Logical Computational Linguistics

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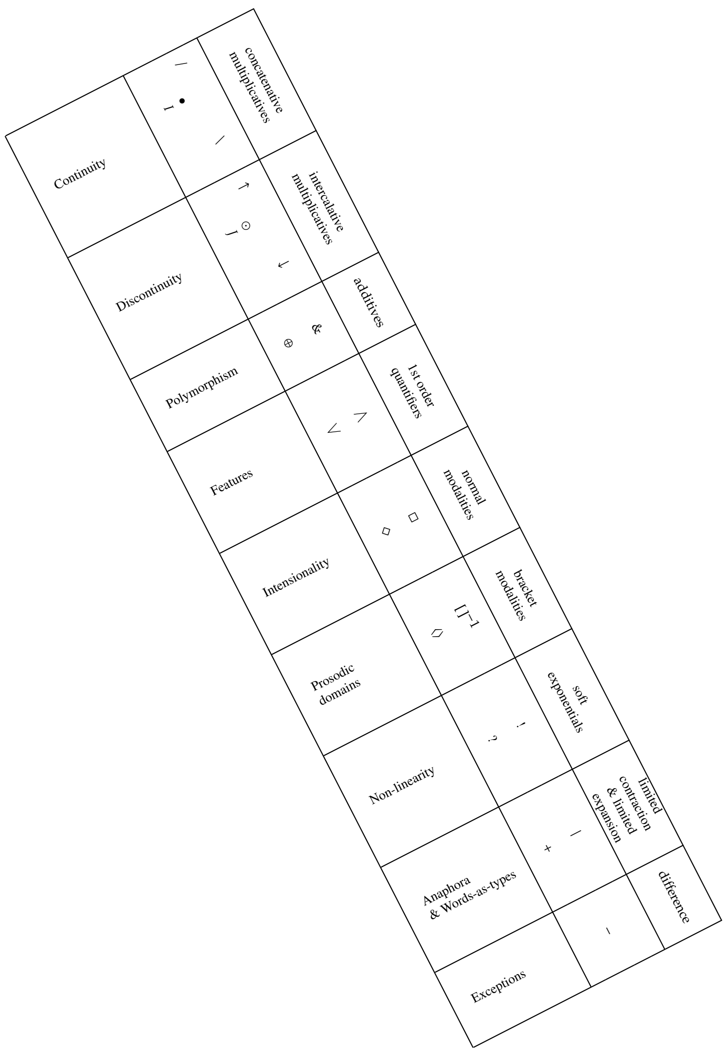
17th April 2026[3]



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Caveat lector

"This study deals with syntactic structure both in the broad sense (as opposed to semantics) and the narrow sense (as opposed to phonemics and morphology). It forms part of an attempt to construct a formalized general theory of linguistic structure and to explore the foundations of such a theory. The search for rigorous formulation in linguistics has a much more serious motivation than mere concern for logical niceties or the desire to purify well-established methods of linguistic analysis. Precisely constructed models for linguistic structure can play an important role, both negative and positive, in the process of discovery itself. By pushing a precise but inadequate formulation to an unacceptable conclusion, we can often expose the exact source of this inadequacy and, consequently, gain a deeper understanding of the linguistic data. More positively, a formalized theory may automatically provide solutions for many problems other than those for which it was explicitly designed. Obscure and intuition-bound notions can neither lead to absurd conclusions nor provide new and correct ones, and hence they fail to be useful in two important respects. I think that some of those linguists who have questioned the value of precise and technical development of linguistic theory may have failed to recognize the productive potential in the method of rigorously stating a proposed theory and applying it strictly to linguistic material with no attempt to avoid unacceptable conclusions by ad hoc adjustments or loose formulation. The results reported below were obtained by a conscious attempt to follow this course systematically. Since this fact may be obscured by the informality of the presentation, it is important to emphasize it here."

Noam Chomsky (1957) *Syntactic Structures*, Preface

"I reject the contention that an important theoretical difference exists between formal and natural languages. On the other hand, I do not regard as successful the formal treatments of natural languages attempted by certain contemporary linguists. Like Donald Davidson[[1]] I regard the construction of a theory of truth—or rather, of the more general notion of truth under an arbitrary interpretation—as the basic goal of serious syntax and semantics; and the developments emanating from the Massachusetts Institute of Technology offer little promise towards that end."

Richard Montague (1970) "English as a Formal Language", opening paragraph

Preface

The book of nature is written in the language of mathematics.

Galileo Galilei

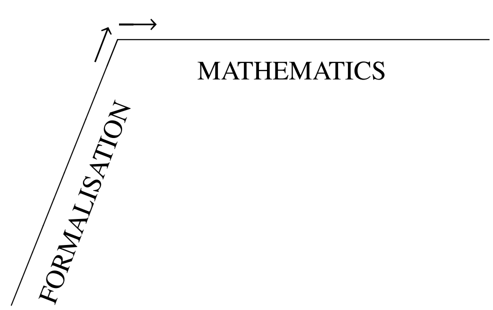
We agree with Montague that the construction of a theory of truth under an interpretation is the basic goal of syntax and semantics, and we think that Chomsky has not adhered to the formal methodology so eloquently advocated in his Preface to *Syntactic Structures*. But we disagree with Montague that there is no important difference between formal and natural languages: the latter have sentimental qualities which the former do not. This book on syntax and semantics extends to logical syntax and formal grammar Montague's original program of logical semantics and extends to logical semantics and formal grammar Chomsky's original program of formal syntax.

Logic, Computation, and Linguistics form a triangle of disciplines of informatics. Logic and theoretical computer science, ever since their modern conception, about a century ago, have formed branches of mathematics. But linguistics, until now, has had a poorer relation with mathematics; rather, it has struggled to be even formal. There has been some aspiration to rigour, but the result has fallen short of generating structures which are of interest mathematically: there has lacked this sign that the science has come of age.

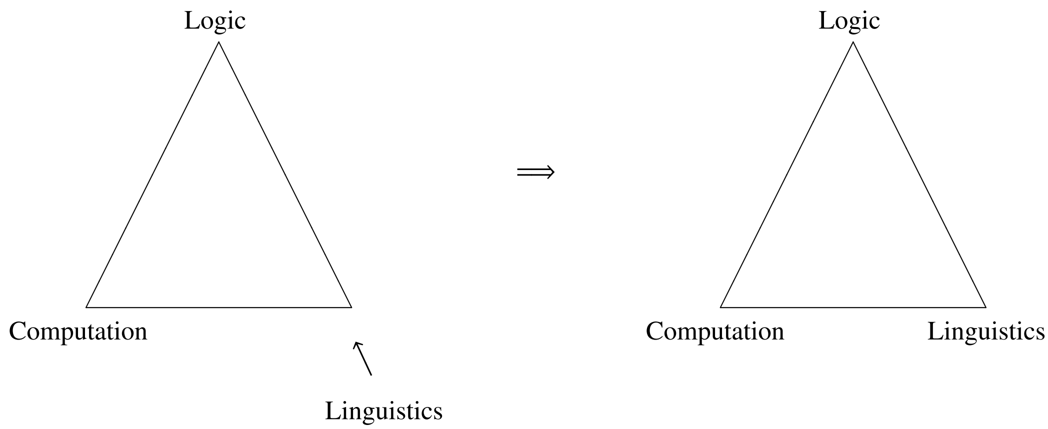
Chomsky (1957[15]) made an articulate plea for formal syntax. He later reneged on this methodology, for example in his Mimimalist Program (Chomsky 1996[14]), with desparaging remarks about formalisation for formalisation's sake and formalisation being premature, but the seeds of formal grammar had been planted. Montague (1973[51]) inaugurated formal semantics, and formal syntax and semantics enjoyed a golden era in the 1980s and 1990s. At that time there was a strong emphasis on interdisciplinary informatics at institutions like the Centre for Cognitive Science in Edinburgh and the ILLC in Amsterdam and in projects such as the DYANA and DYANA-2 European Basic Research Actions. Our goal now is to promote *mathematical* linguistics. Our thesis is that what we present here is not just *interdisciplinary*, but is promising of a *paradigm shift*.

In 1992 one of us (G.V.M.) asked Noam Chomsky what he thought of Montague Grammar. He replied that Montague believed grammar was a branch of mathematics and that mathematicians were interested in theorems, and he asked, rhetorically, whether there were any interesting theorems in Montague Grammar; of course there were none.

Logical syntax and semantics, however, *has* generated interesting theorems, for example the proof by Pentus (1992[83]) of context-free equivalence of Lambek calculus or, we suggest, Valentín's (2012[89]) analysis of Cut-elimination for displacement calculus. On our view, from a steep climb of formalisation, this branch of linguistics has hauled itself up onto the plateau of mathematics:



Thus we feel that recent years tend towards a paradigm shift whereby (categorial) linguistics becomes a fully-fledged branch of mathematics in the informatics triangle:



If we are right, linguistics is becoming mathematics categorially. The present work pursues and develops this tendency. The book is divided into five parts. Part I, SYNTAX, presents a fragment of categorial logic with algebraic semantics, explaining how it forms a type logic of syntax. Part II, SEMANTICS, presents this fragment of categorial logic with the so-called Curry-Howard formulas-as-types and proofs-as-programs semantic labelling, explaining how the type logic of syntax is also a logic of semantics. Part III, PROCESSING, explains focusing and count-invariance for our categorial logic, these forming the basis of the efficient computer implementation CatLog3 of parsing/theorem proving for the categorial logic. Part IV, GRAMMAR, gives examples of grammatical analysis in the framework, with derivations produced by CatLog3. Part V, CONCLUSION, looks into future prospects.

Chapter 5 of Part I is based on the communications of Valentín *Algebraic Semantics for Full Displacement Calculus with Linguistic Subexponentials and Bracket Modalities*, Logical Aspects of Computational Linguistics LACL21 (Montpellier 2021), and Valentín *Algebraic Completeness and Semantic Cut-Elimination of a Multimodal Formulation of Morrrill's Categorial Logic*, NCL'24 Non-Classical Logics: Theory and Applications (Łódź 2024). Chapters 9 and 10 of Part III are based on Morrill and Valentín (2018[77]) *Spurious Ambiguity and Focalization* and Kuznetsov, Morrill and Valentín (2017[42]) *Count-Invariance Including Exponentials* respectively. Chapters 13, 15, and 16 of Part IV are based on Morrill and Valentín (2016[74]) *Computational Coverage of Type Logical Grammar: The Montague Test*, Morrill, Valentín and Fadda (2011[79]) *The Displacement Calculus*, Chapter 16 is based on Morrill (2017[66]) *Grammar Logicised: Relativisation*. And Chapter 17 of Part V is partially based on Morrill and Valentín (2017[[76]) *A Reply to Kubota and Levine on Gapping*. We thank the publishers for permission to reuse this material; the rest of the material is new expression of the literature and the folklaw of the field.

G.V.M. & O.V.G.
Barcelona
2026

Primitive Categorial Connectives

Multiplicatives	$/$	$\backslash$	Lambek (1958[46]; 1988[45])
	$\bullet$		
	I		
	$\uparrow$	$\downarrow$	Morrill, Valentín & Fadda (2011[79])
	$\odot$		
	J		
Additives	$\&$	$\oplus$	Lambek (1961[44]), Girard (1987[22])
Quantifiers	$\bigwedge$	$\bigvee$	Morrill (1994[80])
Normal modalities	$\Box$	$\Diamond$	Morrill (1990[59]; 1994[80]), Moortgat (1997[54])
Bracket modalities	$[\,]^{-1}$	$\langle\rangle$	Morrill (1992[60]), Moortgat (1995[53])
(Sub)exponentials	$!$	$?$	Girard (1987[22])
Limited contr. & limited expan.	$\mid$	$+$	Jäger (2005[29]); Morrill & Valentín (2014[72])
Difference	$-$		Morrill & Valentín (2014[71])

Systems of Generalised Displacement Logic GDL

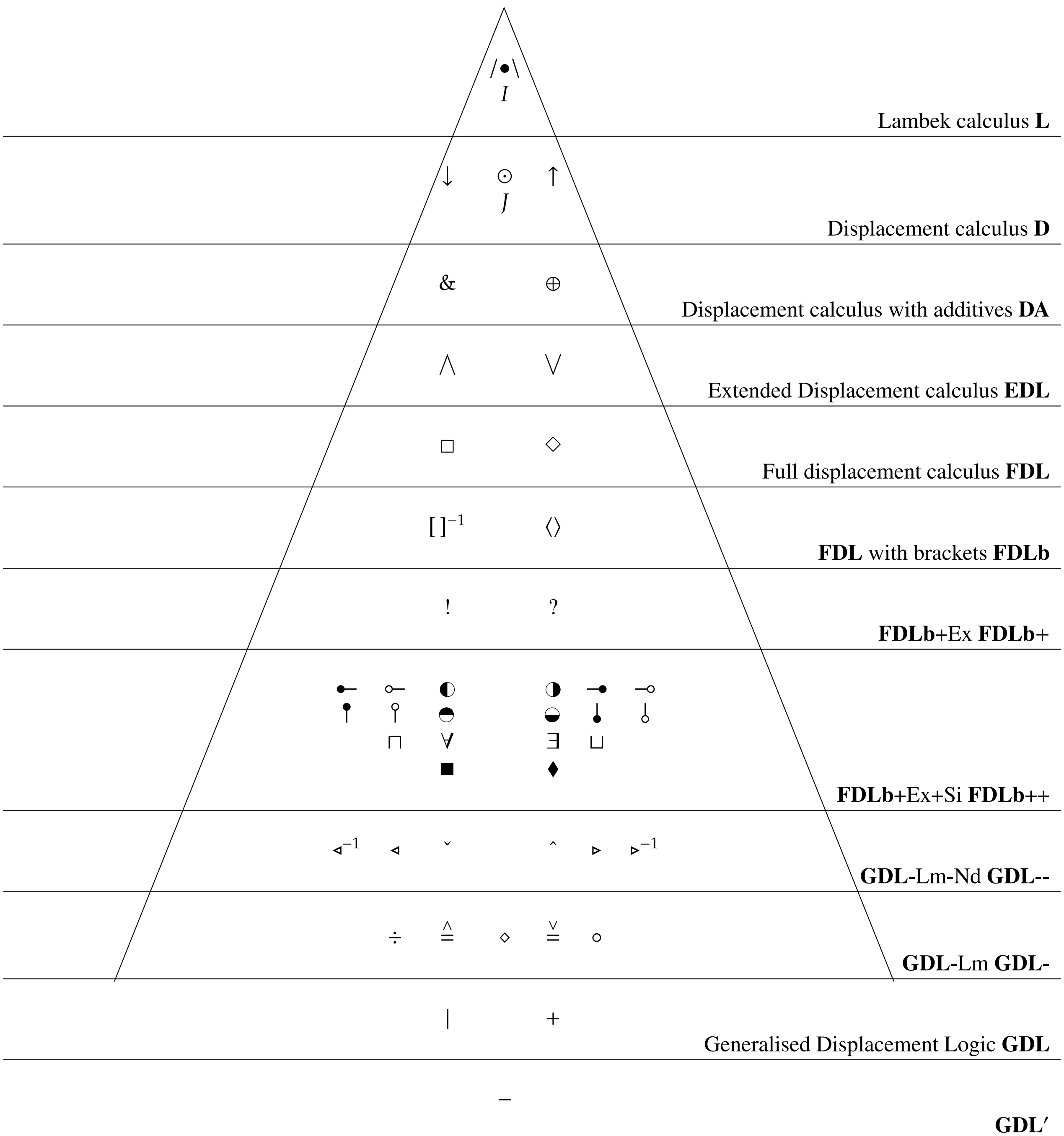


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Table of Numbered Categorial Connectives

	cont. mult.	disc. mult.	add.	qu.	norm. mod.	brack. mod.	subexp.	limited contr. & expan.
primitive	$/_1$ $\backslash_2$ $\bullet_3$ I_4	$\uparrow_5$ $\downarrow_6$ $\odot_7$ J_8	$\&_9$ $\oplus_{10}$	$\bigwedge_{11}$ $\bigvee_{12}$	$\Box_{13}$ $\Diamond_{14}$	$[\,]^{-1}{}_{15}$ $\langle\rangle_{16}$	$!_{17}$ $?_{18}$	$\vert_{48}$ $+_{49}$
sem. inactive variants	•–$_{19}$ –∘$_{20}$ ∘–$_{22}$ –•$_{23}$ ◐$_{21}$ ◑$_{24}$	$_{25}$ $_{26}$ $_{28}$ $_{29}$ ◓$_{27}$ ◒$_{30}$	$\sqcap_{31}$ $\sqcup_{32}$	$\forall_{33}$ $\exists_{34}$	■$_{35}$ ⧫$_{36}$			
det. synth.	$\triangleleft^{-1}{}_{37}$ $\triangleright^{-1}{}_{39}$ $\triangleleft_{38}$ $\triangleright_{40}$	$\check{}_{41}$ $\hat{}_{42}$						diff.
non – det. synth.	$\div_{43}$ $\circ_{44}$	$\stackrel{\wedge}{=}_{45}$ $\stackrel{\vee}{=}_{46}$ $\diamond_{47}$						$-_{50}$

