_n \in L$, if $s_i \in [\![A_i]\!], 1 \leq i \leq n$ then $s_1 + \cdots + s_n \in [\![A]\!]$, and that for all relational interpretations, for all $v_0, \ldots, v_n \in V$, if $\langle v_{i-1}, v_i\rangle \in [\![A_i]\!], 1 \leq i \leq n$ then $\langle v_0, v_n\rangle \in [\![A]\!]$. The valid sequents are those generated by the following sequent calculus ($\Gamma(\Delta)$ indicates a configuration Γ with a distinguished subconfiguration Δ):[3]

$$
\text{a.}\quad A \Rightarrow A \ \ \text{id} \qquad \frac{\Gamma \Rightarrow A \qquad \Delta(A) \Rightarrow B}{\Delta(\Gamma) \Rightarrow B}\text{Cut} \tag{4}
$$

$$
\text{b.}\quad \frac{\Gamma \Rightarrow A \qquad \Delta(B) \Rightarrow C}{\Delta(\Gamma, A\backslash B) \Rightarrow C}\backslash\text{L} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A\backslash B}\backslash\text{R}
$$

$$
\text{c.}\quad \frac{\Gamma \Rightarrow A \qquad \Delta(B) \Rightarrow C}{\Delta(B/A, \Gamma) \Rightarrow C}/\text{L} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B/A}/\text{R}
$$

$$
\text{d.}\quad \frac{\Gamma(A, B) \Rightarrow C}{\Gamma(A\bullet B) \Rightarrow C}\bullet\text{L} \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A\bullet B}\bullet\text{R}
$$

Each connective has a rule of *use* in which it appears in the antecedent of the conclusion sequent, and a rule of *proof* in which it appears in the succedent of the conclusion sequent; in every instance of these *logical* rule schemata there is exactly one more connective occurrence in the conclusion than in the premises so that backward chaining proof steps involving these rules are complexity-reducing: trying to prove conclusions by proving the premises generates strictly simpler subgoals. The *identity* rule schemata id and Cut reflect respectively the reflexivity and transitivity of set containment. The id rule schema has zero premises, i.e. it is an axiom schema; the instances where A is a compound formula are derivable by the other rules from atomic instances, hence id can be restricted to apply to atoms without altering the set of theorems generated. In the Cut rule schema the Cut formula A is duplicated in the premises and the rule fails to be complexity-reducing in the sense of the logical rules. However, the

[2] Officially the antecedent is non-empty, a detail we gloss over.

[3] Regarding completeness with respect to semigroup, free semigroup, and relational interpretation see Buszkowski (1986), Pentus (1994) and Andréka and Mikulás (1994) respectively; see also Kurtonina (1995). Buszkowski (1996) gives a survey.

calculus has the property of Cut-elimination: for every proof there is an equivalent Cut-free proof. This means that naive Cut-free backward chaining proof search constitutes a decision procedure for theoremhood. The Cut-elimination result has as a corollary the *subformula property* that every theorem has a proof containing only its subformulas — namely any Cut-free proof.

Lambek calculus provides a classificatory framework for subcategorisation which synchronizes naturally with Fregean semantics of incompleteness and compositionality. It provides for some proper treatment of quantification, and for some action-at-a-distance. Still, from a linguistic point of view the possibilities of the Lambek calculus are extremely limited since it is a logic of only concatenation, i.e. continuity, whereas language exhibits discontinuity.

1.2 Discontinuity

By way of examples of discontinuity beyond the reach of **L** we consider medial extraction, and in situ binding. In (5) the relative pronoun binds a position which is medial in the relative clause.

$$\text{(the dog) that}_i \text{ John gave } t_i \text{ to Mary} \tag{5}$$

Defining the relative pronoun as R/(N\S) or R/(S/N) (where R is CN\CN) allows it to bind only left or right peripheral positions: (5) is not generated. To deal with such cases, Moortgat (1988) defines as follows a binary operator which we write $\uparrow_e$:

$$[\![B{\uparrow_e}A]\!] = \{s_1{+}s_2 | \forall s \in [\![A]\!], s_1{+}s{+}s_2 \in [\![B]\!]\} \tag{6}$$

Assigning the relative pronoun to category R/(S$\uparrow_e$N) allows both medial and peripheral extraction, via the rule of proof (7).

$$\frac{\Gamma_1, A, \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B{\uparrow_e}A}\uparrow_e\mathrm{R} \tag{7}$$

Such a treatment potentially accommodates obligatory extraction valencies:[4]

a. (the man) that$_i$ John assured Mary t_i to be reliable (8)
b. ∗John assured Mary Bill to be reliable.

If the extraction valency of "assured" is marked by $\uparrow_e$, a sequent corresponding to (8a) is valid while that for (8b) is invalid, as required. But Moortgat (1988: 121–2) observes that a satisfactory sequent rule of use cannot be formulated, and, as pointed out by I. Sag (p.c.), in the absence of a rule of use it is impossible to actually derive all cases like (8a) since when the obligatory extraction valency

[4] Cf. e.g. Oehrle (1990). Morrill (1994) considers medial and obligatory extraction in terms of S/△N rather than S$\uparrow_e$N, where △ is a modality licensing permutation (Barry, Hepple, Leslie and Morrill 1991). It remains to explore whether the current methods can be applied to such a modality.

verb is subordinate to some functor, one needs to *make use* of the operator in the course of the derivation.

Regarding in situ binding, in (9) the quantifier phrase and reflexive are in situ binders, taking scope at the sentence and verb phrase levels respectively.

a. John bought someone Fido. (9)
b. John bought himself Fido.

Moortgat (1996) gives a ternary operator Q which we may interpret:

$$[\![Q(B, A, C)]\!] = \{s | \forall s_1, s_3, [\forall s_2 \in [\![A]\!], s_1{+}s_2{+}s_3 \in [\![B]\!]] \rightarrow s_1{+}s{+}s_3 \in [\![C]\!]\} \tag{10}$$

Moortgat categorises quantifier phrases and reflexives as sentence and verb phrase in situ binders: $Q(\mathrm{S}, \mathrm{N}, \mathrm{S})$ and $Q(\mathrm{N}\backslash\mathrm{S}, \mathrm{N}, \mathrm{N}\backslash\mathrm{S})$ respectively. Cases such as (9) are generated by means of the rule of use (11).

$$\frac{\Gamma(A) \Rightarrow B \qquad \Delta(C) \Rightarrow D}{\Delta(\Gamma(Q(B, A, C))) \Rightarrow D}Q\mathrm{L} \tag{11}$$

However, this time no satisfactory rule of proof can be given. Therefore, as pointed out by H. Hendriks (p.c.), a valid sequent such as (12), showing that a sentence in situ binder is also a verb phrase in situ binder, cannot actually be derived.

$$Q(\mathrm{S}, \mathrm{N}, \mathrm{S}) \Rightarrow Q(\mathrm{N}\backslash\mathrm{S}, \mathrm{N}, \mathrm{N}\backslash\mathrm{S}) \tag{12}$$

However, it may be that a proof net syntax can still be provided for such systems.

2 Proof nets

Surveys of proof nets include Lamarche and Retoré (1996) and Retoré (1996). Works on (possibly) non-commutative or partially commutative proof nets includes Retoré (1993, 1997), Bellin and van de Wiele (1995), Ruet (1997), Abrusci and Ruet (1998), de Groote (1999) and Moot and Puite (1999). Here we consider the possibility of a systematic correspondence between combined algebraic and relational interpretation and paths in proof nets, for which it is convenient to describe, in the following two subsections, some aspects of classical linear logic, and proof nets for classical linear logic, and in subsection 2.3 proof nets for the Lambek-van Benthem categorial calculus. In section 3 we consider paths in proof nets for the Lambek calculus, and in subsections 3.1, 3.2 and 3.3, paths in proof nets for the medial divisor the in situ binder, and discontinuity more generally.

2.1 Classical linear logic

Formulas for classical linear logic can be defined as follows:

$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{F}_1 \otimes \mathcal{F}_2 \mid \mathcal{F}_1 \wp \mathcal{F}_2 \mid \mathcal{F}_1 \multimap \mathcal{F}_2 \mid \mathcal{F}^{\perp} \tag{13}$$

In the sequent calculus (14), sequents are of the form $\Gamma \Rightarrow \Delta$ where configurations Γ and Δ are finite sequences of formulas.

(14)

a. $$\frac{}{A \Rightarrow A}\text{id} \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\text{Cut}$$

b. $$\frac{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \Rightarrow \Delta}\text{P}_L \qquad \frac{\Gamma \Rightarrow \Delta_1, A, B, \Delta_2}{\Gamma \Rightarrow \Delta_1, B, A, \Delta_2}\text{P}_R$$

c. $$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \otimes B \Rightarrow \Delta}\otimes\text{L} \qquad \frac{\Gamma_1 \Rightarrow A, \Delta_1 \quad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \otimes B, \Delta_1, \Delta_2}\otimes\text{R}$$

d. $$\frac{A, \Gamma_1 \Rightarrow \Delta_1 \quad B, \Gamma_2 \Rightarrow \Delta_2}{A \wp B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\wp\text{L} \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \wp B}\wp\text{R}$$

e. $$\frac{\Gamma_1 \Rightarrow A, \Delta_1 \quad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, A \multimap B, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\multimap\text{L} \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \multimap B, \Delta}\multimap\text{R}$$

f. $$\frac{\Gamma \Rightarrow A, \Delta}{\Gamma, A^{\perp} \Rightarrow \Delta}{}^{\perp}\text{L} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A^{\perp}, \Delta}{}^{\perp}\text{R}$$

We recognize for $\otimes$ ("times"), $\wp$ ("par"), $\multimap$ ("linear implication"), and $^{\perp}$ ("perp") classical sequent rules for conjunction, disjunction, implication and negation respectively. Indeed, the only difference with respect to classical logic is that the structural rules of contraction and weakening are not included. This calculus, *multiplicative classical linear logic*, has the property of Cut-elimination.