Part I

SYNTAX

In this part we present the syntactic calculus of our logical grammar. In Chapter 1 we comment on the conceptual and historical contexts of syntactic types and syntactic calculus: Lambek calculus **L** and displacement calculus **D**. In Chapter 2 we define the types of the latter, establishing the core connectives for the rest of the book, and we define the syntactical models (phase semantics) for **D**. In Chapter 3 we present the sequent calculus for *DC*, note soundness, and illustrate linguistically. We go on to define further connectives, syntactical models, (sound) sequent calculus, and linguistic illustration in Chapter 4. In Chapter 5 we present completeness results, which have as a corollary semantic Cut-elimination.

Chapter 1

Introduction

1.1 Logical grammar

'Logic' is synonymous with the organisation of information and verification, and forms a basis of management of the large amounts of information making up natural language grammar. In the course of the 20th century the term has acquired a theoretical and technical refinement of the colloquial use. The relevance of such mathematical logic to the logical semantics of natural language seems clear enough when the logic is the semantic logic of the object natural language itself. However, why should *syntax* be logical?

A grammar characterises as a subset of a set of expressions those which are grammatical. A logic characterises as a subset of a set of statements those which are logical. Equate grammar with logic, and grammatical derivations are proofs. Standard logic deals with resources which are mental: ideas, which are persistent; they have no weight and are timeless, and such logic is appropriate for logical semantics, the mental semantics of the object natural language. For logical syntax we require a logic of resources which are physical and temporal: things, which are material; that have multiplicity and location in time, the physical prosodics of the object natural language. Once we accept the possibility of such resource-conscious logic, a logic of mental ideas can be complemented by a logic of physical things in grammar, facilitating the description of the vast mind/body association that is language.

If this explains how grammar *could* be logical, can we say something about why it *should*? The following two views were expressed to one of us (G.V.M.) by Andre Scedrov and Stefan Kuznetsov, p.c. Firstly, if we model grammar as logic *and* prove Cut-elimination then we have not just modelled grammar as a theory in a logic technically but have also identified the logic of the empirical domain modelled, grammar, itself in reality (A. Scedrov). Secondly, we think that logic holds out the prospect of *verification* of grammar; imagine that natural language processing was life-critical (S. Kuznetsov).

1.2 Syntactic calculus

The syntactic and semantic structures of logic are generally well-nested. But from the inception of modern linguistics it has been emphasized that the central question in grammar is how to treat the pervasive syntactic/semantic mismatch in natural language, a question to which Chomsky proposed the answer of transformations. We think that although this is the right question, transformations are not the right answer. Rather we propose to treat syntactic/semantic 'disorder' by logical syntax and semantics based on displacement calculus, a discontinuous sublinear intuitionistic logic in which 'displacement' emerges from an underlyingly well-nested structure. Thus apparent disorder is, eventually, order again.

Lambek (1958[46]) defined a calculus of syntactic types. This *Lambek calculus* is free of structural rules, and Lambek proved (constructively) that it enjoys the Cut-elimination theorem, i.e. that every theorem has a Cut-free proof. This Cut-Elimination property has as corollaries the subformula property (that every theorem has a proof containing only its subformulas — in this case any one of its Cut-free proofs), the finite proof property (that every theorem has only a finite number of Cut-free proofs, since the Cut-free proof search-space is finite) and decidability (that whether or not a statement is provable can be determined by an effective procedure — for example Cut-free backward chaining proof search).

Retrospectively the Lambek calculus can be recognised as the multiplicative fragment of intuitionistic non-commutative linear logic (Girard 1987[22]; Abrusci 1990[3], 1991[1]) without empty antecedents. The Lambek calculus was predated by Ajdukiewicz (1935[4]) and Bar-Hillel (1953[7]) but was developed independently of these; Bar-Hillel, Gaifman and Shamir (1961[6]) introduced the term 'categorial grammar' to refer to such systems of *logical syntax*.

Montague (1973[51]) defined a *logical semantics* for a significant formal fragment of English, a remarkable achievement at a time when it was widely believed that semantics was beyond the reaches of formalisation. As a demonstration of the power of our approach we have given a computational cover grammar of the Montague fragment in Chapter 13, and we propose that doing so be adopted as a standard criterion or 'Montague test' in computational syntax and semantics.

The Lambek calculus remained largely unknown until its rediscovery in the 1980s when it was observed that there is a perfect matching of the logical syntax of Lambek and the logical semantics of Montague; see van Benthem (1983[92]). (In this context the finite proof property has a further corollary: the finite reading property whereby every expression is at most finitely ambiguous, a finiteness which appears to be an empirical universal of the natural languages of the world; it follows from semantic Cut-elimination, Hendriks (1993[25]).) Monograph and reference article studies developing this *logical grammar* include Moortgat (1988[52]; 1997[54]), Morrill (1994[80]; 2011[81]; 2012[65]), Carpenter (1997[11]), Jäger (2005[29]) and Moot and Retoré (2012[56]).

Probably one reason why Lambek calculus **L** was largely ignored for so long was the suspicion that it was context-free in generative power together with the belief, deriving from Chomsky (1957[15]), that context-free grammar is observationally inadequate for natural language. And indeed, Shieber (1985[85]) proves that Swiss-German is non-context free, and Pentus (1993[83]) proves that the Lambek calculus generates exactly the context free languages without the empty string. Thus although the categorial syntax/semantics interface is philosophically and technically impeccable, there remained a justified doubt about it's linguistic adequacy. Morrill, Valentín and Fadda (2011[79]) address this concern by defining a conservative extension of the Lambek calculus which allows displacement while remaining free of structural rules and preserving the syntax/semantics interface and Cut-elimination of the Lambek calculus, and all of its metatheoretic corollaries: the subformula property, the finite proof property, decidability, and the finite reading property. The displacement calculus characterises, for example, both covert movement such as quantifier scoping and overt movement such as cross-serial dependencies. This book develops logical grammar built over the displacement calculus **D**. Linguistics motivates additional sensitivity and expressivity.

The *displacement calculus with additives*, **DA**, is the displacement calculus plus the additives (&, ⊕) 9–10, which have application to polymorphism. *Extended displacement calculus*, **EDL**, is **DA** extended with 11–12: 1st order quantifiers ($\bigwedge$, $\bigvee$), which have application to features. *Full displacement calculus* **FDL** is Extended displacement calculus **EDL** plus normal modalities. *Full displacement calculus with brackets* **FDLb** is Full displacement calculus **FDL** plus brackets; this also requires a change in the notion of sequent, to include brackets in antecedents.

Full displacement calculus with brackets and soft subexponentials **FDLb+** is Full displacement calculus with brackets **FDLb** plus soft subexponentials (!, ?) (term of Kanovich et al. (2018[33]).

Full displacement calculus with brackets and soft subexponentials and semantically inactive variants **FDLb + +** is Full displacement calculus with brackets and soft subexponentials **FDLb+** plus the semantically inactive variants 19–47. We call the connectives of **FDLb + +** the *negation-free primary* connectives.

Generalised displacement logic minus limited contraction and expansion and minus non-determinism **GDL − −** is *Generalised displacement logic minus limited contraction and expansion* **FDLb + +** without non-determinism. We call the connectives of **GDL − −** the *negation-free secondary* connectives. *Generalised displacement calculus without limited contraction and expansion* **GDL−** is Generalised displacement calculus **GDL** without limited contraction and expansion plus the 'synthetic' *(*defined) connectives . We call these the tercary *negation-free primitive* connectives.

Generalised displacement calculus with difference **GDL′** is Generalised displacement calculus **GDL** is plus the 'metalogical negation' 50 difference.

Chapter 2

The types of Displacement Logic DL

2.1 Syntactic types of DL

The syntactic types $\{\mathbf{Tp}_i\}_{i\in\mathcal{N}}$ of our categorial logic are sorted according to the number of points $i \in \mathcal{N}$ of discontinuity that their expressions contain. $i' = i + 1$. Each *type predicate letter* P will have a sort and an arity which are naturals, and a corresponding semantic type. Assuming to be already given ordinary (feature) terms interpreted in a domain F, where P is a type predicate letter of sort i, arity n, and semantic type σ and $t_1, \ldots, t_n$ are feature terms, $Pt_1 \ldots t_n$ is an atomic type of sort i and of semantic type σ. Compound types of Displacement Logic **DL** are formed from these by connectives as indicated in Figure 2.1, and the structure preserving prosodic type map s associates these with sorts. The sort $s(A)$ (sA) of a type A is the $i \in \mathcal{N}$ such that $A \in \mathbf{Tp}_i$. For example, if $s(N) = s(S) = 0$, we have $s((S{\uparrow_1}N){\uparrow_0}N) = s((S{\uparrow_0}N){\uparrow_1}N) = 2$, and if $s(VP) = 1$, $s((J\backslash(N\backslash VP)){\uparrow_0}N) = 2$.

0.	$\mathbf{Tp}_i$	::=	$Pt_1 \ldots t_n$	$s(A)$	=	σ	
1.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_{i+j}/\mathbf{Tp}_j$	$s(C/B)$	=	$s(C) - s(B)$	over [46]
2.	$\mathbf{Tp}_j$	::=	$\mathbf{Tp}_i\backslash\mathbf{Tp}_{i+j}$	$s(A\backslash C)$	=	$s(C) - s(A)$	over [46]
3.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_i{\bullet}\mathbf{Tp}_j$	$s(A{\bullet}B)$	=	$s(A) + s(B)$	continuous product [46]
4.	$\mathbf{Tp}_0$	::=	I	$s(I)$	=	0	continuous unit [45]
$5, k.$	$\mathbf{Tp}_{i'}$	::=	$\mathbf{Tp}_{i+j}{\uparrow_k}\mathbf{Tp}_j, 0 < k < i'$	$s(C{\uparrow_k}B)$	=	$s(C)' - s(B)$	circumfix [79]
$6, k.$	$\mathbf{Tp}_j$	::=	$\mathbf{Tp}_{i'}{\downarrow_k}\mathbf{Tp}_{i+j}, 0 < k < i'$	$s(A{\downarrow_k}C)$	=	$s(C)' - s(A)$	infix [79]
$7, k.$	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i'}{\odot_k}\mathbf{Tp}_j, 0 < k < i'$	$s(A{\odot_k}B)$	=	$s(A) + s(B) - 1$	discontinuous product [79]
8.	$\mathbf{Tp}_1$	::=	J	$s(J)$	=	1	discontinuous unit [79]
9.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i\&\mathbf{Tp}_i$	$s(A\&B)$	=	$s(A) = s(B)$	additive conjunction [44, 57]
10.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i\oplus\mathbf{Tp}_i$	$s(A\oplus B)$	=	$s(A) = s(B)$	additive disjunction [44, 57]
11.	$\mathbf{Tp}_i$	::=	$\bigwedge V\mathbf{Tp}_i$	$s(\bigwedge vA)$	=	$s(A)$	1st order univ. qu. [80]
12.	$\mathbf{Tp}_i$	::=	$\bigvee V\mathbf{Tp}_i$	$s(\bigvee vA)$	=	$s(A)$	1st order exist. qu. [80]
13.	$\mathbf{Tp}_i$	::=	$\Box\mathbf{Tp}_i$	$s(\Box A)$	=	$s(A)$	univ. modality [59]
14.	$\mathbf{Tp}_i$	::=	$\Diamond\mathbf{Tp}_i$	$s(\Diamond A)$	=	$s(A)$	exist. modality [54]
15.	$\mathbf{Tp}_i$	::=	$[\,]^{-1}\mathbf{Tp}_i$	$s([\,]^{-1}A)$	=	$s(A)$	univ. bracket modality [60, 53]
16.	$\mathbf{Tp}_i$	::=	$\langle\rangle\mathbf{Tp}_i$	$s(\langle\rangle A)$	=	$s(A)$	exist. bracket modality [60, 53]
17.	$\mathbf{Tp}_0$	::=	$!\mathbf{Tp}_0$	$s(!A)$	=	$s(A) = 0$	univ. subexponential [9]
18.	$\mathbf{Tp}_0$	::=	$?\mathbf{Tp}_0$	$s(?A)$	=	$s(A) = 0$	exist. subexponential [80]

Figure 2.1: The types of Displacement Logic **DL**

The family of types 1–4: $\{/, \backslash, \bullet, I\}$ are the Lambek connectives of Lambek 1958[46] and Lambek 1988[45]. They are defined in relation to concatenation, with left residual $\backslash$ ('under'), right residual $/$ ('over'), language concatenation $\bullet$ ('product') and language concatenation unit I ('product unit'). The structure $(\mathbf{Tp}, \backslash, \bullet, /, I; \subseteq)$ forms a residuated monoid. The calculus with only these connectives is the *Lambek calculus (with unit)*, **L**.

The family of types 5–8: $\{\uparrow_k, \downarrow_k, \odot_k, J\}$ are the displacement connectives of Morrill 2011[81], Morrill and Valentín 2010[70] and Morrill, Valentín and Fadda 2011[79]. They are defined in relation to intercalation, i.e. the replacement by a second operand of a point of discontinuity ('separator') in a first operand; the subscript k indexes which separator is replaced: 0 for the first from the left, 1 for the second from the left, 2 for the third from the left, as so on. There is the left residual $\downarrow_k$ ('infix'), right residual $\uparrow_k$ ('circumfix'), language intercalation $\odot_k$ ('discontinuous product') and language intercalation unit J ('discontinuous product unit'). The structure $(\mathbf{Tp}, \downarrow_k, \odot_k, \uparrow_k, J; \subseteq)$ also forms a (sorted) residuated monoid. The calculus with the connectives 1–8 is the *displacement calculus*, **D**.

Displacement calculus **D**, like Lambek calculus **L**, is a 'multiplicative' system in the terminology of linear logic, meaning that resources (like words) have multiplicity of occurrence and do not preserve grammatical properties like well-formedness and meaning under either fission (contraction) or fusion (expansion), but must be perfectly balanced between production and consumption in reasoning.

2.2 Syntactical models of DL

In standard logic information does not have multiplicity. Thus where + is the notion of addition of information and $\leq$ is the notion of inclusion of information we have 'sharing' $x{+}x \leq x$ and $x \leq x{+}x$; together these two properties amount to *idempotency*: $x{+}x = x$. The properties are expressed by the rules of inference Contraction and Expansion:

(1) $$\frac{\Delta(A, A) \Rightarrow B}{\Delta(A) \Rightarrow B}\text{Contraction} \qquad \frac{\Delta(A) \Rightarrow B}{\Delta(A, A) \Rightarrow B}\text{Expansion}$$

Expansion is a special case of Weakening:

(2) $$\frac{\Delta() \Rightarrow B}{\Delta(A) \Rightarrow B}\text{Weakening}$$

Linguistic resources do not have these properties: grammaticality is not generally preserved under addition or removal of copies of words or expressions. However, there are some constructions manifesting something similiar. Parasitic gaps allow a kind of controlled contraction:

(3) a. patient that$_i$ John convinced the friends of Mary to visit t_i
b. patient that$_i$ John convinced the friends of t_i to visit Mary
c. patient that$_i$ John convinced the friends of t_i to visit t_i

And iterated coordination allows a kind of controlled expansion:

(4) John, Fred, Bill, ... and Mary.

The left hand conjunct can be expanded indefinitely by like-type conjuncts, preserving grammaticality.

Anaphora and expletive pronouns allow kinds of limited contraction and limited expansion:

(5) a. Near him/Dan, Dan/he saw a snake.
b. It rains.

That is, in logical grammar idempotency, or sharing, is the exception rather than the rule.

Idempotency introduces complexity in the syntactical model theory of categorial grammar. We include here syntactical interpretation for controlled and limited contraction and expansion.

Chapter 3

Hedge sequent calculus

In this chapter we present Gentzen sequent calculus for Displacement Logic. Gentzen sequent format is a kind of linguafranca of logic, in which all rules are of the form

$$\frac{\Sigma_1 \ \ldots \ \Sigma_n}{\Sigma_0}$$

where the Σ_i are statements of consequence called *sequents*. Typically in (sub)linear logics, once the *Cut-elimination* theorem is proved, Gentzen sequent calculus yields a finite search space and hence a decision procedure for theoremhood.[1] We employ a particular form of sequent calculus to deal with the discontinuity in displacement logic which we will refer to as *hedge* sequent calculus. In addition, the sequent calculus contains bracketing to deal with the bracket modalities ('structural inhibition') and *stoups* (Girard 2011[23]) to deal with subexponentials ('structural facilitation'). A stoup is a reserved location containing types which can undergo structural rules such as contraction.

3.1 Hedge sequent calculus

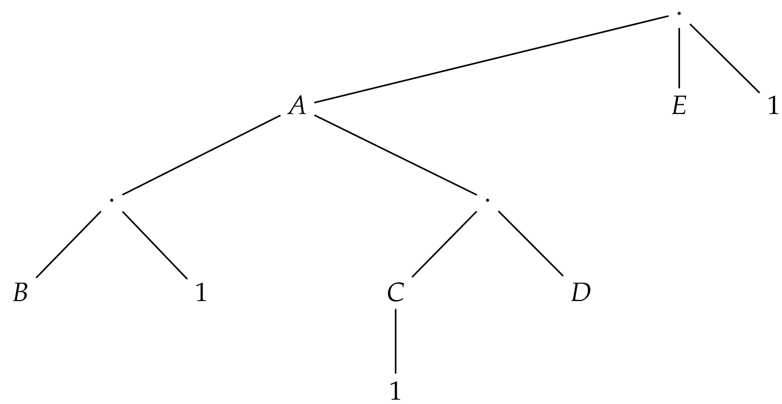
The sets **Zone** of *zones*, **Stoup** of *stoups*, **Config** of *configurations* and **TreeTerm** of *tree terms* of hedge sequent calculus **hDL** for Displacement Logic are defined by mutual recursion as follows, where $\emptyset$ is the metalinguistic empty stoup, Λ is the metalinguistic empty configuration and 1 is a metalinguistic placeholder called the *separator*:

(6)
$$\begin{array}{rcl}
\mathbf{Zone} & ::= & \mathbf{Stoup}; \mathbf{Config} \\
\mathbf{Stoup} & ::= & \emptyset \mid \mathbf{Tp}_0, \mathbf{Stoup} \\
\mathbf{Config} & ::= & \Lambda \mid \mathbf{TreeTerm}, \mathbf{Config} \\
\mathbf{TreeTerm} & ::= & 1 \mid \mathbf{Tp}_0 \mid \mathbf{Tp}_{i>0}\{\underbrace{\mathbf{Config} : \ldots : \mathbf{Config}}_{i\ \mathbf{Config}\text{'s}}\} \mid [\mathbf{Zone}]
\end{array}$$

For example, there is the configuration $\gamma = A\{B, 1 : C\{1\}, D\}, E, 1$ where $s(A) = 2$ and $s(C) = 1$ and $s(B) = s(D) = s(E) = 0$. Diagramatically this configuration γ is:

[1] Other formats, such as natural deduction, do not necessarily do the same, and usually adapting them to yield a decision procedure amounts, in effect, to converting them to Gentzen sequent format.

(7)



The intuition is the following. Dotted nodes signify unbounded arity concatenations and a type labelling a mother node signifies a discontinuous type intercalated by its daughter configurations. Leaf types are continuous, and a leaf 1 marks a point of discontinuity. The sort $s(\Gamma)$ of a configuration Γ is the number of separators 1 that Γ contains. For example, $s(\gamma) = 3$.

Note that a stoup can only contain types of sort 0; types of other sort would not preserve sort-equality under contraction.[2] The sort of a zone is the sort of its configuration. A *sequent* is of the form:

(8) $\mathbf{Zone} \Rightarrow \mathbf{Tp} \qquad \text{where } s(\mathbf{Zone}) = s(\mathbf{Tp})$

The *figure* $\overrightarrow{A}$ of a type A is defined by:

(9) $$\overrightarrow{A} = \begin{cases} A & \text{if } s(A) = 0 \\ A\{\underbrace{1 : \ldots : 1}_{sA\ 1's}\} & \text{if } s(A) > 0 \end{cases}$$

For example, $\overrightarrow{(S{\uparrow_1}N){\uparrow_2}N} = (S{\uparrow_1}N){\uparrow_2}N\{1 : 1\}$. Where Γ is a configuration of sort i and $\Delta_1, \ldots, \Delta_i$ are configurations, the *fold* $\Gamma \otimes \langle \Delta_1 : \ldots : \Delta_i \rangle$ is the result of replacing the successive 1's in Γ by $\Delta_1, \ldots, \Delta_i$ respectively. Thus for our running example, if γ_1, γ_2 and γ_3 are configurations, the fold $\gamma \otimes \langle \gamma_1 : \gamma_2 : \gamma_3 \rangle = A\{C, \gamma_1 : B\{\gamma_2\}, D\}, F, \gamma_3$. Where Δ is a configuration of sort $i > 0$ and Γ is a configuration, the kth metalinguistic wrap, $1 \leq k \leq i$, $\Delta |_k \Gamma$ is given by

(10) $$\Delta |_k \Gamma =_{df} \Delta \otimes \langle \underbrace{1 : \ldots : 1}_{k-1\ 1\text{'s}} : \Gamma : \underbrace{1 : \ldots : 1}_{i-k\ 1\text{'s}} \rangle$$

i.e. $\Delta |_k \Gamma$ is the configuration resulting from replacing by Γ the kth separator in Δ. Thus for our running example, where γ' is a configuration, $\gamma |_2 \gamma' = A\{C, 1 : B\{\gamma'\}, D\}, F, 1$.

$\Delta(\Gamma)$ signifies a configuration or zone Δ with a distinguished subconfiguration or subzone Γ. Where Γ is a configuration of sort i, $\langle\Gamma\rangle$ signifies $\Gamma \otimes \langle \Delta_1, \ldots, \Delta_n \rangle$; and $\Delta\langle\Gamma\rangle$ signifies $\Delta_0(\langle\Gamma\rangle)$ i.e. $\Delta_0(\Gamma \otimes \langle \Delta_1, \ldots, \Delta_i \rangle)$, i.e. a configuration Δ with a potentially discontinuous distinguished subconfiguration Γ.

The hedge sequent calculus **hDL** for the **DL** of Chapter 2 has the following identity axiom and Cut rule; Ξ denotes a zone; Γ and Δ, possibly with subscripts, denote configurations; ζ, possibly with subscripts, denotes a stoup.

(11) $$\frac{}{\emptyset; \overrightarrow{P} \Rightarrow P}\ id \qquad\qquad \frac{\zeta_1; \Gamma \Rightarrow A \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{A}\rangle, \Delta_2) \Rightarrow B}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\Gamma\rangle, \Delta_2) \Rightarrow B}\ Cut$$

Note that the conclusion stoup is empty in the axiomatic id rule and is partitioned in the binary Cut rule. The identity axiom asserts the reflexivity of the derivability relation for atomic types P. In general it will be the case that for all types A, $\overrightarrow{A} \Rightarrow A$. The Cut rule asserts the transitivity of the derivability relation, which is a property that is usually desired. However Cut is problematic computationally because the Cut formula A is a new unknown reading from conclusion to premises. A part of the art of Gentzen sequent calculus

[2] For sort i a natural, $i = i + i$ only when $i = 0$.

is to formulate rules which are partially executed with respect to Cut in such a way that all theorems have Cut-free proofs, that is: to have the effect of Cut without using Cut. Gentzen called the proof of such Cut-elimination his *Haupsatz*.

3.2 Rules for the primary connectives of hDL

The logical rules are listed in what follows. Linguistic applications are given.

3.2.1 Multiplicatives

The continuous multiplicatives $\{/, \backslash, \bullet, I\}$ the Lambek connectives of Lambek (1958[46]; 1988[45]), are the basic means of categorial (sub)categorization. Their Gentzen sequent rules are as in Figure 3.1.

$$1. \quad \frac{\zeta_1; \Gamma \Rightarrow B \qquad \Xi(\zeta_2; \Delta_1, \langle \overrightarrow{C} \rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle \overrightarrow{C/B}, \Gamma \rangle, \Delta_2) \Rightarrow D} /L \qquad \frac{\zeta; \Gamma, \overrightarrow{B} \Rightarrow C}{\zeta; \Gamma \Rightarrow C/B} /R$$

$$2. \quad \frac{\zeta_1; \Gamma \Rightarrow A \qquad \Xi(\zeta_2; \Delta_1, \langle \overrightarrow{C} \rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle \Gamma, \overrightarrow{A\backslash C} \rangle, \Delta_2) \Rightarrow D} \backslash L \qquad \frac{\zeta; \overrightarrow{A}, \Gamma \Rightarrow C}{\zeta; \Gamma \Rightarrow A\backslash C} \backslash R$$

$$3. \quad \frac{\Xi\langle \overrightarrow{A}, \overrightarrow{B} \rangle \Rightarrow D}{\Xi\langle \overrightarrow{A\bullet B} \rangle \Rightarrow D} \bullet L \qquad \frac{\zeta_1; \Gamma_1 \Rightarrow A \qquad \zeta_2; \Gamma_2 \Rightarrow B}{\zeta_1 \uplus \zeta_2; \Gamma_1, \Gamma_2 \Rightarrow A\bullet B} \bullet R$$

$$4. \quad \frac{\Xi\langle \Lambda \rangle \Rightarrow A}{\Xi\langle \overrightarrow{I} \rangle \Rightarrow A} IL \qquad \frac{}{\emptyset; \Lambda \Rightarrow I} IR$$

Figure 3.1: Continuous multiplicative rules

Notice how in the multiplicative rules the conclusion stoup is partitioned between premises in binary rules, copied to the premise in unary rules, and is empty in the axiomatic *IR* rule. When stoups are empty, both the empty stoup and '$;$' may be omitted in derivations.

The directional divisions "over", $/$, and "under", $\backslash$, are exemplified by assignments such as **the**: N/CN for the noun phrase **the man**: N and **sings**: $N\backslash S$ for the sentence **John sings**: S and **loves**: $(N\backslash S)/N$ for the sentence **John loves Mary**: S. Hence, for **the man**:

$$(12) \quad \frac{CN \Rightarrow CN \qquad N \Rightarrow N}{N/CN, CN \Rightarrow N} /L$$

And for **John sings** and **John loves Mary**:

$$(13) \quad \frac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S} \backslash L \qquad \frac{N \Rightarrow N \qquad \dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S} \backslash L}{N, (N\backslash S)/N, N \Rightarrow S} /L$$

The continuous product "times", $\bullet$, is exemplified by a 'small clause' assignment like **considers**: $(N\backslash S)/(N\bullet(CN/CN))$ for **John considers Mary socialist**: S:

$$(14) \quad \frac{\dfrac{N \Rightarrow N \qquad \dfrac{\dfrac{CN \Rightarrow CN \qquad CN \Rightarrow CN}{CN/CN, CN \Rightarrow CN} /L}{CN/CN \Rightarrow CN/CN} /R}{N, CN/CN \Rightarrow N\bullet(CN/CN)} \bullet R \qquad \dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S} \backslash L}{N, (N\backslash S)/(N\bullet(CN/CN)), N, CN/CN \Rightarrow S} /L$$

Of course this positive/succedent/right use of times is not essential: we could just as well have used $((N\backslash S)/(CN/CN))/N$ since in general we have both

$$A/(C\bullet B) \Rightarrow (A/B)/C \text{ ('currying')}$$

and

$$(A/B)/C \Rightarrow A/(C\bullet B) \text{ ('uncurrying')}.$$

For a negative/antecedent/left essential use of times, for past participles, see Morrill (2000[61], section 2)).