Those properties of classical logic which do not depend on contraction and weakening are inherited by classical linear logic. For example, the negation is involutive, $A^{\perp\perp} \Leftrightarrow A$:

(15)

a. $$\frac{\dfrac{A \Rightarrow A}{A, A^{\perp} \Rightarrow}{}^{\perp}\text{L}}{A \Rightarrow A^{\perp\perp}}{}^{\perp}\text{R} \qquad \text{b. } \frac{\dfrac{A \Rightarrow A}{\Rightarrow A^{\perp}, A}{}^{\perp}\text{R}}{A^{\perp\perp} \Rightarrow A}{}^{\perp}\text{L}$$

And there are the following proofs of the two sides of the de Morgan law

$(A \otimes B)^{\perp} \Leftrightarrow A^{\perp} \wp B^{\perp}$:

$$\text{a.} \quad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A, A^{\perp}}{}^{\perp}\mathrm{R} \quad \dfrac{B \Rightarrow B}{\Rightarrow B, B^{\perp}}{}^{\perp}\mathrm{L}}{\Rightarrow A \otimes B, A^{\perp}, B^{\perp}}\otimes\mathrm{R}}{\Rightarrow A \otimes B, A^{\perp} \wp B^{\perp}}\wp\mathrm{R}}{(A \otimes B)^{\perp} \Rightarrow A^{\perp} \wp B^{\perp}}{}^{\perp}\mathrm{L} \qquad \text{b.} \quad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A^{\perp}, A \Rightarrow}{}^{\perp}\mathrm{L} \quad \dfrac{B \Rightarrow B}{B^{\perp}, B \Rightarrow}{}^{\perp}\mathrm{L}}{A^{\perp} \wp B^{\perp}, A, B \Rightarrow}\wp\mathrm{L}}{A^{\perp} \wp B^{\perp}, A \otimes B \Rightarrow}\otimes\mathrm{L}}{A^{\perp} \wp B^{\perp} \Rightarrow (A \otimes B)^{\perp}}{}^{\perp}\mathrm{R} \tag{16}$$

The other de Morgan law, $(A \wp B)^{\perp} \Leftrightarrow A^{\perp} \otimes B^{\perp}$, is obtained similarly, and also the equivalence $A \multimap B \Leftrightarrow A^{\perp} \wp B$. Consequently, all formulas have a negation normal form for which they may be regarded as metalinguistic abbreviations; that is the way classical linear logic is usually presented but, for expository reasons, we do otherwise here.[5]

2.2 Proof nets for classical linear logic

In sequent calculus each formula is situated with respect to an opposition, antecedent-succedent. In proof nets, each formula A will be correspondingly situated by signing it as of either input polarity, $A^{\bullet}$, or as of output polarity, A°. In order to define proof nets we first define a class of proof structures of which they are a subset. A proof structure is a connected graph with nodes labelled by signed formulas, assembled out of the *proof links* given in figure 1; in the identity links, X and $\overline{X}$ are $A^{\bullet}$ and A° (in either order). Each formula in a proof link (and a proof structure) is also labelled implicitly as either a premise or a conclusion, or else as internal. We draw edges in such a way that premises always look upwards and conclusions always look downwards; the logical links each have two premises and one conclusion; the id axiom link has two conclusions and no premises, the Cut link two premises and no conclusions.[6]

We define a *signed formula tree* to be a finite tree with leaves labelled by signed atoms, each local tree of which is a logical link. A *proof frame* is a finite sequence[7] of signed formula trees. A *proof structure* is obtained from a proof frame by connecting complementary leaves with axiom links, and complementary roots with Cut links, in such a way that each leaf is connected to exactly one other, and each root to at most one other. Alternatively viewed, proof structures are assembled by identifying premises and conclusions of proof links which are of the same signed formula; see figure 2.

A proof structure with input conclusions $A_1{}^{\bullet}, \ldots, A_n{}^{\bullet}$ and output conclusions $B_1{}^{\circ}, \ldots, B_m{}^{\circ}$ is read as asserting that $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$ is valid.

[5] Furthermore, since e.g. $\Gamma, A \Rightarrow \Delta$ if and only if $\Gamma \Rightarrow A^{\perp}, \Delta$ one may convert every sequent to an equivalent one-sided sequent, and work with a one-sided calculus; but for comparison with later calculi, we retain the (more cumbersome) two-sided view.

[6] We consider the premises of proof links to be ordered, left and right, in the way they are drawn; to maintain a purely graph-theoretic view we should say that there is an implicit directed edge between the premises of logical links.

[7] Again, regarding this ordering see the previous note.

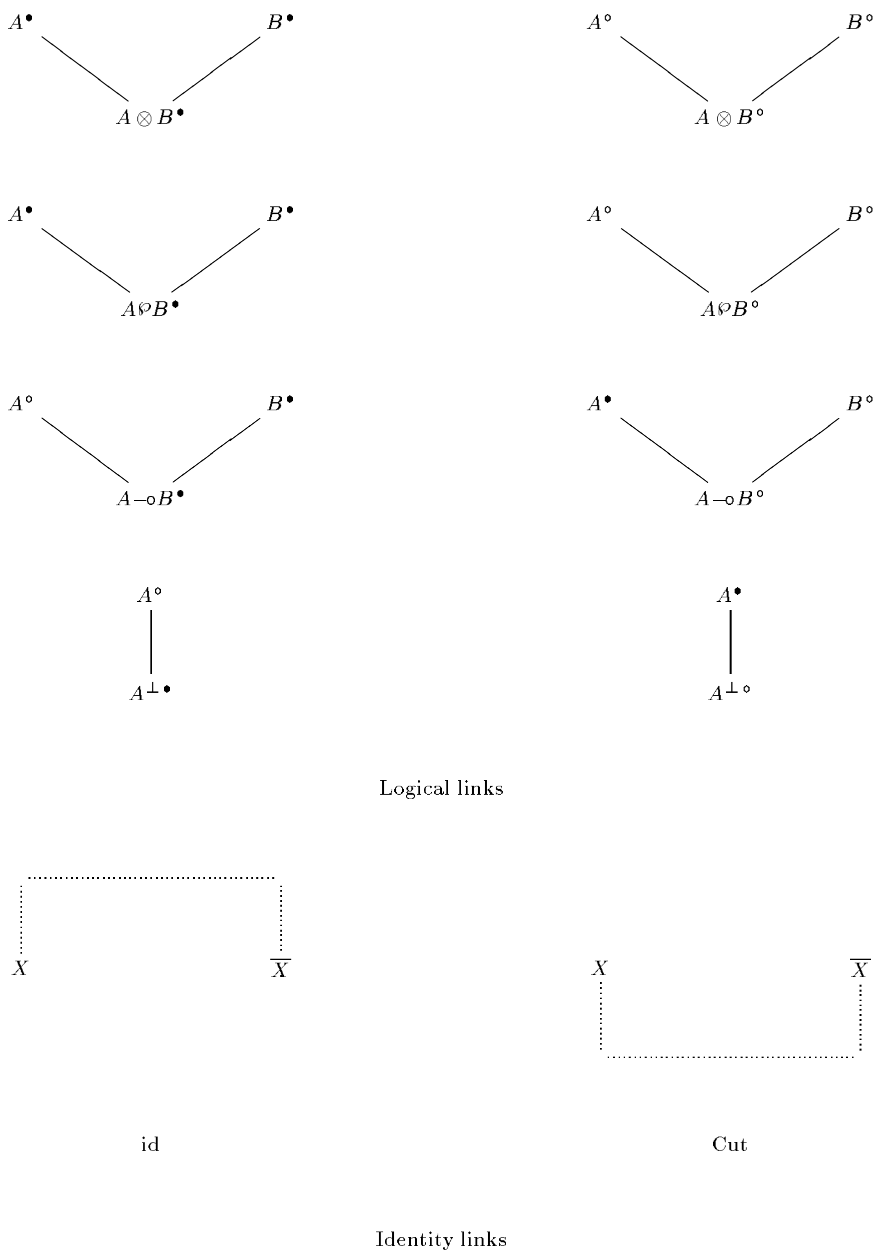

Figure 1: Proof links of classical linear logic

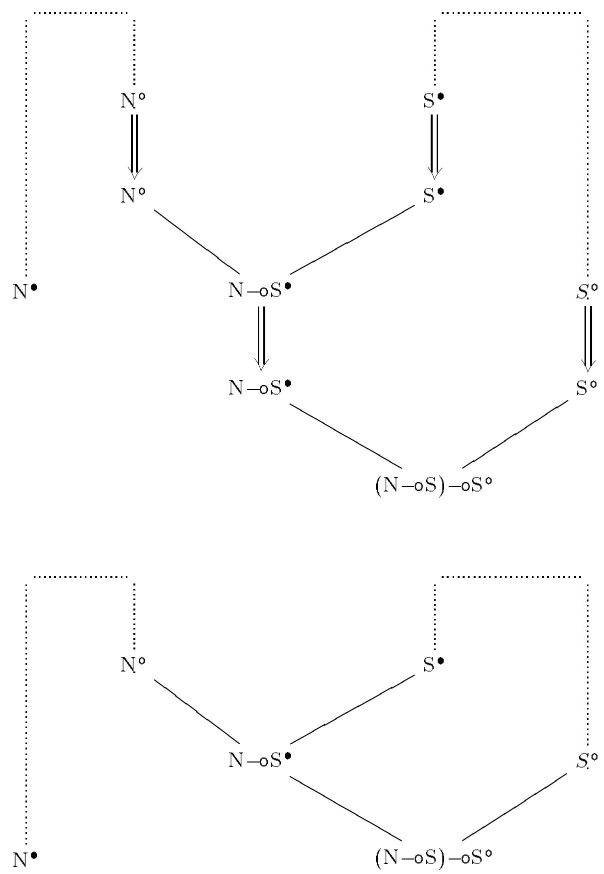

Figure 2: Assembly of a proof structure

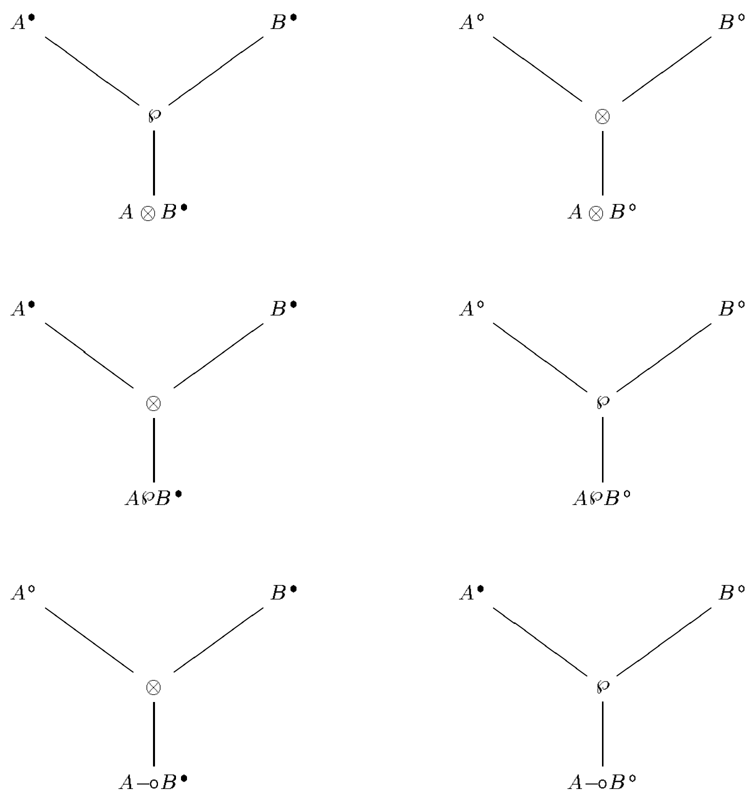

Figure 3: Expanded logical links of classical linear logic

Thus, the proof structure of figure 2 asserts N $\Rightarrow$ (N$\multimap$S)$\multimap$S, which is in fact true, but not all proof structures are correct; indeed $\otimes$ and $\wp$ are not distinguished: the splitting of contexts by binary sequent rules is not represented.

We shall define correctness conditions on proof structures in terms of what we call *expanded* proof links/frames/structures. The conditions are easily applied to proof structures themselves in virtue of their (unique) expansion, but reference to the expanded level allows for a perhaps tidier statement. The links which alter on expansion are given in figure 3. In the $\otimes$- and $\wp$-output links the central node is the principal connective of the conclusion. In the $\otimes$- and $\wp$-input links the central node is the de Morgan dual of the principal connective of the conclusion; this is because we regard input polarity as *negating*.[8] In the $\multimap$-output link we see the disjunction and polarity propagation of the equivalence $A \multimap B \Leftrightarrow A^\perp \wp B$, and in the $\multimap$-input link we see the conjunction and polarity

[8] That is, we adopt the point of view of one-sided sequents in which the antecedent is empty, which is the usual perspective of linear logic; but one could equally adopt the point of view of one-sided sequents in which the succedent is empty, which is the usual point of view of refutation, in which case we would regard output polarity as negating.

propagation of the equivalence $(A \multimap B)^\perp \Leftrightarrow A \otimes B^\perp$.