The discontinuous multiplicatives $\{\uparrow, \downarrow, \odot, J\}$, the displacement connectives, of Morrill and Valentín (2010[70]), Morrill, Valentín & Fadda (2011[79]) and Morrill (2011[81]), are defined in relation to intercalation (wrapping). Their Gentzen sequent rules are given in Figure 3.2.

$$5,k.\quad \frac{\zeta_1;\Gamma \Rightarrow B \qquad \Xi(\zeta_2;\Delta_1,\langle\overrightarrow{C}\rangle,\Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2;\Delta_1,\langle\overrightarrow{C{\uparrow_k}B}\,|_k\,\Gamma\rangle,\Delta_2) \Rightarrow D}\uparrow_k L \qquad \frac{\zeta;\Gamma\,|_k\,\overrightarrow{B} \Rightarrow C}{\zeta{:}\,\Gamma \Rightarrow C{\uparrow_k}B}\uparrow_k R$$

$$6,k.\quad \frac{\zeta_1;\Gamma \Rightarrow A \qquad \Xi(\zeta_2;\Delta_1,\langle\overrightarrow{C}\rangle,\Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2;\Delta_1,\langle\Gamma\,|_k\,\overrightarrow{A{\downarrow_k}C}\rangle,\Delta_2) \Rightarrow D}\downarrow_k L \qquad \frac{\zeta;\overrightarrow{A}\,|_k\,\Gamma \Rightarrow C}{\zeta;\Gamma \Rightarrow A{\downarrow_k}C}\downarrow_k R$$

$$7,k.\quad \frac{\Xi\langle\overrightarrow{A}\,|_k\,\overrightarrow{B}\rangle \Rightarrow D}{\Xi\langle\overrightarrow{A{\odot_k}B}\rangle \Rightarrow D}\odot_k L \qquad \frac{\zeta_1;\Gamma_1 \Rightarrow A \qquad \zeta_2;\Gamma_2 \Rightarrow B}{\zeta_1 \uplus \zeta_2;\Gamma_1\,|_k\,\Gamma_2 \Rightarrow A{\odot_k}B}\odot_k R$$

$$8.\quad \frac{\Xi\langle 1\rangle \Rightarrow A}{\Xi\langle\overrightarrow{J}\rangle \Rightarrow A}JL \qquad \frac{}{\emptyset;1 \Rightarrow J}JR$$

Figure 3.2: Discontinuous multiplicative rules

Notice how the discontinuous multiplicative rules have exactly the same form as the continuous multiplicative rules but with metalinguistic intercalation "$|_k$" in place of the metalinguistic concatenation ","; the stoups distribute as before.

When the value of the k subscript is one it may be omitted, i.e. it defaults to one. "Circumfixation", $\uparrow$, is exemplified by a particle verb assignment **calls**+1+**up**: $(N\backslash S)\uparrow N$ for **Mary calls the man up** : S:

$$(15)\quad \frac{\dfrac{CN \Rightarrow CN \qquad N \Rightarrow N}{N/CN, CN \Rightarrow N}/L \qquad \dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\backslash L}{N, (N\backslash S){\uparrow}N\{N/CN, CN\} \Rightarrow S}\uparrow L$$

'Infixation', $\downarrow$, and circumfixation together are exemplified by a quantifier assignment **everyone**: $(S\uparrow N)\downarrow S$ simulating Montague's S14 quantifying in:

$$(16)\quad \frac{\dfrac{\ldots, N, \ldots \Rightarrow S}{\ldots, 1, \ldots \Rightarrow S{\uparrow}N}\uparrow R \qquad \dfrac{}{S \Rightarrow S}id}{\ldots, (S{\uparrow}N){\downarrow}S, \ldots \Rightarrow S}\downarrow L$$

Circumfixation and discontinuous product, "wrap", $\odot$, are illustrated in an assignment to a relative pronoun **that**: $(CN\backslash CN)/((S\uparrow N)\odot I)$ allowing both peripheral and medial extraction, **that John likes**: $CN\backslash CN$ and **that John saw today**: $CN\backslash CN$:

$$(17)\quad \frac{\dfrac{\dfrac{N, (N\backslash S)/N, N \Rightarrow S}{N, (N\backslash S)/N, 1 \Rightarrow S{\uparrow}N}\uparrow R \qquad \dfrac{}{\Rightarrow I}IL}{N, (N\backslash S)/N \Rightarrow (S{\uparrow}N){\odot}I}\odot R \qquad CN\backslash CN \Rightarrow CN\backslash CN}{(CN\backslash CN)/((S{\uparrow}N){\odot}I), N, (N\backslash S)/N \Rightarrow CN\backslash CN}/L$$

$$(18)\quad \dfrac{\dfrac{\dfrac{N,(N\backslash S)/N,N,S\backslash S\Rightarrow S}{N,(N\backslash S)/N,1,S\backslash S\Rightarrow S\uparrow N}\uparrow R \qquad \dfrac{}{\Rightarrow I}IL}{N,(N\backslash S)/N,S\backslash S\Rightarrow (S\uparrow N)\odot I}\odot R \qquad CN\backslash CN\Rightarrow CN\backslash CN}{(CN\backslash CN)/((S\uparrow N)\odot I),N,(N\backslash S)/N,S\backslash S\Rightarrow CN\backslash CN}/L$$

3.2.2 Additives

The additive conjunction and disjunction $\{\&, \oplus\}$ of Lambek (1961[44]), Morrill (1990[58]), and Kanazawa (1992[32]), capture polymorphism. Their rules are given in Figure 3.3.

$$9.\quad \dfrac{\Xi\langle\overrightarrow{A}\rangle\Rightarrow C}{\Xi\langle\overrightarrow{A\&B}\rangle\Rightarrow C}\&L_1 \qquad \dfrac{\Xi\langle\overrightarrow{B}\rangle\Rightarrow C}{\Xi\langle\overrightarrow{A\&B}\rangle\Rightarrow C}\&L_2$$

$$\dfrac{\Xi\Rightarrow A \qquad \Xi\Rightarrow B}{\Xi\Rightarrow A\&B}\&R$$

$$10.\quad \dfrac{\Xi\langle\overrightarrow{A}\rangle\Rightarrow C \qquad \Xi\langle\overrightarrow{B}\rangle\Rightarrow C}{\Xi\langle\overrightarrow{A\oplus B}\rangle\Rightarrow C}\oplus L$$

$$\dfrac{\Xi\Rightarrow A}{\Xi\Rightarrow A\oplus B}\oplus R_1 \qquad \dfrac{\Xi\Rightarrow B}{\Xi\Rightarrow A\oplus B}\oplus R_2$$

Figure 3.3: Additive rules

Notice how the stoup is shared between premises and conclusion in the additive rules. By way of example of the additives, the additive conjunction "with", &, can be used for the polymorphism of a mass noun **rice**: $N\&CN$ as in **rice grows**: S and **the rice grows**: S:

$$(19)\quad \dfrac{\dfrac{N\Rightarrow N}{N\&CN\Rightarrow N}\&L_1 \qquad S\Rightarrow S}{N\&CN,N\backslash S\Rightarrow S}\backslash L \qquad\qquad \dfrac{N/CN,CN,N\backslash S\Rightarrow S}{N/CN,N\&CN,N\backslash S\Rightarrow S}\&L_2$$

The additive disjunction "plus", $\oplus$, can be used for the polymorphism of the copula

is: $(N\backslash S)/(N\oplus(CN/CN))$

as in **Tully is Cicero**: S and **Tully is humanist**: S:

$$(20)\quad \dfrac{\dfrac{N\Rightarrow N}{N\Rightarrow N\oplus(CN/CN)}\oplus R_1 \qquad N\backslash S\Rightarrow N\backslash S}{(N\backslash S)/(N\oplus(CN/CN)),N\Rightarrow N\backslash S}/L \qquad \dfrac{\dfrac{CN/CN\Rightarrow CN/CN}{CN/CN\Rightarrow N\oplus(CN/CN)}\oplus R_2 \qquad N\backslash S\Rightarrow N\backslash S}{(N\backslash S)/(N\oplus(CN/CN)),CN/CN\Rightarrow N\backslash S}/L$$

3.2.3 Quantifiers

The 1st order quantifiers $\{\bigwedge, \bigvee\}$ of Morrill (1994[80]), have application to features. The rules are given in Figure 3.4.

11. $$\frac{\Xi\langle\overrightarrow{A[t/v]}\rangle \Rightarrow B}{\Xi\langle\overrightarrow{\bigwedge vA}\rangle \Rightarrow B}\bigwedge L \qquad \frac{\Xi \Rightarrow A[a/v]}{\Xi \Rightarrow \bigwedge vA}\bigwedge R^{\dagger}$$

12. $$\frac{\Xi\langle\overrightarrow{A[a/v]}\rangle \Rightarrow B}{\Xi\langle\overrightarrow{\bigvee vA}\rangle \Rightarrow B}\bigvee L^{\dagger} \qquad \frac{\Xi \Rightarrow A[t/v]}{\Xi \Rightarrow \bigvee vA}\bigvee R$$

Figure 3.4: Quantifier rules, where † indicates that there is no a in the conclusion

Notice how the stoup is identical on premises and conclusions in the quantifier rules. By way of example of the quantifiers, we can generalise over singular and plural number in **sheep**: $\bigwedge nCNn$ for **the sheep grazes**: S and **the sheep graze**: S:

(21) $$\frac{CNsg \Rightarrow CNsg}{\bigwedge nCNn \Rightarrow CNsg}\bigwedge L \qquad \frac{CNpl \Rightarrow CNpl}{\bigwedge nCNn \Rightarrow CNpl}\bigwedge L$$

And we can express a past, present or future tense finite sentence complement: **said**: $(N\backslash S)/\bigvee tSf(t)$ in **John said Mary walked**: S, **John said Mary walks**: S and **John said Mary will walk**: S:

(22) $$\frac{Sf(past) \Rightarrow Sf(past)}{Sf(past) \Rightarrow \bigvee tSf(t)}\bigvee R \qquad \frac{Sf(pres) \Rightarrow Sf(pres)}{Sf(pres) \Rightarrow \bigvee tSf(t)}\bigvee R \qquad \frac{Sf(fut) \Rightarrow Sf(fut)}{Sf(fut) \Rightarrow \bigvee tSf(t)}\bigvee R$$

3.2.4 Modalities

Normal modalities

With respect to the normal modalities $\{\Box, \Diamond\}$ of Morrill (1990[59], 1994[80]), Hepple 1990[27], and Moortgat (1996[53], 1997[54]), the universal has application to semantic intensionality (Morrill [59]). The rules are given in Figure 3.5.

13. $$\frac{\Xi\langle\overrightarrow{A}\rangle \Rightarrow B}{\Xi\langle\overrightarrow{\Box A}\rangle \Rightarrow B}\Box L \qquad \frac{\boxtimes\Xi \Rightarrow A}{\boxtimes\Xi \Rightarrow \Box A}\Box R$$

14. $$\frac{\boxtimes\Xi\langle\overrightarrow{A}\rangle \Rightarrow \diamondplus B}{\boxtimes\Xi\langle\overrightarrow{\Diamond A}\rangle \Rightarrow \diamondplus B}\Diamond L \qquad \frac{\Xi \Rightarrow A}{\Xi \Rightarrow \Diamond A}\Diamond R$$

Figure 3.5: Normal modality rules; $\boxtimes/\diamondplus$ marks a structure all the types of which have principal connective a box/diamond

Note how the stoup is identical in premises and conclusions in the normal modality rules. With respect to the (**S4**) normal modalities the universal (Morrill 1990[59]) has application to intensionality. For example, for a propositional attitude verb such as **believes** we can assign type $\Box((N\backslash S)/\Box S)$ with a modality outermost since the word has a sense, and a modality on the first argument but not the second, since the sentential complement is an intensional domain, but not the subject. The modalities are in the categorial type, distinctly from, but in relation to, the logical interpretation of the propositional attitude verb.

Bracket modalities

The bracket modalities $\{[\,]^{-1}, \langle\rangle\}$ of Morrill (1992[60]) and Moortgat (1995[53]), have application to prosodic/syntactic domains such as prosodic phrases and extraction islands or intonational domains. The rules are given in Figure 3.6.

15. $$\frac{\Xi\langle\overrightarrow{A}\rangle \Rightarrow B}{\Xi\langle[\overrightarrow{[\,]^{-1}A}]\rangle \Rightarrow B}[\,]^{-1}L \qquad \frac{[\Xi] \Rightarrow A}{\Xi \Rightarrow [\,]^{-1}A}[\,]^{-1}R$$

16. $$\frac{\Xi\langle[\overrightarrow{A}]\rangle \Rightarrow B}{\Xi\langle\overrightarrow{\langle\rangle A}\rangle \Rightarrow B}\langle\rangle L \qquad \frac{\Xi \Rightarrow A}{[\Xi] \Rightarrow \langle\rangle A}\langle\rangle R$$

Figure 3.6: Bracket modality rules

Notice how the stoup is identical in conclusions and premises of bracket modality rules.

By way of example of bracket modalities, we may assign **walks**: $\langle\rangle N\backslash S$ for the subject condition (Chomsky 1973[13]), and **before**: $[\,]^{-1}(VP\backslash VP)/VP$ for the adverbial island constraint, which are weak islands, and can contain parasitic gaps; for a strong island such as a coordinate structure, which cannot contain a parasitic gap, we define doubly bracketed strong islands — **and**: $(S\backslash[\,]^{-1}[\,]^{-1}S)/S$.

(23) $$\frac{\dfrac{N \Rightarrow N}{[N] \Rightarrow \langle\rangle N}\langle\rangle R \quad S \Rightarrow S}{[N], \langle\rangle N\backslash S \Rightarrow S}\backslash L$$

$$\frac{S \Rightarrow S \quad \dfrac{S \Rightarrow S \quad \dfrac{\dfrac{S \Rightarrow S}{[[\,]^{-1}S] \Rightarrow S}[\,]^{-1}L}{[[[\,]^{-1}[\,]^{-1}S]] \Rightarrow S}[\,]^{-1}L}{[[S, S\backslash[\,]^{-1}[\,]^{-1}S]] \Rightarrow S}\backslash S}{[[S, (S\backslash[\,]^{-1}[\,]^{-1}S)/S, S]] \Rightarrow S}/S$$

3.2.5 Subexponentials

The subexponentials $\{!, ?\}$ of Girard (1987[22]), Barry et al. (1991[9]), Morrill (1994[80]), Morrill (2011[81]), Morrill and Valentín (2015[73]), Morrill and Valentín (2016[75]), and Morrill (2017[66] have application to nonlinearity. The rules are given in Figure 3.7. Note that the rule for $?L$ is infinitary (we know of no linguistic need for it).

17. $$\frac{\Xi(\zeta \uplus \{A\}; \Gamma_1, \Gamma_2) \Rightarrow B}{\Xi(\zeta; \Gamma_1, !A, \Gamma_2) \Rightarrow B}!L \qquad \frac{\zeta; \Rightarrow A}{\zeta; \Rightarrow !A}!R$$

$$\frac{\Xi(\zeta; \Gamma_1, A, \Gamma_2) \Rightarrow B}{\Xi(\zeta \uplus \{A\}; \Gamma_1, \Gamma_2) \Rightarrow B}!P \qquad \frac{\Xi(\zeta \uplus \{A\}; \Gamma_1, [\{A\}; \Gamma_2], \Gamma_3) \Rightarrow B}{\Xi(\zeta \uplus \{A\}; \Gamma_1, \Gamma_2, \Gamma_3) \Rightarrow B}!C$$

18. $$\frac{\Xi(A) \Rightarrow D \qquad \Xi(A, A) \Rightarrow D \qquad \cdots}{\Xi(?A) \Rightarrow D}?L \qquad \frac{\Xi \Rightarrow A}{\Xi \Rightarrow ?A}?R$$

$$\frac{\Xi \Rightarrow A \qquad \Xi' \Rightarrow ?A}{\Xi, \Xi' \Rightarrow ?A}?M$$

Figure 3.7: Subexponential rules

Using the universal subexponential, !, we can assign a relative pronoun type

that: $(CN\backslash CN)/(S/!N)$

allowing both medial extraction (via the permutation rule) and parasitic extraction (via the contraction rule), Morrill (2011[81]), Morrill and Valentín (2015[73]), and Morrill (2017[66]), such as **paper that John filed without reading**: CN, where parasitic gaps can appear only in (weak) islands, but can be iterated in (weak) subislands, subsubislands, and so on. Using the existential exponential, ?, we can assign a coordinator type **and**: $(?N\backslash N)/N$ allowing iterated coordination as in **John, Bill, Mary and Suzy**: N.

3.3 Soundness of hDL for DL

Chapter 4

Further Types

In this chapter we present further types. There are semantically inactive primitive types and defined synthetic connectives, limited contraction and limited expansion, and the primitive difference operator. Including the **DL** primitive connectives from the previous chapter, all the connectives of Generalised Displacement Logic **GDL** are given in Figure 4.1.

1.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_{i+j}/\mathbf{Tp}_j$	$s(C/B)$	=	$s(C) - s(B)$	over [46]
2.	$\mathbf{Tp}_j$	::=	$\mathbf{Tp}_i\backslash\mathbf{Tp}_{i+j}$	$s(A\backslash C)$	=	$s(C) - s(A)$	under [46]
3.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_i\bullet\mathbf{Tp}_j$	$s(A\bullet B)$	=	$s(A) + s(B)$	times [46]
4.	$\mathbf{Tp}_0$	::=	I	$s(I)$	=	0	continuous unit [45]
5, k.	$\mathbf{Tp}_{i'}$	::=	$\mathbf{Tp}_{i+j}\uparrow_k\mathbf{Tp}_j, 1 \leq k \leq i'$	$s(C\uparrow_k B)$	=	$s(C)' - s(B)$	circumfix [79]
6, k.	$\mathbf{Tp}_j$	::=	$\mathbf{Tp}_{i'}\downarrow_k\mathbf{Tp}_{i+j}, 1 \leq k \leq i'$	$s(A\downarrow_k C)$	=	$s(C)' - s(A)$	infix [79]
7, k.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i'}\odot_k\mathbf{Tp}_j, 1 \leq k \leq i'$	$s(A\odot_k B)$	=	$s(A) + s(B) - 1$	wrap[79]
8.	$\mathbf{Tp}_1$	::=	J	$s(J)$	=	1	discontinuous unit [79]
9.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i\&\mathbf{Tp}_i$	$s(A\&B)$	=	$s(A) = s(B)$	with [44, 57]
10.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i\oplus\mathbf{Tp}_i$	$s(A\oplus B)$	=	$s(A) = s(B)$	plus [44, 57]
11.	$\mathbf{Tp}_i$	::=	$\bigwedge V\mathbf{Tp}_i$	$s(\bigwedge vA)$	=	$s(A)$	1st order univ. qu. [80]
12.	$\mathbf{Tp}_i$	::=	$\bigvee V\mathbf{Tp}_i$	$s(\bigvee vA)$	=	$s(A)$	1st order exist. qu. [80]
13.	$\mathbf{Tp}_i$	::=	$\Box\mathbf{Tp}_i$	$s(\Box A)$	=	$s(A)$	univ. norm. modality [59]
14.	$\mathbf{Tp}_i$	::=	$\Diamond\mathbf{Tp}_i$	$s(\Diamond A)$	=	$s(A)$	exist. norm. modality [54]
15.	$\mathbf{Tp}_i$	::=	$[\,]^{-1}\mathbf{Tp}_i$	$s([\,]^{-1}A)$	=	$s(A)$	univ. bracket modality [60, 53]
16.	$\mathbf{Tp}_i$	::=	$\langle\rangle\mathbf{Tp}_i$	$s(\langle\rangle A)$	=	$s(A)$	exist. bracket modality [60, 53]
17.	$\mathbf{Tp}_0$	::=	$!\mathbf{Tp}_0$	$s(!A)$	=	$s(A) = 0$	universal subexponential [9]
18.	$\mathbf{Tp}_0$	::=	$?\mathbf{Tp}_0$	$s(?A)$	=	$s(A) = 0$	existential subexponential [80]
19.	$\mathbf{Tp}_i$	:=	$\mathbf{Tp}_{i+j}\bullet\!\!-\mathbf{Tp}_j$	$s(C\bullet\!\!-B)$	=	$s(C) - s(B)$	left sem. inactive over [72]
20.	$\mathbf{Tp}_j$	:=	$\mathbf{Tp}_i\multimap\mathbf{Tp}_{i+j}$	$s(A\multimap C)$	=	$s(C) - s(A)$	left sem. inactive under [72]
21.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_i◐\mathbf{Tp}_j$	$s(A◐B)$	=	$s(A) + s(B)$	left sem. inactive times [72]
22.	$\mathbf{Tp}_i$	:=	$\mathbf{Tp}_{i+j}\circ\!\!-\mathbf{Tp}_j$	$s(C\circ\!\!-B)$	=	$s(C) - s(B)$	right sem. inactive over [72]
23.	$\mathbf{Tp}_j$	:=	$\mathbf{Tp}_i-\!\!\bullet\mathbf{Tp}_{i+j}$	$s(A-\!\!\bullet C)$	=	$s(C) - s(A)$	right sem. inactive under [72]
24.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_i◑\mathbf{Tp}_j$	$s(A◑B)$	=	$s(A) + s(B)$	right sem. inactive times [72]
25, k.	$\mathbf{Tp}_{i'}$	:=	$\mathbf{Tp}_{i+j}\,⫯_k\mathbf{Tp}_j, 1 \leq k \leq i+j$	$s(C\,⫯_k B)$	=	$s(C)' - s(B)$	upper sem. inactive circumfix [72]
26, k.	$\mathbf{Tp}_j$	:=	$\mathbf{Tp}_{i'}\,⫰_k\mathbf{Tp}_{i+j}, 1 \leq k \leq i'$	$s(A\,⫰_k C)$	=	$s(C)' - s(A)$	upper sem. inactive infix [72]
27, k.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i'}◓_k\mathbf{Tp}_j, 1 \leq k \leq i'$	$s(A◓_k B)$	=	$s(A) + s(B) - 1$	upper sem. inactive wrap [72]
28, k.	$\mathbf{Tp}_{i'}$	:=	$\mathbf{Tp}_{i+j}\,⫯_k\mathbf{Tp}_j, 1 \leq k \leq i'$	$T(C\,⫯_k B)$	=	$s(C)' - s(B)$	lower sem. inactive circumfix [72]
29, k.	$\mathbf{Tp}_j$	:=	$\mathbf{Tp}_{i'}\,⫰_k\mathbf{Tp}_{i+j}, 1 \leq k \leq i'$	$s(A\,⫰_k C)$	=	$s(C)' - s(A)$	lower sem. inactive infix [72]
30, k.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i'}◒_k\mathbf{Tp}_j, 1 \leq k \leq i'$	$s(A◒_k B)$	=	$s(A) + s(B) - 1$	lower sem. inactive wrap[72]
31.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i\sqcap\mathbf{Tp}_i$	$s(A\sqcap B)$	=	$s(A) = s(B)$	sem. inactive with [80]
32.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i\sqcup\mathbf{Tp}_i$	$s(A\sqcup B)$	=	$s(A) = s(B)$	sem. inactive plus [80]
33.	$\mathbf{Tp}_i$	::=	$\forall V\mathbf{Tp}_i$	$s(\forall vA)$	=	$s(A)$	sem. inactive 1st order univ. qu. [80]
34.	$\mathbf{Tp}_i$	::=	$\exists V\mathbf{Tp}_i$	$s(\exists vA)$	=	$s(A)$	sem. inactive 1st order exist. qu. [80]
35.	$\mathbf{Tp}_i$	::=	$\blacksquare\mathbf{Tp}_i$	$s(\blacksquare A)$	=	$s(A)$	sem. inactive univ. norm. modality [59]
36.	$\mathbf{Tp}_i$	::=	$\blacklozenge\mathbf{Tp}_i$	$s(\blacklozenge A)$	=	$s(A)$	sem. inactive exist. norm. modality [80]
37.	$\mathbf{Tp}_i$	::=	$\triangleleft^{-1}\mathbf{Tp}_{i'}$	$s(\triangleleft^{-1}A)$	=	$s(A) - 1$	left projection [78]
38.	$\mathbf{Tp}_{i'}$	::=	$\triangleleft\mathbf{Tp}_i$	$s(\triangleleft A)$	=	$s(A)'$	left injection [78]
39.	$\mathbf{Tp}_i$	::=	$\triangleright^{-1}\mathbf{Tp}_{i'}$	$s(\triangleright^{-1}A)$	=	$s(A) - 1$	right projection [78]
39.	$\mathbf{Tp}_{i'}$	::=	$\triangleleft\mathbf{Tp}_i$	$s(\triangleleft A)$	=	$s(A)'$	left injection [78]
40.	$\mathbf{Tp}_{i'}$	::=	$\triangleright\mathbf{Tp}_i$	$s(\triangleright A)$	=	$s(A)'$	right injection [78]
41, k.	$\mathbf{Tp}_{i'}$	::=	$\check{}^k\mathbf{Tp}_i$	$s(\check{}^k A)$	=	$s(A)'$	split [69]
42, k.	$\mathbf{Tp}_i$	::=	$\hat{}^k\mathbf{Tp}_{i'}$	$s(\hat{}^k A)$	=	$s(A) - 1$	bridge [69]
43.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_{i+j}\div\mathbf{Tp}_j^{\overline{p}}$	$s(B\div A)$	=	$s(B) - s(A)$	non-det. division [79]
44.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_i\circ\mathbf{Tp}_j$	$s(A\circ B)$	=	$s(A) + s(B)$	non-det. times [79]
45.	$\mathbf{Tp}_{i'}$	::=	$\mathbf{Tp}_{i+j}\stackrel{\wedge}{=}\mathbf{Tp}_j^{\overline{p}}$	$s(C\stackrel{\wedge}{=}B)$	=	$s(C)' - s(B)$	non-det. circumfix [79]
46.	$\mathbf{Tp}_j$	::=	$\mathbf{Tp}_{i'}^{\overline{p}}\stackrel{\vee}{=}\mathbf{Tp}_{i+j}$	$s(A\stackrel{\vee}{=}C)$	=	$s(C)' - s(A)$	non-det. infix [79]
47.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i'}\diamond\mathbf{Tp}_j$	$s(A\diamond B)$	=	$s(A) + s(B) - 1$	non-det. wrap [79]
48.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i+j}\vert\mathbf{Tp}_j$	$s(B\vert A)$	=	$s(B) \geq T(A)$	limited contraction [29]
49.	$\mathbf{Tp}_0$	::=	w	$s(w)$	=	0	limited expansion [72]
50.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i - \mathbf{Tp}_i$	$s(A - B)$	=	$s(A) = s(B)$	difference [71]

Figure 4.1: The types of Generalised Displacement Logic **GDL**

4.1 Semantically inactive primitives of EDL

The semantically inactive connectives divide into the semantically inactive multiplicatives (continuous and discontinuous), additives, quantifers, and (normal) modalities:

Semantically inactive multiplicatives	•− −∘	∘− −•	Morrill & Valentín (2014[72])
	◐	◑	
	$\uparrow_k$ $\downarrow_k$ (filled / hollow)	$\uparrow_k$ $\downarrow_k$ (hollow / filled)	
	◓$_k$	◒$_k$	
Semantically inactive additives	⊓	⊔	Morrill (1994[80])
Semantically inactive quantifiers	∀	∃	Morrill (1994[80])
Semantically inactive normal modalities	■	⧫	Hepple (1990[27]), Morrill (1994[80])

Multiplicatives

Syntactically, the rules for the semantically inactive connectives are exactly the same as for their semantically active counterparts; it is in their semantic labelling, given in Part II, that the semantically active and inactive rules differ. Thus the semantically inactive continuous multiplicative rules are as given in Figure 4.2 and the semantically inactive discontinuous multiplicative rules are as given in Figure 4.3.

19. $$\frac{\zeta_1; \Gamma \Rightarrow B \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\overrightarrow{C\bullet\!\!-B}, \Gamma\rangle, \Delta_2) \Rightarrow D}\,\bullet\!\!-L \qquad \frac{\zeta; \Gamma, \overrightarrow{B} \Rightarrow C}{\zeta; \Gamma \Rightarrow C\bullet\!\!-B}\,\bullet\!\!-R$$

20. $$\frac{\zeta_1; \Gamma \Rightarrow A \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\Gamma, \overrightarrow{A\multimap C}\rangle, \Delta_2) \Rightarrow D}\,\multimap L \qquad \frac{\zeta; \overrightarrow{A}, \Gamma \Rightarrow C}{\zeta; \Gamma \Rightarrow A\multimap C}\,\multimap R$$

21. $$\frac{\Xi\langle\overrightarrow{A}, \overrightarrow{B}\rangle \Rightarrow D}{\Xi\langle\overrightarrow{A\LEFTcircle B}\rangle \Rightarrow D}\,\LEFTcircle L \qquad \frac{\zeta_1; \Delta \Rightarrow A \qquad \zeta_2; \Gamma \Rightarrow B}{\zeta_1 \uplus \zeta_2; \Delta, \Gamma \Rightarrow A\LEFTcircle B}\,\LEFTcircle R$$

22. $$\frac{\zeta_1; \Gamma \Rightarrow B \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\overrightarrow{C\circ\!\!-B}, \Gamma\rangle, \Delta_2) \Rightarrow D}\,\circ\!\!-L \qquad \frac{\zeta; \Gamma, \overrightarrow{B} \Rightarrow C}{\zeta; \Gamma \Rightarrow C\circ\!\!-B}\,\circ\!\!-R$$

23. $$\frac{\zeta_1; \Gamma \Rightarrow A \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\Gamma, \overrightarrow{A-\!\!\bullet C}\rangle, \Delta_2) \Rightarrow D}\,-\!\!\bullet L \qquad \frac{\zeta; \overrightarrow{A}, \Gamma \Rightarrow C}{\zeta; \Gamma \Rightarrow A-\!\!\bullet C}\,-\!\!\bullet R$$

24. $$\frac{\Xi\langle\overrightarrow{A}, \overrightarrow{B}\rangle \Rightarrow D}{\Xi\langle\overrightarrow{A\RIGHTcircle B}\rangle \Rightarrow D}\,\RIGHTcircle L \qquad \frac{\zeta_1; \Delta \Rightarrow A \qquad \zeta_2; \Gamma \Rightarrow B}{\zeta_1 \uplus \zeta_2; \Delta, \Gamma \Rightarrow A\RIGHTcircle B}\,\RIGHTcircle R$$

Figure 4.2: Semantically inactive continuous multiplicative rules

25, k. $$\frac{\zeta_1; \Gamma \Rightarrow B \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\overrightarrow{C\overset{\bullet}{|}_k B} |_k \Gamma\rangle, \Delta_2) \Rightarrow D}\,\overset{\bullet}{|}_k L \qquad \frac{\zeta; \Gamma |_k \overrightarrow{B} \Rightarrow C}{\zeta; \Gamma \Rightarrow C\overset{\bullet}{|}_k B}\,\overset{\bullet}{|}_k R$$