The original correctness criterion of Girard (1987), the *long trip condition*, is as follows. Each $\otimes$- and $\wp$-fork in an expanded proof structure is considered a switch which determines travel instructions according to which of two states it is in: open to the left (and closed to the right) or open to the right (and closed to the left). Entering an open premise, we always exit through the conclusion, but the other two cases depend on the connective. Entering the closed premise of $\otimes$ we exit through the other (open) premise, but entering the closed premise of $\wp$ we *bounce*, returning immediately out of the same (closed) premise back the way we came. Entering the conclusion of $\otimes$ we go out through the closed premise, but entering the conclusion of $\wp$ we go out through the open premise. Finally, when we arrive at a conclusion, we also bounce, returning immediately in the direction from which we just came.

A trip is a path through a proof structure according to a switching; note that once begun a trip extends deterministically. A trip is *long* if and only if it returns to its starting point having traversed each edge exactly once in each direction. A switching defines a long trip if and only if there is some long trip for the switching; in view of determinism and periodicity, a switching defines some long trip if and only if starting anywhere results in a long trip. A proof structure is correct, that is it is a *proof net*, if and only if every switching defines a long trip. A sequent $\Gamma \Rightarrow \Delta$ is a theorem of the sequent calculus if and only if there is a proof net with input conclusions Γ and output conclusions Δ.

The proof nets, like the sequent calculus, have the property of Cut-elimination: for every proof net there is an equivalent Cut-free proof net — having the same bindings in identity links of the atoms of conclusions. This means that there is the following decision procedure for determining theoremhood via proof nets. Given a sequent $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$, construct the proof frame with conclusions ${A_1}^\bullet, \ldots, {A_n}^\bullet, {B_1}^\circ, \ldots, {B_m}^\circ$ comprising the sequence of signed formula trees given by the following recursive unfolding:

$$\frac{A^\bullet \quad B^\bullet}{A \otimes B^\bullet}\wp \qquad \frac{A^\circ \quad B^\circ}{A \otimes B^\circ}\otimes \qquad \frac{A^\bullet \quad B^\bullet}{A \wp B^\bullet}\otimes \qquad \frac{A^\circ \quad B^\circ}{A \wp B^\circ}\wp \tag{17}$$

$$\frac{A^\circ \quad B^\bullet}{A \multimap B^\bullet}\otimes \qquad \frac{A^\bullet \quad B^\circ}{A \multimap B^\circ}\wp \qquad \frac{A^\circ}{A^{\perp\bullet}} \qquad \frac{A^\bullet}{A^{\perp\circ}}$$

Then test whether the long trip condition is satisfied for some Cut-free proof structure (there are a finite number) that can be built by putting axiom links on the proof frame.

Testing the long trip condition as it stands is not attractive computationally since in a proof structure with i $\wp$-links and j $\otimes$-links there are 2^{i+j} switchings to be tried. The situation is improved with the correctness criterion as formulated by Danos and Regnier (1989), which considers only switchings of $\wp$-links. For any given switching, a certain graph results by removing from an expanded proof net the edges between each $\wp$-conclusion and its closed premise. The result of Danos and Regnier is that a proof structure is a proof net if and only if for every switching of $\wp$-links, the result of removing these edges is acyclic and connected. A direct application of this simplified criterion requires only 2^i switchings to be tried.

2.3 Lambek-van Benthem calculus

Consider formulas defined as follows.

$$\mathcal{F} ::= \mathcal{A} \mid \mathcal{F}_1 \otimes \mathcal{F}_2 \mid \mathcal{F}_1 \multimap \mathcal{F}_2 \tag{18}$$

In the calculus (19) sequents are of the form $\Gamma \Rightarrow A$ where the antecedent configuration is a sequence of formulas, but the succedent comprises exactly one formula.

$$\text{a.}\quad \frac{}{A \Rightarrow A}\text{id} \qquad \frac{\Gamma_1 \Rightarrow A \quad A, \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B}\text{Cut} \tag{19}$$

$$\text{b.}\quad \frac{\Gamma_1, A, B, \Gamma_2 \Rightarrow C}{\Gamma_1, B, A, \Gamma_2 \Rightarrow C}\text{P}$$

$$\text{c.}\quad \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \otimes B \Rightarrow C}\otimes\text{L} \qquad \frac{\Gamma_1 \Rightarrow A \quad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \otimes B}\otimes\text{R}$$

$$\text{d.}\quad \frac{\Gamma_1 \Rightarrow A \quad B, \Gamma_2 \Rightarrow C}{\Gamma_1, A \multimap B, \Gamma_2 \Rightarrow C}\multimap\text{L} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B}\multimap\text{R}$$

We recognize positive intuitionistic sequent rules for conjunction and implication; indeed, the only difference with respect to positive intuitionistic logic is that the structural rules of contraction and weakening are not included. This is the Lambek-van Benthem categorial calculus **LP**: a multiplicative fragment of intuitionistic linear logic; it has the property of Cut-elimination. Compared to classical linear logic, we see that there is now only one (left-sided) permutation rule, since there are never two formulas in the succedent to which a right permutation rule could apply. All the rules are instances of rules of the classical calculus, so intuitionistic proof nets are a special case of classical proof nets, and every intuitionistic linear theorem is also a classical linear theorem.

We give the proof links in figure 4. An **LP** signed formula tree is a finite tree with atomic (signed) leaves each local tree of which is an **LP** logical link. An **LP** proof frame is a finite sequence of **LP** signed formula trees. An **LP** proof structure is obtained by connecting complementary leaves with axiom links and complementary roots with Cut links in such a way that each leaf is connected to exactly one other and each root is connected to at most one other, and which has exactly one conclusion of output polarity. A proof structure with input conclusions Γ and output conclusion A is read as asserting that $\Gamma \Rightarrow A$ is valid. Being special cases of classical proof nets, intuitionistic proof nets must

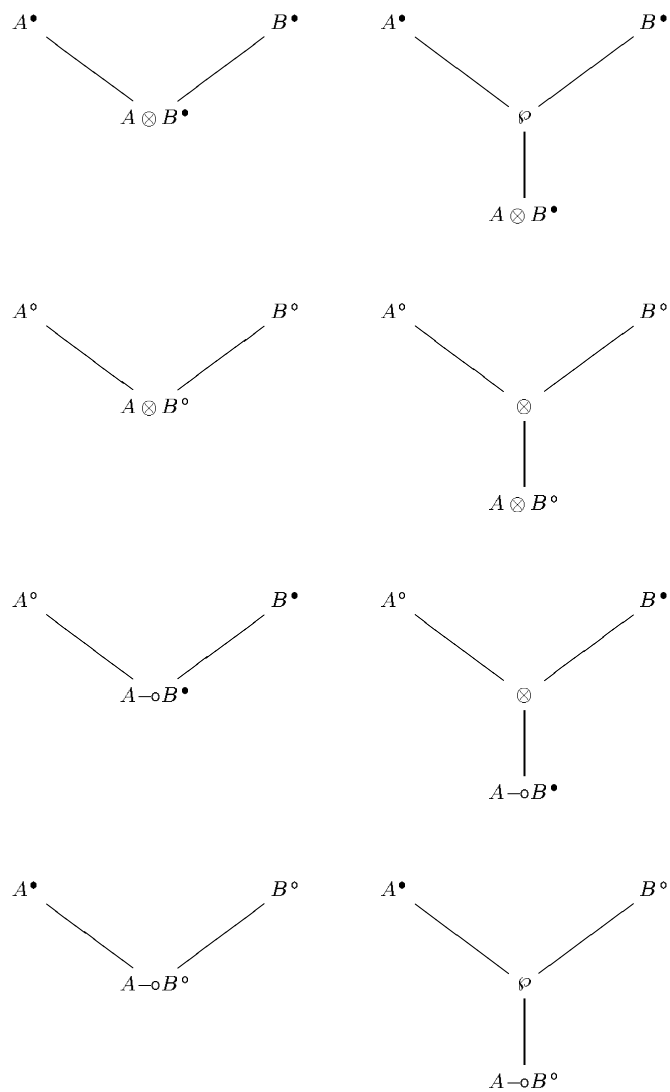

Figure 4: Logical proof links of **LP** and their expansions

satisfy the following, otherwise there would be cyclicity on some Danos-Regnier switching:

Acyclicity condition. Every cycle must cross both edges of some $\wp$-link. (20)

Now, so far as we are aware, an intuitionistic sequent, i.e. a single conclusion sequent of $\{\otimes, \multimap\}$-formulas, is an intuitionistic theorem if and only if it is a classical theorem[9], and (20) is even sufficient for **LP** correctness. Then as **LP** proof nets satisfy Cut-elimination, there is the following decision procedure for determining **LP** theoremhood by searching for Cut-free proof nets. Given a sequent $A_1, \ldots, A_n \Rightarrow A$ construct the proof frame with conclusions $A_1{}^\bullet, \ldots, A_n{}^\bullet, A^\circ$ comprising the sequence of signed formula trees given by the following recursive unfolding:

$$\frac{A^\bullet \quad B^\bullet}{A \otimes B^\bullet}\wp \qquad \frac{A^\circ \quad B^\circ}{A \otimes B^\circ}\otimes \qquad \frac{A^\circ \quad B^\bullet}{A \multimap B^\bullet}\otimes \qquad \frac{A^\bullet \quad B^\circ}{A \multimap B^\circ}\wp \tag{21}$$

Then test whether there is some proof structure that can be built by putting axiom links on the proof frame, which satisfies the Acyclicity condition.

Since **LP** is a restriction of intuitionistic logic, each proof can be read as an intuitionistic proof. The intuitionistic natural deduction proof, encoded as a linear term of λ-calculus with function and pair types, is extracted from a proof net as follows (cf. de Groote and Retoré 1996). First, one associates distinct variables with each output implication link and distinct constants with each input conclusion. Then, one starts travelling upwards at the unique output conclusion: going up into an output division (i.e. implication) link, λ-abstract over the associated variable the result of going up into the output premise; going up into an output product (i.e. conjunction) link, pair the result of going up into the premise for the first subformula with the result of going up into the premise for the second subformula; going up into one premise of an id link, go down into the other premise; going down into one conclusion of a Cut link, go up into the other conclusion; going down into an input division link, functionally apply the result of going down into its conclusion to the result of going up into the other premise; going down into the premise for the first subformula of an input product link, take the first projection of the result of going down into its conclusion; going down into the premise for the second subformula of an input product link, take the second projection of the result of going down into its conclusion; going down into an output division link, return the associated variable; and going down into an input root, return the associated constant. This extraction procedure is the same for all categorial products and divisions.

[9] Attributed by Bellin and Scott (1994) to J. van de Wiele; cf. also Lamarche (1994, 1995).

3 Lambek calculus and extensions

The (associative) Lambek calculus **L**, a multiplicative fragment of intuitionistic non-commutative linear logic,[10] has the formulas and sequent calculus of (1) and (4). When we read $\bullet$ as $\otimes$ and both $A\backslash B$ and B/A as $A \multimap B$, each rule is **LP**-derivable, so every theorem of **L** is also a theorem of **LP** when read in this way, and for a proof structure to be a proof net it is necessary that there be no vicious circle in the sense before. But this is no longer sufficient since in the absence of permutation, order must be taken into account.