26, k. $$\frac{\zeta_1; \Gamma \Rightarrow A \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\Gamma |_k \overrightarrow{A\underset{\circ}{|}_k C}\rangle, \Delta_2) \Rightarrow D}\,\underset{\circ}{|}_k L \qquad \frac{\zeta; \overrightarrow{A} |_k \Gamma \Rightarrow C}{\zeta; \Gamma \Rightarrow A\underset{\circ}{|}_k C}\,\underset{\circ}{|}_k R$$

27, k. $$\frac{\Xi\langle\overrightarrow{A} |_k \overrightarrow{B}\rangle \Rightarrow D}{\Xi\langle\overrightarrow{A\text{◓}_k B}\rangle \Rightarrow D}\,\text{◓}_k L \qquad \frac{\zeta_1; \Delta \Rightarrow A \qquad \zeta_2; \Gamma \Rightarrow B}{\zeta_1 \uplus \zeta_2; \Delta |_k \Gamma \Rightarrow A\text{◓}_k B}\,\text{◓}_k R$$

28, k. $$\frac{\zeta_1; \Gamma \Rightarrow B \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\overrightarrow{C\overset{\circ}{|}_k B} |_k \Gamma\rangle, \Delta_2) \Rightarrow D}\,\overset{\circ}{|}_k L \qquad \frac{\zeta; \Gamma |_k \overrightarrow{B} \Rightarrow C}{\zeta; \Gamma \Rightarrow C\overset{\circ}{|}_k B}\,\overset{\circ}{|}_k R$$

29, k. $$\frac{\zeta_1; \Gamma \Rightarrow A \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\Gamma |_k \overrightarrow{A\underset{\bullet}{|}_k C}\rangle, \Delta_2) \Rightarrow D}\,\underset{\bullet}{|}_k L \qquad \frac{\zeta; \overrightarrow{A} |_k \Gamma \Rightarrow C}{\zeta; \Gamma \Rightarrow A\underset{\bullet}{|}_k C}\,\underset{\bullet}{|}_k R$$

30, k. $$\frac{\Xi\langle\overrightarrow{A} |_k \overrightarrow{B}\rangle \Rightarrow D}{\Xi\langle\overrightarrow{A\text{◒}_k B}\rangle \Rightarrow D}\,\text{◒}_k L \qquad \frac{\zeta_1; \Delta \Rightarrow A \qquad \zeta_2; \Gamma \Rightarrow B}{\zeta_1 \uplus \zeta_2; \Delta |_k \Gamma \Rightarrow A\text{◒}_k B}\,\text{◒}_k R$$

Figure 4.3: Semantically inactive discontinuous multiplicative rules

Additives, quantifiers and normal modalities

The semantically inactive additives, quantifiers and normal modalities are as shown in Figures 4.4, 4.5, and 4.6 respectively.

$$31. \quad \frac{\Xi\langle\overrightarrow{A}\rangle \Rightarrow C}{\Xi\langle\overrightarrow{A\sqcap B}\rangle \Rightarrow C}\sqcap L_1 \qquad \frac{\Xi\langle\overrightarrow{B}\rangle \Rightarrow C}{\Xi\langle\overrightarrow{A\sqcap B}\rangle \Rightarrow C}\sqcap L_2$$

$$\frac{\Xi \Rightarrow A \qquad \Xi \Rightarrow B}{\Xi \Rightarrow A\sqcap B}\sqcap R$$

$$32. \quad \frac{\Xi\langle\overrightarrow{A}\rangle \Rightarrow C \qquad \Xi\langle\overrightarrow{B}\rangle \Rightarrow C}{\Xi\langle\overrightarrow{A\sqcup B}\rangle \Rightarrow C}\sqcup L$$

$$\frac{\Xi \Rightarrow A}{\Xi \Rightarrow A\sqcup B}\sqcup R_1 \qquad \frac{\Xi \Rightarrow B}{\Xi \Rightarrow A\sqcup B}\sqcup R_2$$

Figure 4.4: Semantically inactive additive rules

$$33. \quad \frac{\Xi\langle\overrightarrow{A[t/v]}\rangle \Rightarrow B}{\Xi\langle\overrightarrow{\forall vA}\rangle \Rightarrow B}\forall L \qquad \frac{\Xi \Rightarrow A[a/v]}{\Xi \Rightarrow \forall vA}\forall R^{\dagger}$$

$$34. \quad \frac{\Xi\langle\overrightarrow{A[a/v]}\rangle \Rightarrow B}{\Xi\langle\overrightarrow{\exists vA}\rangle \Rightarrow B}\exists L^{\dagger} \qquad \frac{\Xi \Rightarrow A[t/v]}{\Xi \Rightarrow \exists vA}\exists R$$

Figure 4.5: Semantically inactive quantifier rules, where † indicates that there is no a in the conclusion

$$35. \quad \frac{\Xi\langle\overrightarrow{A}\rangle \Rightarrow B}{\Xi\langle\overrightarrow{\blacksquare A}\rangle \Rightarrow B}\blacksquare L \qquad \frac{\boxtimes\Xi \Rightarrow A}{\boxtimes\Xi \Rightarrow \blacksquare A}\blacksquare R$$

$$36. \quad \frac{\boxtimes\Xi\langle\overrightarrow{A}\rangle \Rightarrow \diamondplus B}{\boxtimes\Xi\langle\overrightarrow{\blacklozenge A}\rangle \Rightarrow \diamondplus B}\blacklozenge L \qquad \frac{\Xi \Rightarrow A}{\Xi \Rightarrow \blacklozenge A}\blacklozenge R$$

Figure 4.6: Semantically inactive normal modality rules; $\boxtimes/\diamondplus$ marks a structure all the types of which have principal connective a box/diamond

4.2 Synthetic connectives of FDL

We consider in the following subsection unary defined synthetic connectives and binary defined synthetic connectives.

4.2.1 Unary synthetic multiplicatives

The unary (or deterministic) synthetic multiplicatives divide into projection and injection, which are continuous, and split and bridge, which are discontinuous:

Left and right projection and injection	$\triangleleft^{-1}$	$\triangleleft$	$\triangleright$	$\triangleright^{-1}$	Morrill, Valentín & Fadda (2009[78])
Split and bridge		$\check{}_k$	$\hat{}_k$		Morrill & Merenciano (1996[69])

They are defined as shown in Figure 4.7.

$$\begin{array}{rlllll}
\triangleright^{-1}A & =_{df} & J\multimap A & s(\triangleright^{-1}A) & = & s(A)-1 & \text{right projection [78]}\\
\triangleleft^{-1}A & =_{df} & A\multimapinv J & s(\triangleleft^{-1}A) & = & s(A)-1 & \text{left projection [78]}\\
\triangleright A & =_{df} & J\mathbin{\LEFTcircle}A & s(\triangleright A) & = & s(A)' & \text{right injection [78]}\\
\triangleleft A & =_{df} & A\mathbin{\RIGHTcircle}J & s(\triangleleft A) & = & s(A)' & \text{left injection [78]}\\
\check{}_k A & =_{df} & A\uparrow_k I & s(\check{}_k A) & = & s(A)' & \text{leftmost and rightmost split [69]}\\
\hat{}_k & =_{df} & A\odot_k I & s(\hat{}_k A) & = & s(A)-1 & \text{leftmost and rightmost bridge [69]}
\end{array}$$

Figure 4.7: Unary synthetic multiplicatives

Rules for unary synthetic multiplicatives

37. $$\frac{\Xi\langle\overrightarrow{A}\rangle \Rightarrow B}{\Xi\langle\overrightarrow{\triangleleft^{-1}A},1\rangle \Rightarrow B}\triangleleft^{-1}L \qquad \frac{\zeta;\Gamma,1\Rightarrow A}{\zeta;\Gamma\Rightarrow \triangleleft^{-1}A}\triangleleft^{-1}R$$

38. $$\frac{\Xi\langle\overrightarrow{A}\rangle \Rightarrow B}{\Xi\langle 1,\overrightarrow{\triangleright^{-1}A}\rangle \Rightarrow B}\triangleright^{-1}L \qquad \frac{\zeta;1,\Gamma\Rightarrow A}{\zeta;\Gamma\Rightarrow \triangleright^{-1}A}\triangleright^{-1}R$$

39. $$\frac{\Xi\langle\overrightarrow{A},1\rangle \Rightarrow B}{\Xi\langle\overrightarrow{\triangleleft A}\rangle \Rightarrow B}\triangleleft L \qquad \frac{\zeta;\Gamma\Rightarrow A}{\zeta;\Gamma,1\Rightarrow \triangleleft A}\triangleleft R$$

40. $$\frac{\Xi\langle 1,\overrightarrow{A}\rangle \Rightarrow B}{\Xi\langle\overrightarrow{\triangleright A}\rangle \Rightarrow B}\triangleright L \qquad \frac{\zeta;\Gamma\Rightarrow A}{\zeta;1,\Gamma\Rightarrow \triangleright A}\triangleright R$$

41, k. $$\frac{\Xi\langle\overrightarrow{B}\rangle \Rightarrow C}{\Xi\langle\overrightarrow{\check{}_k B}\,|_k\,\Lambda\rangle \Rightarrow C}\check{}_k L \qquad \frac{\zeta;\Delta\,|_k\,\Lambda\Rightarrow B}{\zeta;\Delta\Rightarrow \check{}_k B}\check{}_k R$$

42, k. $$\frac{\Xi\langle\overrightarrow{B}\,|_k\,\Lambda\rangle \Rightarrow C}{\Xi\langle\overrightarrow{\hat{}_k B}\rangle \Rightarrow C}\hat{}_k L \qquad \frac{\zeta;\Delta\Rightarrow B}{\zeta;\Delta\,|_k\,\Lambda\Rightarrow \hat{}_k B}\hat{}_k R$$

Figure 4.8: Unary synthetic multiplicative rules

4.2.2 Binary synthetic multiplicatives

The binary (or non-deterministic) synthetic connectives divide into non-deterministic continuous division and times, and non-deterministic discontinuous circumfix, infix and wrap:

Non-det. division and times	$\div$		$\circ$	Morrill, Valentín & Fadda (2011[79])
Non-det. infix, wrap and circumfix	$\stackrel{\vee}{=}$	$\diamond$	$\stackrel{\wedge}{=}$	Morrill, Valentín & Fadda (2011[79])

These are defined as shown in Figure 4.9.

$B{\div}A$	$=_{df}$	$(A\backslash B)\sqcap(B/A)$	$\{s\mid \forall s' \in A, s_3, +(s, s', s_3) \Rightarrow s_3 \in B\}$	$s(B{\div}A)$	$=$	$s(B) - s(A)$	non-det. division
$A{\circ}B$	$=_{df}$	$(A{\bullet}B)\sqcup(B{\bullet}A)$	$\{s_3\mid \exists s_1 \in A, s_2 \in B, +(s_1, s_2, s_3)\}$	$s(A{\circ}B)$	$=$	$s(A) + s(B)$	nond-et. times
$A\stackrel{\vee}{=}C$	$=_{df}$	$(A{\downarrow_1}C)\sqcap\cdots\sqcap(A{\downarrow_{s(A)}}C)$	$\{s_2\mid \forall s_1 \in A, s_3, \times(s_1, s_2, s_3) \Rightarrow s_3 \in C\}$	$s(A\stackrel{\vee}{=}C)$	$=$	$s(C)' - s(A)$	non-det. infix
$C\stackrel{\wedge}{=}B$	$=_{df}$	$(C{\uparrow_1}B)\sqcap\cdots\sqcap(C{\uparrow_{s(C)'-s(B)}}B)$	$\{s_1\mid \forall s_2 \in B, s_3, \times(s_1, s_2, s_3) \Rightarrow s_3 \in C\}$	$s(C\stackrel{\wedge}{=}B)$	$=$	$s(C)' - T(B)$	non-det. circumfix
$A{\diamond}B$	$=_{df}$	$(A{\odot_1}B)\sqcup\cdots\sqcup(A{\odot_{s(A)}}B)$	$\{s_3\mid \exists s_1 \in A, s_2 \in B, \times(s_1, s_2, s_3)\}$	$s(A{\diamond}B)$	$=$	$s(A)' - s(B)$	non-det. wrap

Figure 4.9: Binary synthetic multiplicatives

Rules for binary synthetic multiplicatives

43.
$$\frac{\zeta_1; \Gamma \Rightarrow A \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\Gamma, \overrightarrow{C{\div}A}\rangle, \Delta_2) \Rightarrow D}\div L_1 \qquad \frac{\zeta_1; \Gamma \Rightarrow A \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\overrightarrow{C{\div}A}, \Gamma\rangle, \Delta_2) \Rightarrow D}\div L_2$$

$$\frac{\zeta; \overrightarrow{A}, \Gamma \Rightarrow C \qquad \zeta; \Gamma, \overrightarrow{A} \Rightarrow C}{\zeta; \Gamma \Rightarrow C{\div}A}\div R$$

44.
$$\frac{\Xi\langle\overrightarrow{A}, \overrightarrow{B}\rangle \Rightarrow D \qquad \Xi\langle\overrightarrow{B}, \overrightarrow{A}\rangle \Rightarrow D}{\Xi\langle\overrightarrow{A{\circ}B}\rangle \Rightarrow D:}\circ L$$

$$\frac{\zeta_1; \Delta \Rightarrow A \qquad \zeta_2; \Gamma \Rightarrow B}{\zeta_1 \uplus \zeta_2; \Delta, \Gamma \Rightarrow A{\circ}B}\circ R_1 \qquad \frac{\zeta_1; \Delta \Rightarrow B \qquad \zeta_2; \Gamma \Rightarrow A}{\zeta_1 \uplus \zeta_2; \Delta, \Gamma \Rightarrow A{\circ}B}\circ R_2$$

45.
$$\frac{\zeta_1; \Gamma \Rightarrow B \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\overrightarrow{C\stackrel{\wedge}{=}B} |_k \Gamma\rangle, \Delta_2) \Rightarrow D}\stackrel{\wedge}{=}L \qquad \frac{\zeta; \Gamma |_1 \overrightarrow{B}: y \Rightarrow C \quad \cdots \quad \zeta; \Gamma |_{sC'-sB} \overrightarrow{B}: y \Rightarrow C}{\zeta; \Gamma \Rightarrow C\stackrel{\wedge}{=}B}\stackrel{\wedge}{=}R$$

46.
$$\frac{\zeta_1; \Gamma \Rightarrow A \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{C}\rangle, \Delta_2) \Rightarrow D:}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1, \langle\Gamma |_k \overrightarrow{A\stackrel{\vee}{=}C}\rangle, \Delta_2) \Rightarrow D}\stackrel{\vee}{=}L \qquad \frac{\zeta; \overrightarrow{A} |_1 \Gamma \Rightarrow C \quad \cdots \quad \zeta; \overrightarrow{A} |_{sA} \Gamma \Rightarrow C}{\zeta; \Gamma \Rightarrow A\stackrel{\vee}{=}C}\stackrel{\vee}{=}R$$

47.
$$\frac{\Xi\langle\overrightarrow{A} |_1 \overrightarrow{B}\rangle \Rightarrow D \quad \cdots \quad \Xi\langle\overrightarrow{A} |_{sA} \overrightarrow{B}\rangle \Rightarrow D}{\Xi\langle\overrightarrow{A{\diamond}B}\rangle \Rightarrow D}\diamond L \qquad \frac{\zeta_1; \Delta \Rightarrow A \qquad \zeta_2; \Gamma \Rightarrow B}{\zeta_1 \uplus \zeta_2; \Delta |_k \Gamma \Rightarrow A{\diamond}B}\diamond R$$

Figure 4.10: Binary synthetic multiplicative rules

4.3 Limited contraction and limited expansion connectives of GDL

The limited contraction |, of Figure 4.11, Jäger (2005[29]), has application to anaphora, and the limited expansion +, to words as types.

48. $$\frac{\zeta_1; \Gamma \Rightarrow A \qquad \Xi(\zeta_2; \Delta_1, \langle\overrightarrow{A}\rangle, \Delta_2\langle\overrightarrow{B}\rangle) \Rightarrow D}{\Xi(\zeta_1 \uplus \zeta_2; \Delta_1\langle\Gamma\rangle, \Delta_2\langle\overrightarrow{B|A}\rangle) \Rightarrow D}|L \qquad \frac{\Xi\langle\overrightarrow{B_0}; \ldots; \overrightarrow{B_n}\rangle \Rightarrow D}{\Xi\langle\overrightarrow{B_0|A}; \ldots; \overrightarrow{B_n|A}\rangle \Rightarrow D|A}|R$$

49. $$\frac{\Xi(\Lambda) \Rightarrow A}{\Xi(\overrightarrow{0} \Rightarrow A}+L \qquad \frac{}{0; \Lambda \Rightarrow 0}+R$$

$$\frac{\Xi\langle\overrightarrow{v+w}\rangle \Rightarrow D}{\Xi\langle\overrightarrow{v}, \overrightarrow{w}\rangle \Rightarrow D}+L \qquad \frac{\zeta_1; \Delta \Rightarrow v \qquad \zeta_2; \Gamma \Rightarrow w}{\zeta_1 \uplus \zeta_2; \Delta, \Gamma \Rightarrow v + w}+R$$

Figure 4.11: Limited contraction and limited expansion

The limited contraction can be used for anaphora in an assignment like **it**: $(S{\uparrow}N){\downarrow}(S|N)$ for, e.g., **the company**$_i$ **said it**$_i$ **flourished**: S, and it can be used for **such that** relativisation in an assignment **such that**: $(CN\backslash CN)/(S|N)$ for, say, **man such that**$_i$ **he**$_i$ **thinks Mary loves him**$_i$: CN. The interesting thing about this example is that it has a positive (succedent) occurrence of the Jäger bar $S!N$. This necessitates a pronoun in the corresponding antcecedent, and allows more such pronouns, all cobound as required to the head noun resticted by the **such that** relativisation.

The limited wexpansion + of Morrill and Valentín (2014[72]) has application to words as types, for example in relation to the semantically inactive multiplicatives, **is**: (**there**$\multimap S)/N$ for **There is John**. The intended reading of this example is that John exists: **there** makes no semantic contribution and is argument to a semantically inactive under.

4.4 The difference operator of GDL′

The difference operator is a connective for which the Cut-rule does not make sense . Therefore it is considered a metalogical connective added to the system **GDL** without Cut, yielding the system **GDL'**. The rules are as follows:

50. $$\frac{\Xi\langle A\rangle \Rightarrow C}{\Xi\langle A - B\rangle \Rightarrow C}-L \qquad \frac{\Xi \Rightarrow A}{\Xi \Rightarrow A - B}-R, \nvdash \Xi \Rightarrow B$$

Figure 4.12: Difference rules

The difference operator is a means to define exceptions. For example, to avoid generation of **the extremely man** with an intensifier type $(CN/CN)/(CN/CN)$ applying to the empty string of type CN/CN, we can instead assign an intensifier type $(CN/CN)/((CN/CN) - I)$ excluding the empty string.

Chapter 5

Completeness of hDL for syntactical models, and Cut-elimination

Part II

SEMANTICS

In this part we present the Curry-Howard type-logical semantics of Generalised Displacement Logic **GDL**. In Chapter 6 we recall the definition of syntactic types of Generalised Displacement Logic, adding the semantic type map to intuitionistic types. In Chapter 7 we define the semantic representation language. In Chapter 8 we present the semantically labelled sequent calculus for **GDL**.

Chapter 6

Generalised Displacement Logic GDL Types

In this chapter we see the semantic type map for Generalised Displacement Logic, **GDL**.

6.1 Semantic types

Recall the following operations on sets:

(24) a. Functional exponentiation: X^Y = the set of all total functions from Y to X

b. Cartesian product: $X \times Y = \{\langle x, y\rangle | \; x \in X \;\&\; y \in Y\}$

c. Disjoint union: $X \uplus Y = (\{1\} \times X) \cup (\{2\} \times Y)$

d. n-th Cross product, $n \in \mathcal{N}$: $$\begin{array}{rcl} X^0 & = & \{0\} \\ X^{n+1} & = & X^n \times X \end{array}$$

We shall use these to build semantic domains. The set $\mathcal{T}$ of *semantic types* of the semantic representation language is defined on the basis of a set δ of *basic semantic types* as follows:

(25) $\mathcal{T} ::= \delta \mid F \mid \top \mid \mathcal{T}{+}\mathcal{T} \mid \mathcal{T}\&\mathcal{T} \mid \mathcal{T}{\to}\mathcal{T} \mid \mathbf{M}\mathcal{T} \mid \mathbf{L}\mathcal{T} \mid \mathcal{T}^+$

A *semantic frame* comprises a family $\{D_\tau\}_{\tau\in\delta}$ of non-empty *basic type domains* and nonempty sets V of feature values and + of possible worlds. This induces a nonempty *type domain* D_τ for each type τ as follows:

(26)
$$\begin{array}{rcl} D_F & = & V \\ D_\top & = & \{\emptyset\} \\ D_{\tau_1+\tau_2} & = & D_{\tau_1} \uplus D_{\tau_2} \\ D_{\tau_1\&\tau_2} & = & D_{\tau_1} \times D_{\tau_2} \\ D_{\tau_1\to\tau_2} & = & D_{\tau_2}^{D_{\tau_1}} \\ D_{\mathbf{M}\tau} & = & W \times D_\tau \\ D_{\mathbf{L}\tau} & = & D_\tau^W \\ D_{\tau^+} & = & \bigcup_{n>0}(D_\tau)^n \end{array}$$

6.2 Syntactic types

The syntactic types of our categorial logic are sorted according to the number of points of discontinuity their expressions contain. Each *type predicate letter* will have a sort, the sort of it's expressions, and an arity, the number of feature term arguments it takes, which are naturals, and there is primitive semantic type map t. Assuming to be already given ordinary feature terms interpreted in the domain F, where P is a type predicate letter of sort i and arity n and $t_1, \ldots, t_n$ are terms, $Pt_1 \ldots t_n$ is an (atomic) type of sort i of the corresponding semantic type. Compound types are formed by connectives as shown in Figure 6.1, and the homomorphic semantic type map T associates these with semantic types and hence, through D, semantic domains.

1.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_{i+j}/\mathbf{Tp}_j$	$T(C/B)$	=	$T(B)\to T(C)$	over [46]
2.	$\mathbf{Tp}_j$	::=	$\mathbf{Tp}_i\backslash\mathbf{Tp}_{i+j}$	$T(A\backslash C)$	=	$T(A)\to T(C)$	under [46]
3.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_i\bullet\mathbf{Tp}_j$	$T(A\bullet B)$	=	$T(A)\&T(B)$	times [46]
4.	$\mathbf{Tp}_0$	::=	I	$T(I)$	=	$\top$	continuous unit [45]
5, k.	$\mathbf{Tp}_{i'}$	::=	$\mathbf{Tp}_{i+j}\uparrow_k\mathbf{Tp}_j, 1\le k\le i'$	$T(C\uparrow_k B)$	=	$T(B)\to T(C)$	circumfix [79]
6, k.	$\mathbf{Tp}_j$	::=	$\mathbf{Tp}_{i'}\downarrow_k\mathbf{Tp}_{i+j}, 1\le k\le i'$	$T(A\downarrow_k C)$	=	$T(A)\to T(C)$	infix [79]
7, k.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i'}\odot_k\mathbf{Tp}_j, 1\le k\le i'$	$T(A\odot_k B)$	=	$T(A)\&T(B)$	wrap [79]
8.	$\mathbf{Tp}_1$	::=	J	$T(J)$	=	$\top$	discontinuous unit [79]
9.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i\&\mathbf{Tp}_i$	$T(A\&B)$	=	$T(A)\&T(B)$	with [44, 57]
10.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i\oplus\mathbf{Tp}_i$	$T(A\oplus B)$	=	$T(A)+T(B)$	plus [44, 57]
11.	$\mathbf{Tp}_i$	::=	$\bigwedge V\mathbf{Tp}_i$	$T(\bigwedge vA)$	=	$F\to T(A)$	1st order univ. qu. [80]
12.	$\mathbf{Tp}_i$	::=	$\bigvee V\mathbf{Tp}_i$	$T(\bigvee vA)$	=	$F\&T(A)$	1st order exist. qu. [80]
13.	$\mathbf{Tp}_i$	::=	$\Box\mathbf{Tp}_i$	$T(\Box A)$	=	$\mathbf{L}T(A)$	univ. norm. modality [59]
14.	$\mathbf{Tp}_i$	::=	$\Diamond\mathbf{Tp}_i$	$T(\Diamond A)$	=	$\mathbf{M}T(A)$	exist. norm. modality [54]
15.	$\mathbf{Tp}_i$	::=	$[\,]^{-1}\mathbf{Tp}_i$	$T([\,]^{-1}A)$	=	$T(A)$	univ. brack. modality [60, 53]
16.	$\mathbf{Tp}_i$	::=	$\langle\rangle\mathbf{Tp}_i$	$T(\langle\rangle A)$	=	$T(A)$	exist. brack. modality [60, 53]
17.	$\mathbf{Tp}_0$	::=	$!\mathbf{Tp}_0$	$T(!A)$	=	$T(A)$	univ. subexponential [9]
18.	$\mathbf{Tp}_0$	::=	$?\mathbf{Tp}_0$	$T(?A)$	=	$T(A)^+$	exist. subexponential [80]
19.	$\mathbf{Tp}_i$	:=	$\mathbf{Tp}_{i+j}\bullet\!\!-\mathbf{Tp}_j$	$T(C\bullet\!\!-B)$	=	$\top$ if $T(C)=\top$	left sem. inactive over [72]
20.	$\mathbf{Tp}_j$	:=	$\mathbf{Tp}_i-\!\!\circ\mathbf{Tp}_{i+j}$	$T(A-\!\!\circ C)$	=	$T(C)$ if $T(A)=\top$	left sem. inactive under [72]
21.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_i$◐$\mathbf{Tp}_j$	$T(A$◐$B)$	=	$T(B)$ if $T(A)=\top$	left sem. inactive cont. product [72]
22.	$\mathbf{Tp}_i$	:=	$\mathbf{Tp}_{i+j}\circ\!\!-\mathbf{Tp}_j$	$T(C\circ\!\!-B)$	=	$T(C)$ if $T(B)=\top$	right sem. inactive over [72]
23.	$\mathbf{Tp}_j$	:=	$\mathbf{Tp}_i-\!\!\bullet\mathbf{Tp}_{i+j}$	$T(A-\!\!\bullet C)$	=	$\top$ if $T(C)=\top$	right sem. inactive under [72]
24.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_i$◑$\mathbf{Tp}_j$	$T(A$◑$B)$	=	$T(A)$ if $T(B)=\top$	right sem. inactive cont. product [72]
25, k.	$\mathbf{Tp}_{i'}$	:=	$\mathbf{Tp}_{i+j}\overset{\bullet}{\mid}_k\mathbf{Tp}_j, 1\le k\le i'$	$T(C\overset{\bullet}{\mid}_k B)$	=	$\top$ if $T(C)=\top$	upper sem. inactive extract [72]
26, k.	$\mathbf{Tp}_j$	:=	$\mathbf{Tp}_{i'}\underset{\circ}{\mid}_k\mathbf{Tp}_{i+j}, 1\le k\le i'$	$T(A\underset{\circ}{\mid}_k C)$	=	$T(C)$ if $T(A)=\top$	upper sem. inactive infix [72]
27, k.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i'}$◓$_k\mathbf{Tp}_j, 1\le k\le i'$	$T(A$◓$_k B)$	=	$T(B)$ if $T(A)=\top$	upper sem. inactive disc. product [72]
28, k.	$\mathbf{Tp}_{i'}$	:=	$\mathbf{Tp}_{i+j}\overset{\circ}{\mid}_k\mathbf{Tp}_j, 1\le k\le i'$	$T(C\overset{\circ}{\mid}_k B)$	=	$T(C)$ if $T(B)=\top$	lower sem. inactive extract [72]
29, k.	$\mathbf{Tp}_j$	:=	$\mathbf{Tp}_{i'}\underset{\bullet}{\mid}_k\mathbf{Tp}_{i+j}, 1\le k\le i'$	$T(A\underset{\bullet}{\mid}_k C)$	=	$\top$ if $T(C)=\top$	lower sem. inactive infix [72]
30, k.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i'}$◒$_k\mathbf{Tp}_j, 1\le k\le i'$	$T(A$◒$_k B)$	=	$T(A)$ if $T(B)=\top$	lower sem. inactive disc. product [72]
31.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i\sqcap\mathbf{Tp}_i$	$T(A\sqcap B)$	=	$T(A)=T(B)$	sem. inactive with [80]
32.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i\sqcup\mathbf{Tp}_i$	$T(A\sqcup B)$	=	$T(A)=T(B)$	sem. inactive plus [80]
33.	$\mathbf{Tp}_i$	::=	$\forall V\mathbf{Tp}_i$	$T(\forall vA)$	=	$T(A)$	sem. inactive 1st order univ. qu. [80]
34.	$\mathbf{Tp}_i$	::=	$\exists V\mathbf{Tp}_i$	$T(\exists vA)$	=	$T(A)$	sem. inactive 1st order exist. qu. [80]
35.	$\mathbf{Tp}_i$	::=	$\blacksquare\mathbf{Tp}_i$	$T(\blacksquare A)$	=	$T(A)$	sem. inactive universal modality [59]
36.	$\mathbf{Tp}_i$	::=	$\blacklozenge\mathbf{Tp}_i$	$T(\blacklozenge A)$	=	$T(A)$	sem. inactive existential modality [80]
37.	$\mathbf{Tp}_i$	::=	$\triangleleft^{-1}\mathbf{Tp}_{i'}$	$T(\triangleleft^{-1}A)$	=	$T(A)$	left projection [78]
38.	$\mathbf{Tp}_i$	::=	$\triangleright^{-1}\mathbf{Tp}_{i'}$	$T(\triangleright^{-1}A)$	=	$T(A)$	right projection [78]
39.	$\mathbf{Tp}_{i'}$	::=	$\triangleleft\mathbf{Tp}_i$	$T(\triangleleft A)$	=	$T(A)$	left injection [78]
40.	$\mathbf{Tp}_{i'}$	::=	$\triangleright\mathbf{Tp}_i$	$T(\triangleright A)$	=	$T(A)$	right injection [78]
41, k.	$\mathbf{Tp}_{i'}$	::=	$\check{}^{k}\mathbf{Tp}_i$	$T(\check{}^{k}A)$	=	$T(A)$	split [69]
42, k.	$\mathbf{Tp}_i$	::=	$\hat{}^{k}\mathbf{Tp}_{i'}$	$T(\hat{}^{k}A)$	=	$T(A)$	bridge [69]
43.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_{i+j}\div\mathbf{Tp}_j^{\overline{p}}$	$T(B\div A)$	=	$T(A)\to T(B)$	non-det. division [79]
44.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_i\circ\mathbf{Tp}_j$	$T(A\circ B)$	=	$T(A)\&T(B)$	non-det. continuous product [79]
45.	$\mathbf{Tp}_{i'}$	::=	$\mathbf{Tp}_{i+j}\overset{\wedge}{=}\mathbf{Tp}_j^{\overline{p}}$	$T(C\overset{\wedge}{=}B)$	=	$T(B)\to T(C)$	non-det. extract [79]
46.	$\mathbf{Tp}_j$	::=	$\mathbf{Tp}_{i'}^{\overline{p}}\overset{\vee}{=}\mathbf{Tp}_{i+j}$	$T(A\overset{\vee}{=}C)$	=	$T(A)\to T(C)$	non-det. infix [79]
47.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i'}\diamond\mathbf{Tp}_j$	$T(A\diamond B)$	=	$T(A)\&T(B)$	non-det. discontinuous product [79]
48.	$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i+j}\mid\mathbf{Tp}_j$	$T(B\mid A)$	=	$T(A)\to T(B)$	limited contraction [29]
49.	$\mathbf{Tp}_0$	::=	w	$T(w)$	=	$\top$	limited expansion [72]
50.	$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_i-\mathbf{Tp}_i$	$T(A-B)$	=	$T(A)$	difference [71]

Figure 6.1: Types of **GDL**

Chapter 7

Semantic Representation Language

In this chapter we define terms of the semantic representation language, interpretation of the semantic representation language, and conversion laws of the semantic representation language.