Roorda (1991) addresses the ordering component in terms of a directional balance by specifying that in output logical links the subformulas of the conclusion appear with their left/right ordering switched in the premises. Then proof structures are required to be *planar*, and a (planar) proof structure is a proof net if and only if it satisfies acyclicity in the usual manner.[11] Here, however, we will be concerned to admit some flexibility in ordering, and we consider correctness conditions based on unifiability (Morrill 1996). We will maintain the order switching of output unfolding, but do not require proof structures to be planar. Rather, we describe *resolution conditions* corresponding to relational interpretation.

In order to construe **L** in a manner uniform with subsequent extensions, consider the interpretation of **L** formulas that results from combining the algebraic and relational models. Interpretation takes place with respect to a semigroup $(L, +)$ and a set V. Formulas are interpreted as subsets of $L \times V \times V$. Given an interpretation $[\![P]\!]$ for each atom P, each category formula A receives an interpretation $[\![A]\!]$ thus:

$$\begin{array}{lcl} [\![A\backslash B]\!] & = & \{\langle s, v_2, v_3\rangle |\ \forall s', v_1, \langle s', v_1, v_2\rangle \in [\![A]\!] \rightarrow \langle s'+s, v_1, v_3\rangle \in [\![B]\!]\} \\ [\![B/A]\!] & = & \{\langle s, v_1, v_2\rangle |\ \forall s', v_3, \langle s', v_2, v_3\rangle \in [\![A]\!] \rightarrow \langle s+s', v_1, v_3\rangle \in [\![B]\!]\} \\ [\![A\bullet B]\!] & = & \{\langle s_1+s_2, v_1, v_3\rangle |\ \exists v_2, \langle s_1, v_1, v_2\rangle \in [\![A]\!]\ \&\ \langle s_2, v_2, v_3\rangle \in [\![B]\!]\} \end{array} \tag{22}$$

The expansion of proof links will reflect the binary relational quantificational structure. Each node labelled by a formula will have two incident dashed edges referred to as its start and its end parameter edges. For an input formula the start comes on the left and the end comes on the right; for an output formula this is reversed:

$$\text{start } A^{\bullet} \text{ end} \qquad \text{end } A^{\circ} \text{ start} \tag{23}$$

These parameter edges are connected to quantifiers in the expanded proof structures which bind the parameters of formulas regarded as binary predicates. The proof links of **L** are given in figures 5 and 6. The expansions are a systematic reflection of the propositional and quantificational structure of the interpretation (negated in the input case). Just as before, an **L** signed formula

[10] Again, we gloss over minor differences regarding whether or not empty antecedents are admitted.

[11] Planarity, reflecting more fundamentally correct bracketing, only works for Cut-free proof nets; for the general case see Abrusci (1995).

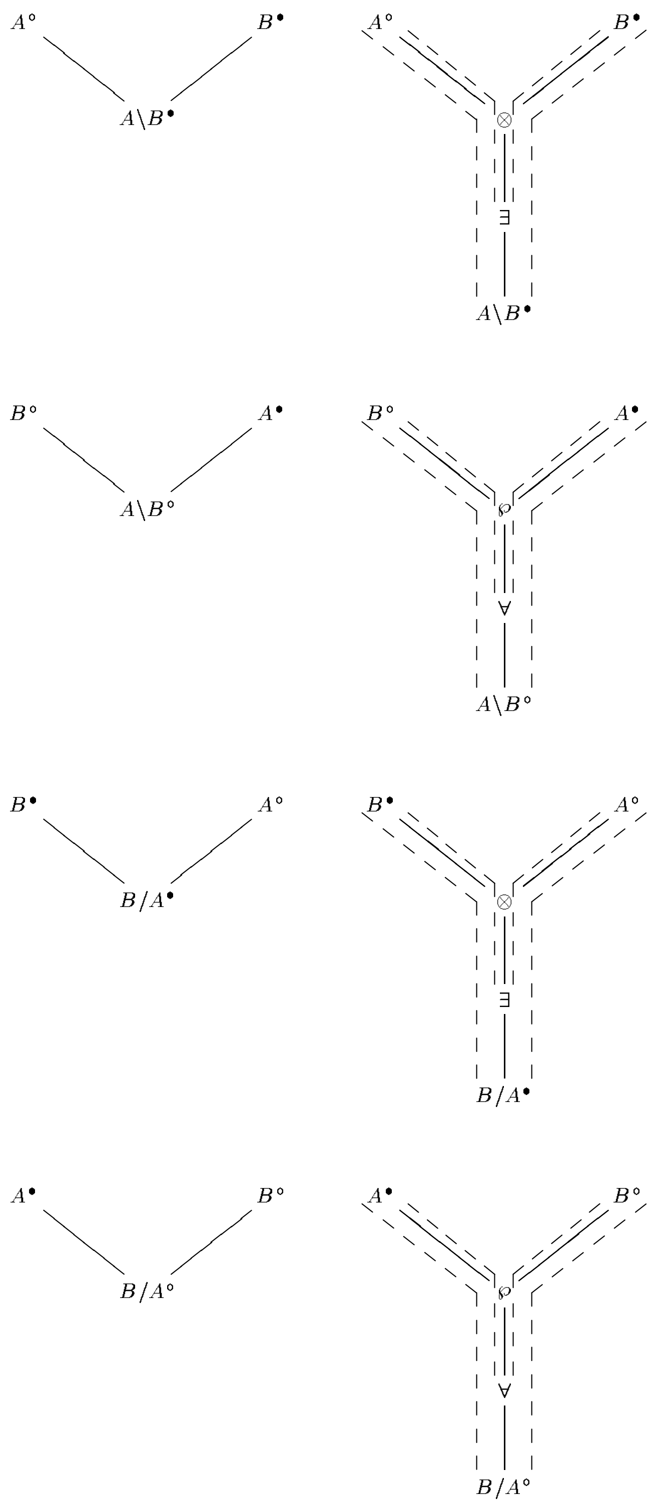

Figure 5: Proof links of **L** and their expansions, I

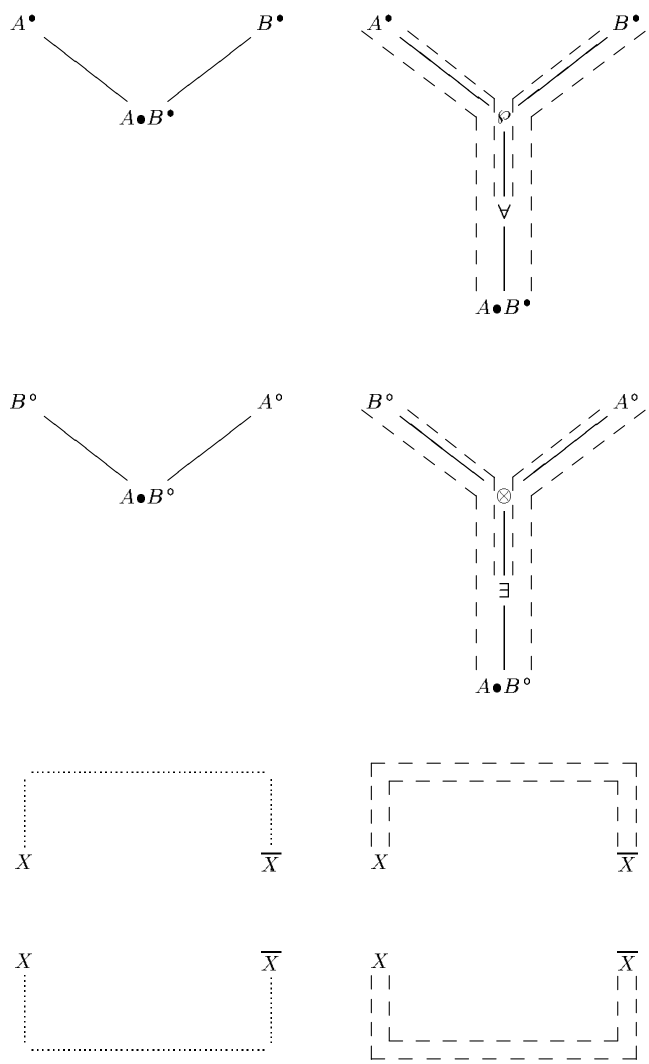

Figure 6: Proof links of **L** and their expansions, II

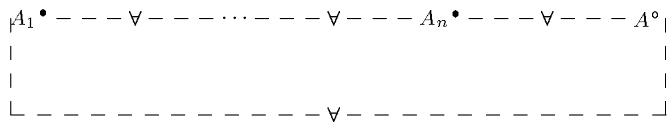

Figure 7: Parameter expansion of conclusions

tree is a finite tree with atomic (signed) leaves each local tree of which is an **L** logical link. An **L** proof frame is a finite sequence of **L** signed formula trees and an **L** proof structure is the result of connecting complementary leaves with axiom links and complementary roots with Cut links in such a way that each leaf is connected to exactly one other, each root is connected to at most one other, and there is exactly one conclusion of output polarity. An *expanded* proof structure has the annotation of figure 7 on its conclusions $A_1{}^\bullet, \ldots, A_n{}^\bullet, A^\circ$ in the proof frame. This corresponds to the meaning of a sequent $A_1, \ldots, A_n \Rightarrow A$ with respect to binary relational interpretation: for all $v_0, \ldots, v_n \in V$, if $\langle v_{i-1}, v_i \rangle \in [\![A_i]\!]$, $1 \leq i \leq n$ then $\langle v_0, v_n \rangle \in [\![A]\!]$.

The complementary atoms linked by axioms in proof nets can be seen as the counterparts of the complementary pairs in a (non-clausal) resolution proof. This gives rise to the following condition for correctness on parameter paths in proof structures. First, to each existential quantifier we associate a new free variable, and to each universal quantifier we associate a Skolem term; note that polarities are the opposite of what is usual since resolution proofs are refutations, i.e. negate succedent formulas, whereas proof nets negate antecedent formulas. A Skolem term is a new constant in the case that the universal quantifier is not descended from any existential; otherwise it comprises a new n-place function symbol with arguments the n variables of the n ancestor existentials. Each axiom link requires the start and end parameters of its two atoms to be unified, and for a proof structure to be correct as a whole, the unification problem defined by its axiom linkings must be solvable.

We can show that the quantificational structure of a proof net is correct by exhibiting a unifier, but we do not need to insist on such a constructive proof of unifiability: the condition only requires than such a unifier exists. Unification fails in two cases, *clash*: if we attempt to match a constant to a different constant, or to match a structured term to a structured term with a different function symbol, or to a constant, or *occurrence*: if we attempt to match a variable to a structured term containing this variable. Let us define a $\forall\exists$-cycle as a cyclic path alternating between universals and dominating existentials as shown in figure 8 (the directionality, shown explicitly, is from descendent to ancestor); thus we can test satisfaction of unifiability of expanded proof structures by the following:

Resolution conditions. No two distinct universals are connected by parameter edges (clash check) and there is no $\forall\exists$-cycle (occurrence check). (24)

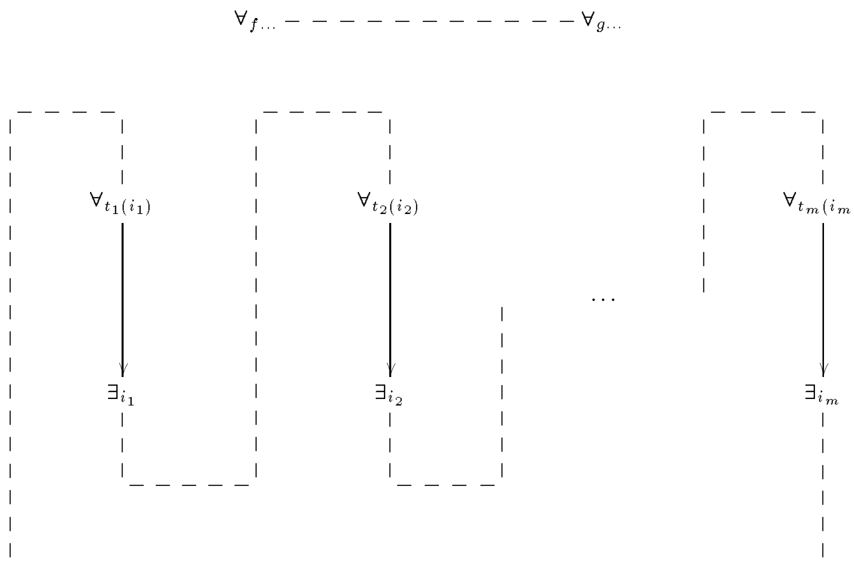

Figure 8: Clash check and occurrence check violations

The idea is that the clash check and occurrence check together take the place of planarity and acyclicity requirements (in particular the notion of $\forall\exists$-cycle is highly similar to that of AE-cycle in Lecomte and Retoré 1995, though the rationale is entirely different) so that we can show that a proof structure is incorrect by identifying either a clash check violation or an occurrence check violation.

That satisfaction of the Resolution conditions is *necessary* is immediate for, if a proof structure is correct, it must be correct as a (non-clausal) resolution proof of classical logic. The question arises as to whether satisfaction of the resolution condition is also *sufficient*. If it is not one must do more to show correctness than just assure solvability of the unification problem defined by a proof structure, but we conjecture that for at least product-free **L** a proof structure is a proof net if and only if it satisfies the Resolution conditions. Assuming this, and given Cut-elimination for **L** proof nets, there is the following algorithm for deciding the validity of an **L** sequent $A_1, \ldots, A_n \Rightarrow A$. Construct the proof frame with conclusions ${A_1}^\bullet, \ldots, {A_n}^\bullet, A^\circ$ comprising the sequence of signed formula trees given by the following recursive unfolding:

$$\frac{A^\circ \quad B^\bullet}{A\backslash B^\bullet}\otimes \qquad \frac{B^\circ \quad A^\bullet}{A\backslash B^\circ}\wp \qquad \frac{B^\bullet \quad A^\circ}{B/A^\bullet}\otimes \qquad \frac{A^\bullet \quad B^\circ}{B/A^\circ}\wp \tag{25}$$

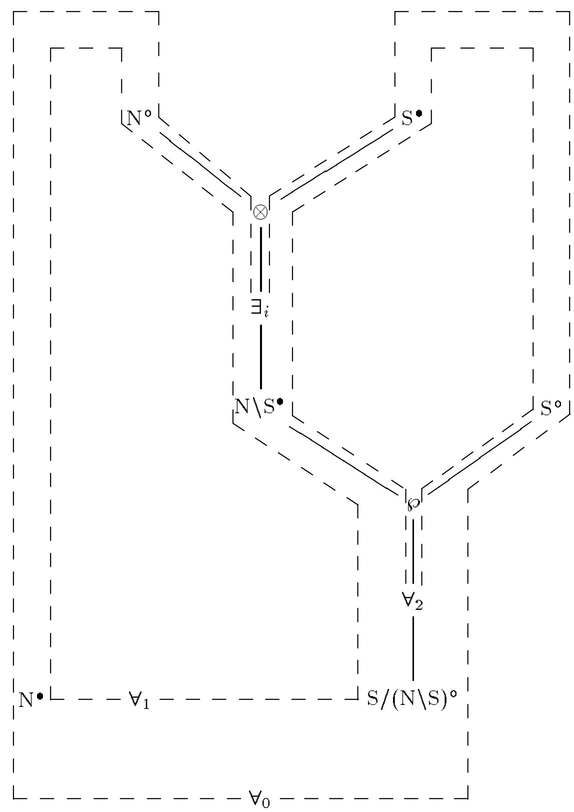

Figure 9: Proof net for lifting

$$\frac{A^\bullet \qquad B^\bullet}{A\bullet B^\bullet}\wp \qquad \frac{B^\circ \qquad A^\circ}{A\bullet B^\circ}\otimes$$

Then test whether some proof structure can be built be adding axiom links which complies with the resolution condition.

In figure 9 we give an expanded proof net for the valid sequent N $\Rightarrow$ S/(N\S), a lifting theorem. It defines the unification problem $\{0 = i, 1 = 1, i = 0, 2 = 2\}$ which has solution $\{0/i\}$. In figure 10 we give an expanded proof structure for the invalid lowering sequent S/(N\S) $\Rightarrow$ N; there is a clash check violation on the outer parameter edges. Figure 11 shows a partial proof structure for the invalid sequent $\Rightarrow$ (S\(N\N))•S, in which the only parameter edge explicitly marked participates in a $\forall\exists$-cycle completed by the two directed edges.

A categorial derivation defines a semantic construction, expressed by the typed λ-term extracted as for **LP** proofs, giving the semantics of the expression derived in terms of the semantics of its lexical signs. In the lifting example of

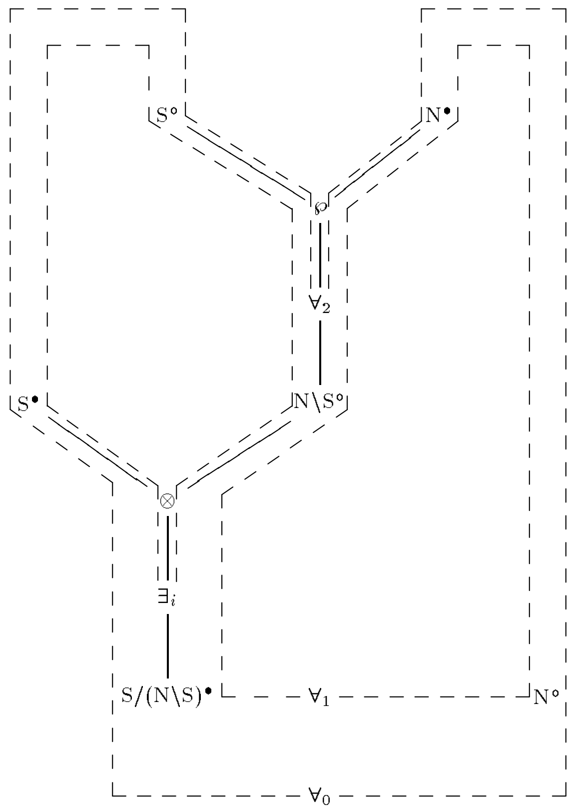

Figure 10: Proof structure for lowering, with clash check violation

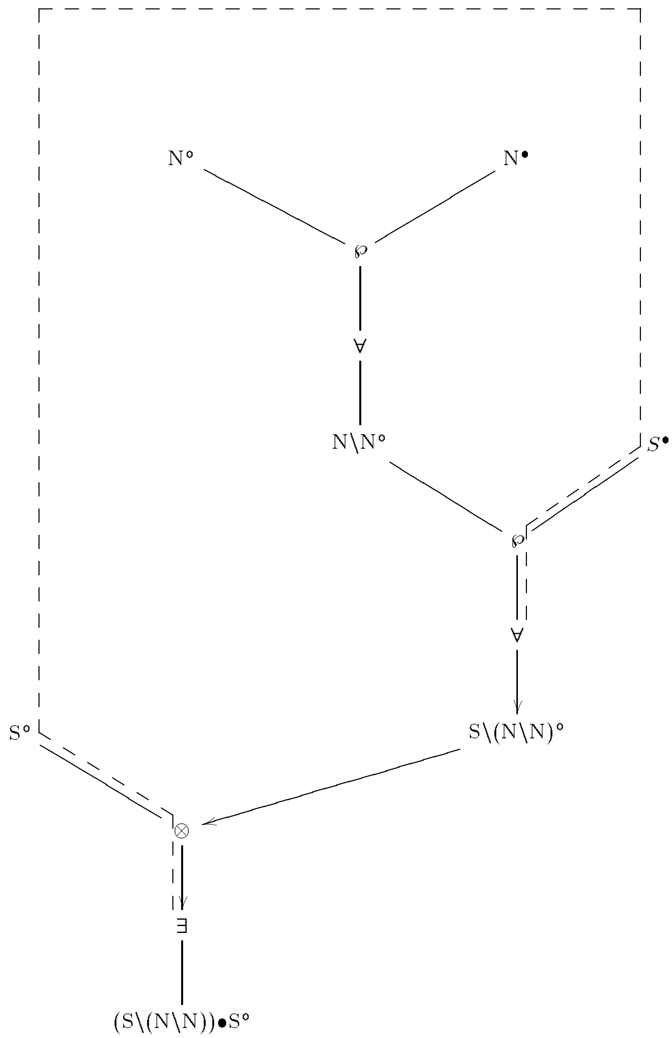

Figure 11: Partial proof structure with ∀∃-cycle

figure 9, the semantic traversal yields the term $\lambda x(x\ a)$ where a is the semantics associated with the $\mathrm{N}^\bullet$ conclusion.

A categorial derivation also defines a prosodic construction giving the word order of the composite expression in terms of its lexical expressions. This is recovered from the parameter edges reflecting relational interpretation by the following *prosodic trip*: begin travelling up at the start parameter of the unique output conclusion; this arrives at the start parameter of the first lexical expression making up the composite; continue travelling up at the end parameter of this input conclusion; this arrives at the start parameter of the second lexical expression making up the composite; continue travelling up at the end parameter of this input conclusion, and so on; the process ends by returning to the end parameter of the unique output conclusion. In the lifting example of figure 9, the prosodic traversal begins at the start parameter of the output conclusion and follows the right outermost parameter edge round to the existential and the left outermost parameter edge round to the start parameter of $\mathrm{N}^\bullet$; travelling up at the end parameter of $\mathrm{N}^\bullet$ we return down to the end parameter of the output conclusion. In fact we write proof nets on the page in such a way that in general this traversal visits the input conclusions in left-to-right order.

3.1 Medial divisor

Medial division involves a kind of non-commutativity. In the combined models the medial divisor $\uparrow_e$ is interpreted:

$$\begin{array}{lcl} [\![B{\uparrow_e}A]\!] & = & \{\langle s_1{+}s_2, v_1, v_2\rangle |\ \exists v \forall s, \langle s, v, v\rangle \in [\![A]\!] \rightarrow \\ & & \quad \langle s_1{+}s{+}s_2, v_1, v_2\rangle \in [\![B]\!]\} \end{array} \tag{26}$$

Proof links for the medial divisor are shown in figure 12. Again the expansion is a systematic reflection of the propositional and relational quantificational structure of the interpretation. For reasons of uniformity we continue the convention of switching the order of subformulas in output links, but the medial divisor can give rise to non-planar proof nets.

In figure 13 we give the expanded proof net (abbreviating N\S to VP) for the medial extraction (5) on the basis of the following assignments:

$$\begin{array}{lcl} \textbf{that} & - & \textit{intersect} \\ & : & \mathrm{R}/(\mathrm{S}{\uparrow_e}\mathrm{N}) \\ \textbf{John} & - & j \\ & : & \mathrm{N} \\ \textbf{gave} & - & \textit{give} \\ & : & ((\mathrm{N}\backslash\mathrm{S})/\mathrm{PP})/\mathrm{N} \\ \textbf{to+Mary} & - & m \\ & : & \mathrm{PP} \end{array} \tag{27}$$

The unification problem defined (omitting repetitions) is $\{0 = 0, i = 4, j = 3, j = k, i = l, 1 = m, 2 = 2, l = 4, k = 3\}$ which has solution $\{4/i, 3/j, 3/k, 4/l, 1/m\}$.

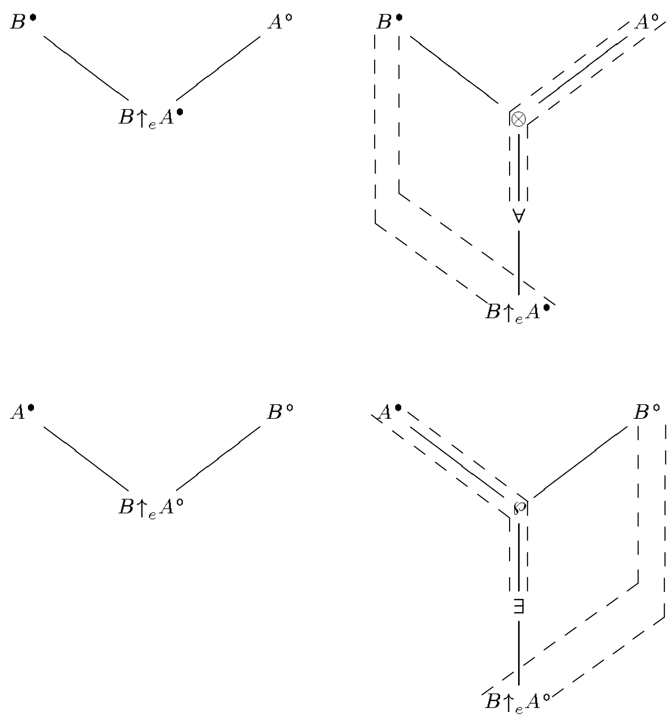

Figure 12: Proof links for the medial divisor and their expansions

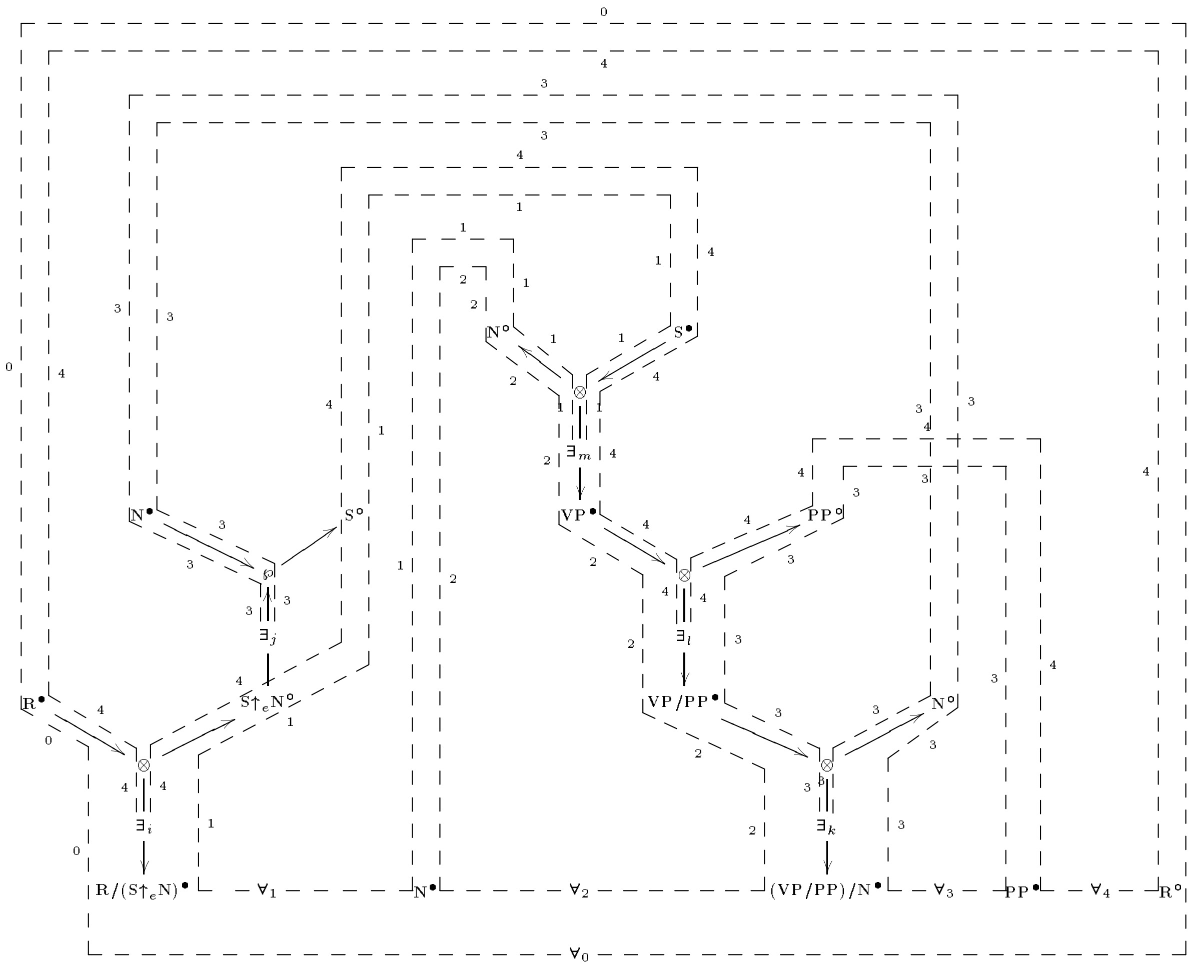

Figure 13: Proof net for '(the dog) that John gave to Mary' via the medial divisor

The edges of successive prosodic traversal are labelled $0, 1, 2, 3, 4$: beginning travelling up at the start of the unique output conclusion, five 0-lines lead to the start of the type for 'that', which is the first word; going up at the end of this type, twelve 1-lines lead to the start of the type for 'John', and so on, yielding in order the words 'gave' and 'to Mary'; hence the prosodic form of the sign is **that+John+gave+to+Mary**.

Arrows mark the directions of semantic traversal; starting with the axiom link going from the outermost right to the outermost left, successive stages of semantic form extraction are as follows:

$$
\begin{array}{l}
(\cdot\ \cdot) \\
(\mathit{intersect}\ \cdot) \\
(\mathit{intersect}\ \lambda x_j \cdot) \\
(\mathit{intersect}\ \lambda x_j(\cdot\ \cdot)) \\
(\mathit{intersect}\ \lambda x_j(\cdot\ \cdot\ j)) \\
(\mathit{intersect}\ \lambda x_j(\cdot\ \cdot\ m\ j)) \\
(\mathit{intersect}\ \lambda x_j(\mathit{give}\ x_j\ m\ j))
\end{array}
\tag{28}
$$

Hence the semantic form of the sign is $(\mathit{intersect}\ \lambda x_j(\mathit{give}\ x_j\ m\ j))$.

In figure 14 we give the expanded proof net (abbreviating (N\S)/VP to XVP) for the obligatory extraction (8a), assuming (additional) type assignments:

$$
\begin{array}{lll}
\textbf{assures} & - & \mathit{assure} \\
 & : & (((N\backslash S)/VP)\uparrow_e N)/N \\
\textbf{Mary} & - & m \\
 & : & N \\
\textbf{to+be+reliable} & - & \mathit{reliable} \\
 & : & VP
\end{array}
\tag{29}
$$

The unification problem defined (omitting repetitions and equations of identical terms) is $\{i = 5, j = 6(k), i = l, 1 = m, l = 5, k = 4\}$ which has solution $\{5/i, 5/l, 4/k, 6(4)/j, 1/m\}$. The sign generated has prosodic form **that+John+-assures+Mary+to+be+reliable** and the semantic form extracted is $(\mathit{intersect}$-$\lambda x_j(\mathit{assure}\ m\ x_j\ \mathit{reliable}\ j)$.

A partial proof structure for the ungrammatical (8b) is given in figure 15; the only parameter edge explicitly marked mediates a clash between two universals.

3.2 In situ binder

In the combined models, the in situ binder Q is interpreted:

$$
\begin{array}{rcl}
[\![Q(A, B, C)]\!] & = & \{\langle s, v_2, v_3\rangle \mid \forall s_1, s_3, v_1, v_4, \\
 & & \quad [\forall s_2, \langle s_2, v_2, v_3\rangle \in [\![A]\!] \rightarrow \langle s_1{+}s_2{+}s_3, v_1, v_4\rangle \in [\![B]\!]] \rightarrow \\
 & & \quad \langle s_1{+}s{+}s_3, v_1, v_4\rangle \in [\![C]\!]\}
\end{array}
\tag{30}
$$

The proof links for the in situ binder are shown in figure 16. Again, the

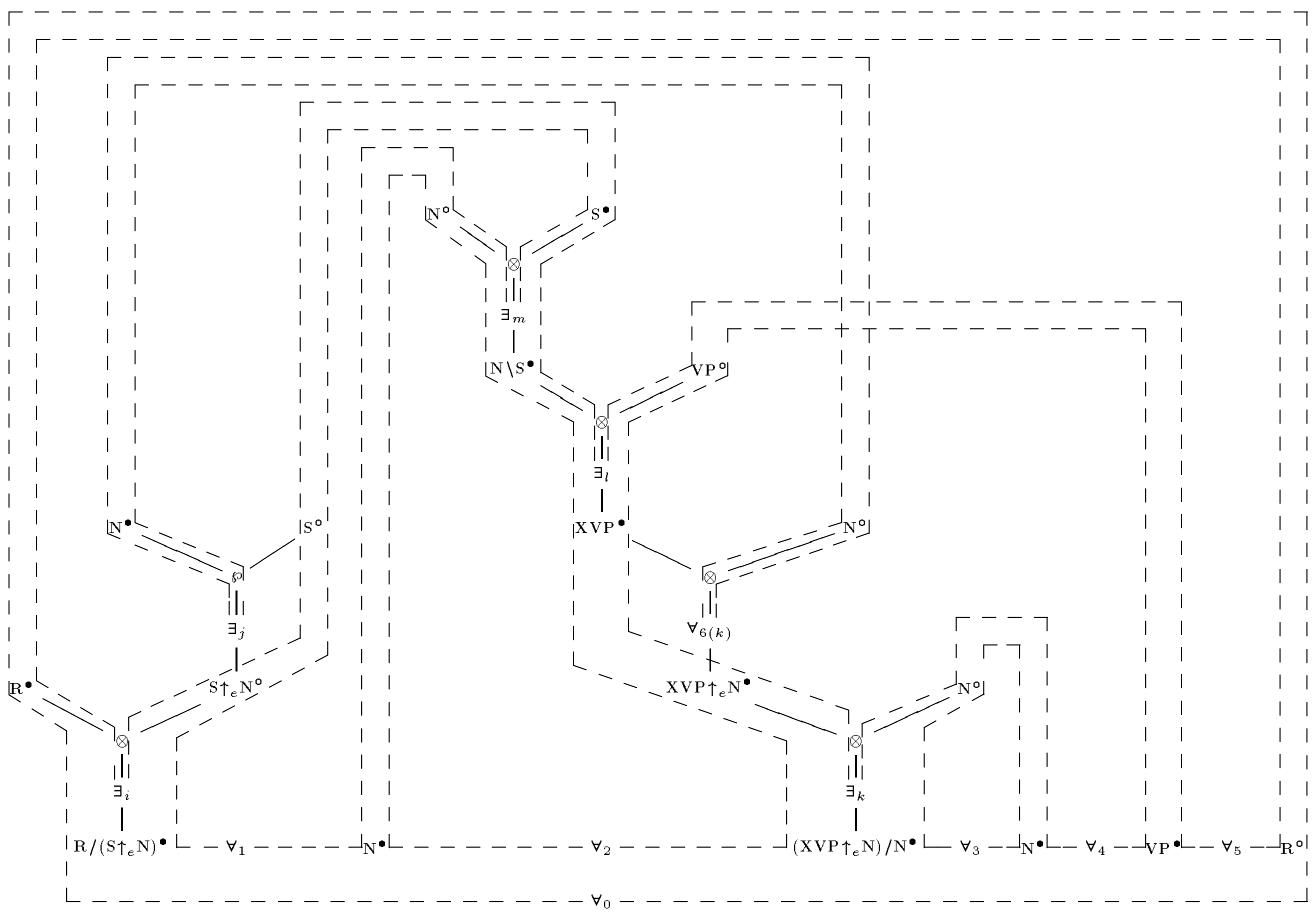

Figure 14: Proof net for '(the man) that John assured Mary to be reliable' via the medial divisor

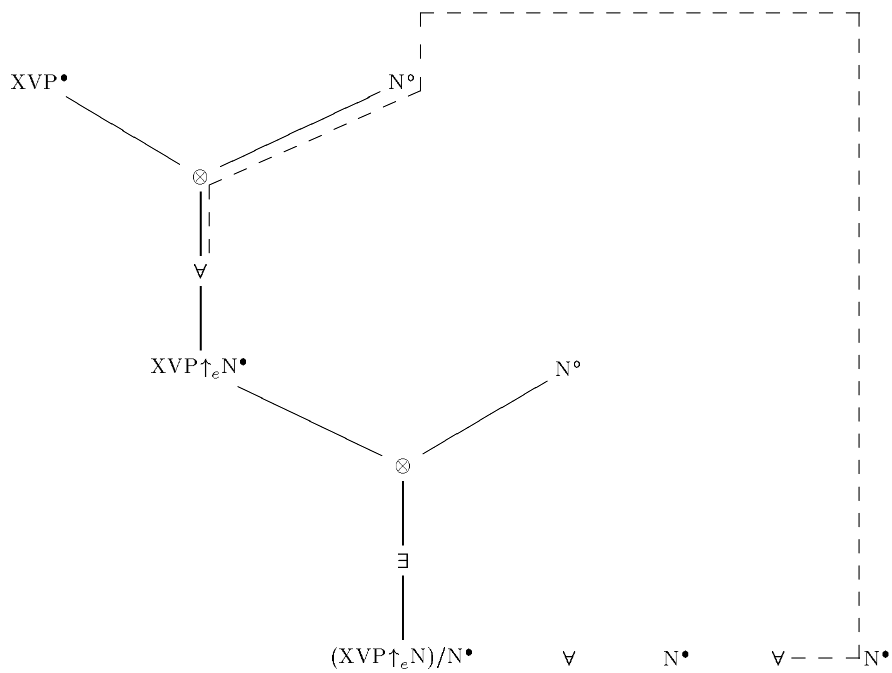

Figure 15: Partial proof structure for 'John assured Mary Bill to be reliable' via the medial divisor, with clash check violation

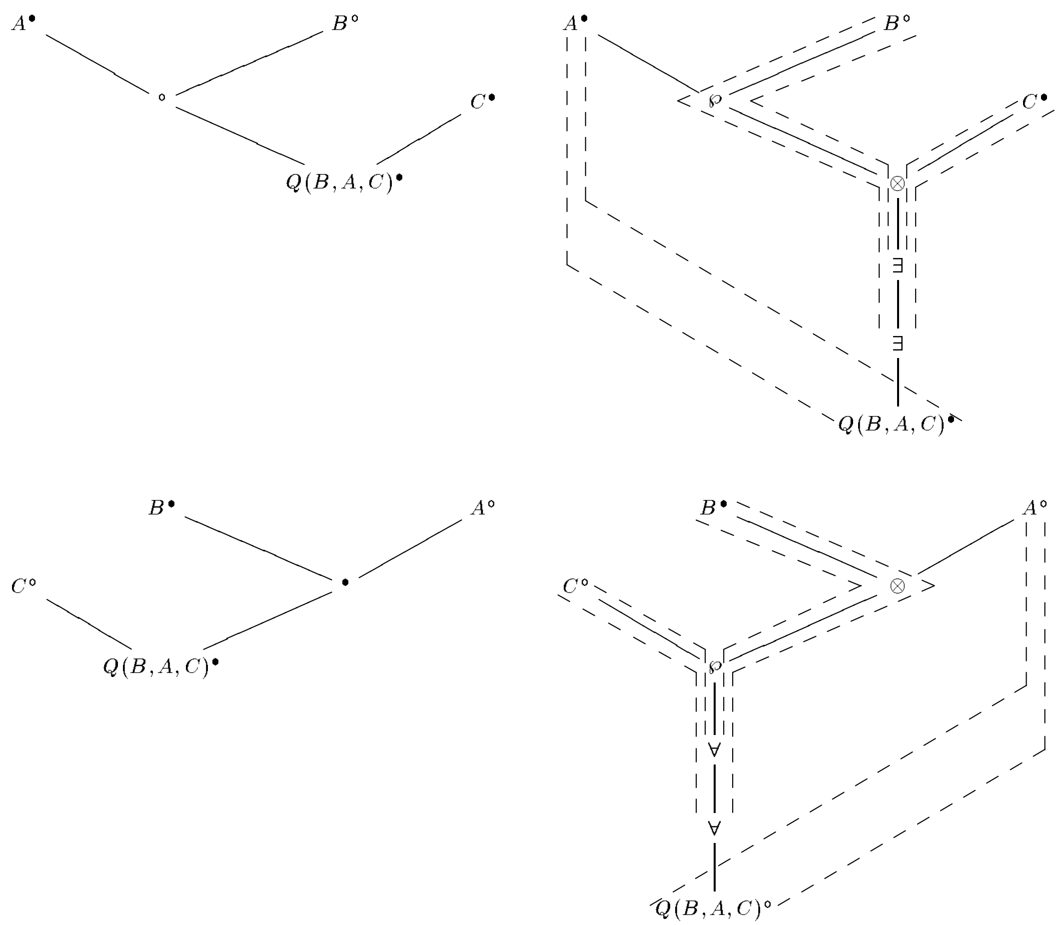

Figure 16: Proof links for the in situ binder and their expansions

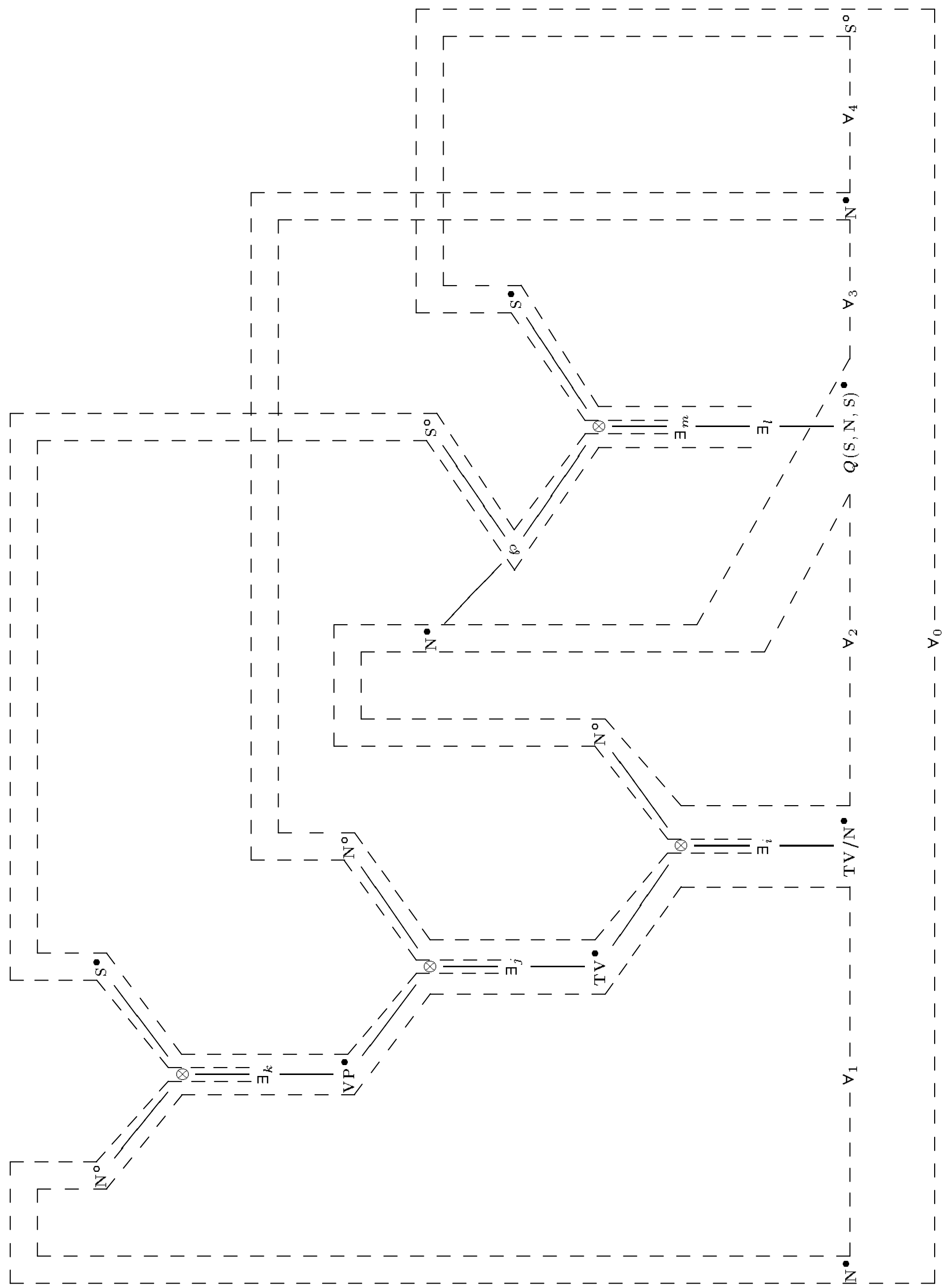

Figure 17: Proof net for ‘John bought someone Fido’ via the in situ binder

expansions are a systematic reflection of the interpretation, and for uniformity orderings of polar opposites are mirror images.

In figure 17 we give the (expanded) proof net (abbreviating (N\S)/N to TV) for the in situ binding (9a) assuming type assignments (31).

$$\begin{array}{lcl} \textbf{bought} & - & \mathit{buy} \\ & : & ((\mathrm{N}\backslash\mathrm{S})/\mathrm{N})/\mathrm{N} \\ \textbf{someone} & - & \lambda x \exists y[(\mathit{person}\ y) \wedge (x\ y)] \\ & : & Q(\mathrm{S}, \mathrm{N}, \mathrm{S}) \\ \textbf{Fido} & - & f \\ & : & \mathrm{N} \end{array} \tag{31}$$

The unification problem defined by the linking is $\{0 = k, k = m, j = l, j = 4, i = 3, m = 0, l = 4\}$ which has solution $\{0/k, 0/m, 4/l, 4/j, 3/i\}$. The result of semantic traversal is $(\mathit{someone}\ \lambda x(\mathit{buy}\ x\ f\ j))$ which on substitution of lexical semantics simplifies to $\exists y[(\mathit{person}\ y) \wedge (\mathit{buy}\ y\ f\ j)]$.

The reader may check the proof net constructions showing that the assignment (32) yields the semantics $(\mathit{buy}\ j\ f\ j)$ for (9b), and showing (12).

$$\begin{array}{lcl} \textbf{himself} & - & \lambda x \lambda y(x\ y\ y) \\ & : & Q(\mathrm{N}\backslash\mathrm{S}, \mathrm{N}, \mathrm{N}\backslash\mathrm{S}) \end{array} \tag{32}$$

3.3 Discontinuity calculus

Much work has gone into development of calculi which do for discontinuity what the Lambek calculus does for continuity (e.g. Solias 1992, 1996; Moortgat 1996a, 1996b; Morrill and Solias 1993; Oehrle 1994; Calcagno 1995; Hendriks 1995; Morrill 1995). For the present purposes it is convenient to consider just continuous strings and strings with exactly one point of discontinuity (Versmissen 1991), and to explicitly regiment the formation and interpretation of types according to these sorts (Morrill and Merenciano 1996): $\mathcal{F}$, interpreted algebraically as subsets of L (and relationally as binary relations), and $\mathcal{F}^2$ interpreted algebraically as subsets of L^2 (and relationally as quaternary relations). Our definition of category formulas becomes (33).

$$\begin{array}{lcl} \mathcal{F} & ::= & \mathcal{A} \mid \mathcal{F}\bullet\mathcal{F} \mid \mathcal{F}\backslash\mathcal{F} \mid \mathcal{F}/\mathcal{F} \mid \mathcal{F}^2\odot\mathcal{F} \mid \mathcal{F}^2\downarrow\mathcal{F} \\ \mathcal{F}^2 & ::= & \mathcal{F}\uparrow\mathcal{F} \end{array} \tag{33}$$

The discontinuous product operator $\odot$ and the divisors $\downarrow$ ("infix") and $\uparrow$ ("extract") are interpreted by "residuation" with respect to an interpolation adjunction W of functionality $L^2, L \to L$, defined by $\langle s_1, s_2\rangle W s = s_1{+}s{+}s_2$, in exactly the same way that the continuity operators are interpreted by residuation with respect to a concatenation adjunction $+$ of functionality $L, L \to L$. In the combined models we have the following:

$$
\begin{array}{rcl}
[\![A{\downarrow}B]\!] & = & \{\langle s, v_2, v_3\rangle |\ \forall s_1, s_2, v_1, v_4, \langle s_1, s_2, v_1, v_2, v_3, v_4\rangle \in [\![A]\!] \rightarrow \\
& & \qquad\qquad \langle s_1{+}s{+}s_2, v_1, v_4\rangle \in [\![B]\!]\} \\
[\![B{\uparrow}A]\!] & = & \{\langle s_1, s_2, v_1, v_2, v_3, v_4\rangle |\ \langle s, v_2, v_3\rangle \in [\![A]\!] \rightarrow \\
& & \qquad\qquad \langle s_1{+}s{+}s_2, v_1, v_4\rangle \in [\![B]\!]\} \\
[\![A{\odot}B]\!] & = & \{\langle s_1{+}s{+}s_2, v_1, v_4\rangle |\ \exists v_2, v_3, \langle s_1, s_2, v_1, v_2, v_3, v_4\rangle \in [\![A]\!]\ \& \\
& & \qquad\qquad \langle s, v_2, v_3\rangle \in [\![B]\!]\}
\end{array}
\tag{34}
$$

We have, then, $Q(B, A, C) = (B{\uparrow}A){\downarrow}C$.[12]

The two incident parameter edges of the binary relational predication of formulas of sort string are notated in expanded proof nets according to (23); the four incident parameter edges of the quaternary relational predication of formulas of sort split string are notated in expanded proof nets as in (35):

$$
\mathrm{start}_1\ \mathrm{end}_2\ A^{\bullet}\ \mathrm{start}_2\ \mathrm{end}_1 \qquad \mathrm{end}_1\ \mathrm{start}_2\ A^{\circ}\ \mathrm{end}_2\ \mathrm{start}_1 \tag{35}
$$

The subscripts refer to the first (left) and second (right) string components of a split string; note that, again, the input and output orderings are mirror-images, which promotes visual symmetry. The expanded proof links for the discontinuity connectives are given in figure 18.

Prosodic traversal visits split string conclusions *twice*. On the first occasion the parameter start_1 of the first component of a split string input conclusion is visited, and travel continues up at the parameter end_1; on the second occasion the parameter start_2 of the second component is visited, and travel continues up at end_2; the material visited meanwhile is interpolated between the two components. Thus the result of prosodic extraction for figure 19 is **John+gave+Mary+the+cold+shoulder**. The result of semantic extraction is (*shun m j*).

4 Conclusion

We have described a method indicating the possibility of a systematic correspondence between combined algebraic and relational interpretation of categorial logic, and the form of proof structures and paths in proof nets. This induces a notion of prosodic traversal, and Resolution conditions which together with acyclicity, or even alone, may be sufficient to define correctness. It is our consideration that the resolution conditions may be sufficient for product-free **L**, and may be sufficient together with acyclicity for wider varieties of relational interpretation including those described here. Technical analysis is due in this respect; even otherwise we hope that it may be positive to delimit the possibilities of the method.

[12] And $B{\uparrow_e}A = (B{\uparrow}A){\odot}I$ where I is the product unit; then the semantic types are not quite identical, but there is a $1{-}1$ correspondence between elements of D and elements of $D{\times}\{1\}$ (and $\{1\}{\rightarrow}D$).

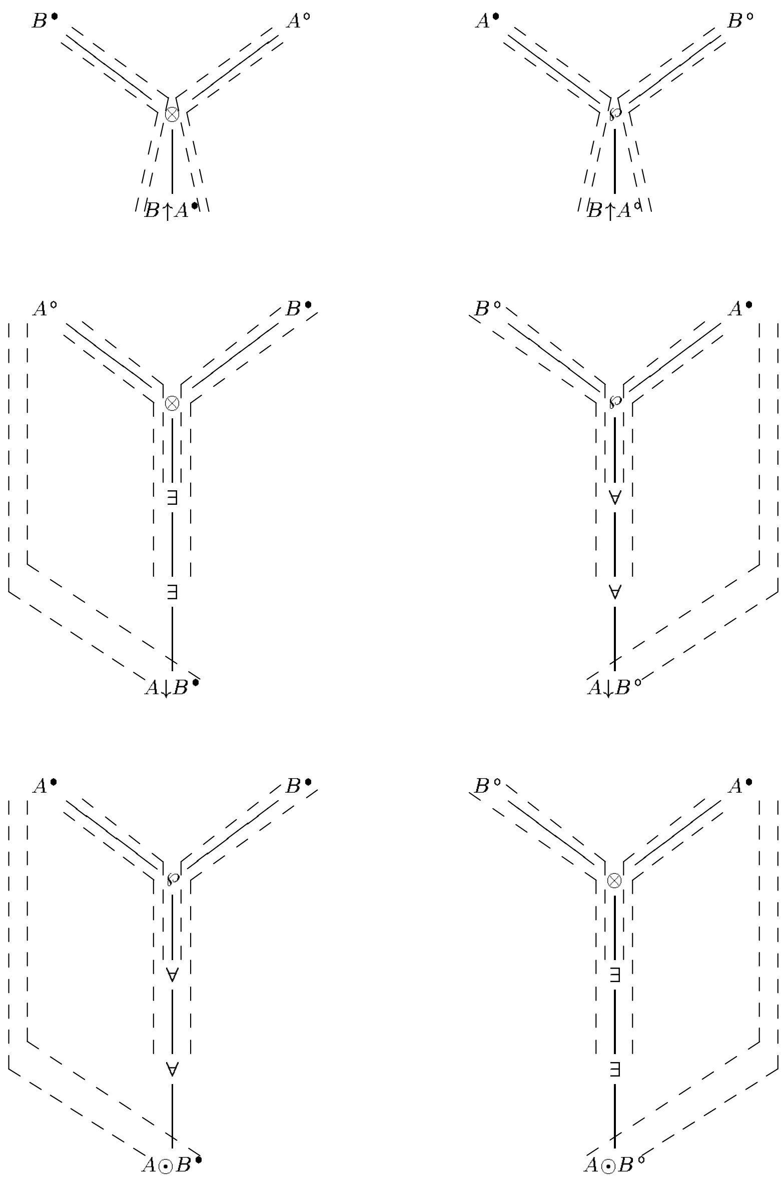

Figure 18: Expanded proof links for the discontinuity connectives

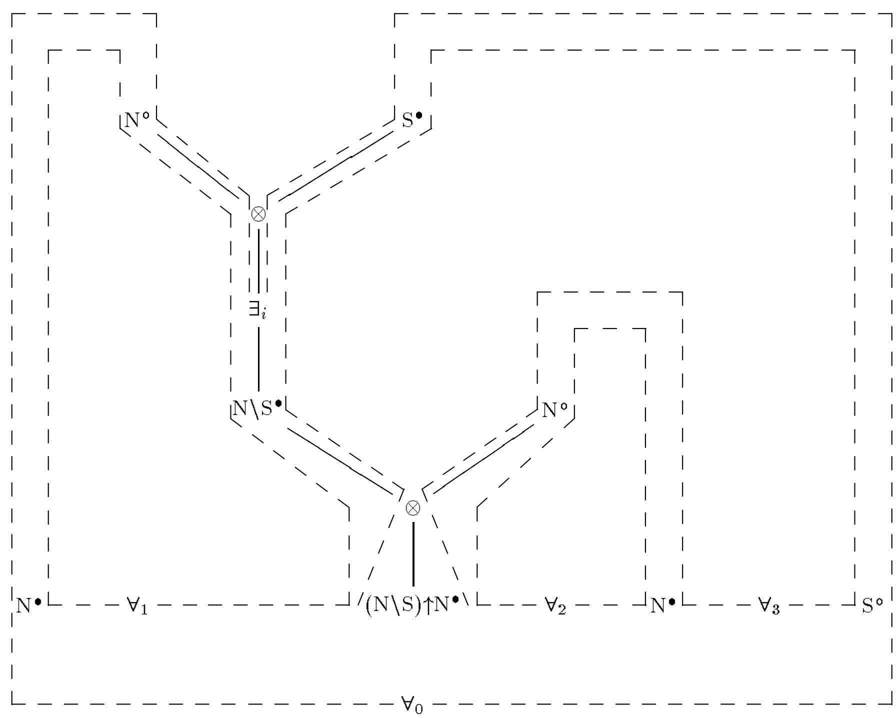

Figure 19: Proof net for ‘John gave Mary the cold shoulder’ via a wrapping functor

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