7.1 Terms

The sets Φ_τ of *terms* of type τ for each semantic type τ are defined on the basis of sets C_τ of constants of type τ and denumerably infinite sets V_τ of variables of type τ for each type τ as follows:

(27)	Φ_τ	::=	C_τ	constants
	Φ_τ	::=	V_τ	variables
	$\Phi_\top$	::=	0	null element
	Φ_τ	::=	$\Phi_{\tau_1+\tau_2}\text{–>}V_{\tau_1}.\Phi_\tau;\ V_{\tau_2}.\Phi_\tau$	case statement
	$\Phi_{\tau+\tau'}$	::=	$\iota_1\Phi_\tau$	first injection
	$\Phi_{\tau'+\tau}$	::=	$\iota_2\Phi_\tau$	second injection
	Φ_τ	::=	$\pi_1\Phi_{\tau\&\tau'}$	first projection
	Φ_τ	::=	$\pi_2\Phi_{\tau'\&\tau}$	second projection
	$\Phi_{\tau\&\tau'}$	::=	$(\Phi_\tau, \Phi_{\tau'})$	ordered pair formation
	Φ_τ	::=	$(\Phi_{\tau'\to\tau}\,\Phi_{\tau'})$	functional application
	$\Phi_{\tau\to\tau'}$	::=	$\lambda V_\tau\Phi_{\tau'}$	functional abstraction
	Φ_τ	::=	${}^\vee\Phi_{\mathbf{L}\tau}$	extensionalization
	$\Phi_{\mathbf{L}\tau}$	::=	${}^\wedge\Phi_\tau$	intensionalization
	Φ_τ	::=	${}^\cup\Phi_{\mathbf{M}\tau}$	projection
	$\Phi_{\mathbf{M}\tau}$	::=	${}^\cap\Phi_\tau$	injection
	Φ_{τ^+}	::=	$[\Phi_\tau] \mid [\Phi_\tau \vert \Phi_{\tau^+}]$	non-empty list construction

7.2 Interpretation

Given a semantic frame, a *valuation* f mapping each constant of type τ into an element of D_τ, an assignment g mapping each variable of type τ into an element of D_τ, and a world $i \in W$, each term ϕ of type τ receives an interpretation $[\phi]^{g,i} \in D_\tau$ as shown in Figure 7.1, where the *update* $g[x := m]$ is $(g - \{(x, g(x))\}) \cup \{(x, m)\}$, i.e. the function which sends x to m and agrees with g elsewhere.

In $x.\phi$, $\lambda x\phi$ or ${}^\wedge\phi$, ϕ is the *scope* of $x.$, λx or ${}^\wedge$. An occurrence of a variable x in a term is called *free* if and only if it does not fall within the scope of any $x.$ or λx; otherwise it is *bound* (by the closest $x.$ or λx within the scope of which it falls). The result $\phi\{\psi_1/x_1, \ldots, \psi_n/x_n\}$ of substituting terms $\psi_1, \ldots, \psi_n$ for variables $x_1, \ldots, x_n$ of the same types respectively in a term ϕ is the result of simultaneously replacing by ψ_i every free occurrence of x_i in ϕ. We say that ψ is *free for x in ϕ* if and only if no variable in ψ becomes bound in $\phi\{\psi/x\}$. We say that a term is *modally closed* if and only if every occurrence of ${}^\vee$ occurs within the scope of an ${}^\wedge$. A modally closed term is denotationally invariant across worlds. We say that a term ψ is

$$
\begin{array}{rcl}
[a]^{g,i} & = & f(a) \text{ for constant } a \in C_\tau \\
[x]^{g,i} & = & g(x) \text{ for variable } x \in V_\tau \\
[0]^{g,i} & = & \emptyset \\
[\phi\text{->}x.\psi; y.\chi]^{g,i} & = & \begin{cases} [\psi]^{g[x:=\mathbf{snd}([\phi]^{g,i})],i} & \text{if } \mathbf{fst}([\phi]^{g,i}) = 1 \\ [\chi]^{g[y:=\mathbf{snd}([\phi]^{g,i})],i} & \text{if } \mathbf{fst}([\phi]^{g,i}) = 2 \end{cases} \\
[\iota_1\phi]^{g,i} & = & \langle 1, [\phi]^{g,i}\rangle \\
[\iota_2\phi]^{g,i} & = & \langle 2, [\phi]^{g,i}\rangle \\
[\pi_1\phi]^{g,i} & = & \mathbf{fst}([\phi]^{g,i}) \\
[\pi_2\phi]^{g,i} & = & \mathbf{snd}([\phi]^{g,i}) \\
[(\phi,\psi)]^{g,i} & = & \langle [\phi]^{g,i}, [\psi]^{g,i}\rangle \\
[(\phi\,\psi)]^{g,i} & = & [\phi]^{g,i}([\psi]^{g,i}) \\
[\lambda x\phi]^{g,i} & = & d \mapsto [\phi]^{g[x:=d],i} \\
[{}^\vee\phi]^{g,i} & = & [\phi]^{g,i}(i) \\
[{}^\wedge\phi]^{g,i} & = & j \mapsto [\phi]^{g,j} \\
[{}^\cup\phi]^{g,i} & = & \mathbf{snd}([\phi]^{g,i}) \\
[{}^\cap\phi]^{g,i} & = & \langle i, [\phi]^{g,i}\rangle \\
[[\phi]]^{g,i} & = & \langle [\phi]^{g,i}, 0\rangle \\
[[\phi|\psi]]^{g,i} & = & \langle [\phi]^{g,i}, [\psi]^{g,i}\rangle
\end{array}
$$

Figure 7.1: Interpretation of the semantic representation language

modally free for x in ϕ if and only if either ψ is modally closed, or no free occurrence of x in ϕ is within the scope of an $^\wedge$.

7.3 Conversion

The laws of conversion in Figure 7.2 obtain. For completeness, the so-called commuting conversions for the case statement are given in Figure 7.3.

$$
\begin{array}{rcl}
\phi{-}{>}y.\psi;z.\chi & \equiv & \phi{-}{>}x.(\psi\{x/y\});z.\chi \\
 & \text{if } x \text{ is not free in } \psi & \\
 & \text{and is free for } y \text{ in } \psi & \\
\phi{-}{>}y.\psi;z.\chi & \equiv & \phi{-}{>}y.\psi;x.(\chi\{x/z\}) \\
 & \text{if } x \text{ is not free in } \chi & \\
 & \text{and is free for } z \text{ in } \chi & \\
\lambda y\phi & \equiv & \lambda x(\phi\{x/y\}) \\
 & \text{if } x \text{ is not free in } \phi & \\
 & \text{and is free for } y \text{ in } \phi & \\
 & \alpha\textit{-conversion} & \\
 & & \\
\iota_1\phi{-}{>}y.\psi;z.\chi & \equiv & \psi\{\phi/y\} \\
 & \text{if } \phi \text{ is free for } y \text{ in } \psi & \\
 & \text{and modally free for } y \text{ in } \psi & \\
\iota_2\phi{-}{>}y.\psi;z.\chi & \equiv & \chi\{\phi/z\} \\
 & \text{if } \phi \text{ is free for } z \text{ in } \chi & \\
 & \text{and modally free for } z \text{ in } \chi & \\
\pi_1(\phi,\psi) & \equiv & \phi \\
\pi_2(\phi,\psi) & \equiv & \psi \\
(\lambda x\phi\,\psi) & \equiv & \phi\{\psi/x\} \\
 & \text{if } \psi \text{ is free for } x \text{ in } \phi, & \\
 & \text{and modally free for } x \text{ in } \phi & \\
{}^{\vee\wedge}\phi & \equiv & \phi \\
{}^{\cup\cap}\phi & \equiv & \phi \\
 & \beta\textit{-conversion} & \\
 & & \\
(\pi_1\phi,\pi_2\phi) & \equiv & \phi \\
\lambda x(\phi\,x) & \equiv & \phi \\
 & \text{if } x \text{ is not free in } \phi & \\
{}^{\wedge\vee}\phi & \equiv & \phi \\
 & \text{if } \phi \text{ is modally closed} & \\
{}^{\cap\cup}\phi & \equiv & \phi \\
 & \eta\textit{-conversion} &
\end{array}
$$

Figure 7.2: Semantic conversion laws

$$\begin{array}{rcl}
\phi\text{-->}x.\iota_1\psi; y.\iota_1\chi & \equiv & \iota_1(\phi\text{-->}x.\psi; y.\chi) \\
\phi\text{-->}x.\iota_2\psi; y.\iota_2\chi & \equiv & \iota_2(\phi\text{-->}x.\psi; y.\chi) \\
\phi\text{-->}x.\pi_1\psi; y.\iota_1\chi & \equiv & \pi_1(\phi\text{-->}x.\psi; y.\chi) \\
\phi\text{-->}x.\pi_2\psi; y.\iota_2\chi & \equiv & \pi_2(\phi\text{-->}x.\psi; y.\chi) \\
\phi\text{-->}x.(\delta, \psi); y.(\delta, \chi) & \equiv & (\delta, \phi\text{-->}x.\psi; y.\chi) \\
\phi\text{-->}x.(\psi, \delta); y.(\chi, \delta) & \equiv & (\phi\text{-->}x.\psi; y.\chi, \delta) \\
\phi\text{-->}x.(\delta\,\psi); y.(\delta\,\chi) & \equiv & (\delta\,\phi\text{-->}x.\psi; y.\chi) \\
\phi\text{-->}x.(\psi\,\delta); y.(\chi\,\delta) & \equiv & (\phi\text{-->}x.\psi; y.\chi\,\delta)
\end{array}$$

$$\phi\text{-->}x.\lambda z\psi; y.\lambda z\chi \quad \equiv \quad \lambda z(\phi\text{-->}x.\psi; y.\chi)$$

if z is not free in ϕ

$$\phi\text{-->}x.{}^{\vee}\psi; y.{}^{\vee}\chi \quad \equiv \quad {}^{\vee}(\phi\text{-->}x.\psi; y.\chi)$$

$$\phi\text{-->}x.{}^{\wedge}\psi; y.{}^{\wedge}\chi \quad \equiv \quad {}^{\wedge}(\phi\text{-->}x.\psi; y.\chi)$$

if ϕ is modally closed

$$\begin{array}{rcl}
\phi\text{-->}x.{}^{\cup}\psi; y.{}^{\cup}\chi & \equiv & {}^{\cup}(\phi\text{-->}x.\psi; y.\chi) \\
\phi\text{-->}x.{}^{\cap}\psi; y.{}^{\cap}\chi & \equiv & {}^{\cap}(\phi\text{-->}x.\psi; y.\chi) \\
\phi\text{-->}x.[\delta|\psi]; y.[\delta|\chi] & \equiv & [\delta|\phi\text{-->}x.\psi; y.\chi] \\
\phi\text{-->}x.[\psi|\delta]; y.[\chi|\delta] & \equiv & [\phi\text{-->}x.\psi; y.\chi|\delta]
\end{array}$$

Figure 7.3: Semantic commuting conversion laws

Chapter 8

Semantically Labelled Sequent Calculus

In this chapter we give semantically labelled Gentzen sequent calculus rules according to Curry-Howard formulas-as-types/proofs-as-programs (Girard, Tayler and Lafont 1989[24]), but we shall do this in relation to the enriched notion of hedge sequent calculus.

8.1 Hedge sequent calculus

We recall the notions of hedge sequent calculus from Chapter 3. In Gentzen sequent calculus for displacement calculus with bracket modalities (structural inhibition) the left hand, antecedent, sides of sequents contain bracket constructors, but subexponentials (structural facilitation) have no special sequent punctuation. There is motivation to provide a special antecedent notation of 'stoups' for structural facilitation as well as that of brackets for structural inhibition. For example, the permutation rules for ! mean that the calculus without stoups does not have the finite proof property: the permutation rules can be applied in a senseless cyclic fashion to generate an infinite number of proofs of the same theorem. The use of stoups restores the finite proof property and it also facilitates the proof-theoretic property of focusing (Chapter 9).

Stoups (cf. the linear logic of Girard 2011[23]) are stores read as multisets for re-usable (non-linear) resources which appear at the left of a configuration marked off by a semicolon (when the stoup is empty the semicolon may be omitted). The stoup of linear logic is for resources which can be contracted (copied) or weakened (deleted). By contrast, our stoup is for a linguistically motivated variant of contraction, and does not allow weakening. Furthermore, whereas linear logic is commutative, our logic is in general noncommutative and the stoup is also used for resources which are commutative.[1]

A configuration together with a stoup is a *zone*. The bracket constructor applies not to a configuration alone but to a configuration with a stoup, i.e a zone: a reusable resource is specific to its bracketed domain.

Zones **Zone** (notated Ξ, possibly with subindices), stoups **Stoup** (notated ζ, possibly with subindices) and configurations **Config** (notated Γ, Δ, possibly with subindices) are defined by ($\emptyset$ is the empty stoup; Λ is the empty configuration; the *separator* 1 marks points of discontinuity.); note that only types of sort 0 can go into the stoup: reusable types of other sorts would not preserve the sequent antecedent-succedent sort equality under contraction:

(28)
$$\begin{array}{rcl}
\textbf{Zone} & ::= & \textbf{Stoup}; \textbf{Config} \\
\textbf{Stoup} & ::= & \emptyset \mid \textbf{Tp}_0, \textbf{Stoup} \\
\textbf{Config} & ::= & \Lambda \mid \textbf{TreeTerm}, \textbf{Config} \\
\textbf{TreeTerm} & ::= & 1 \mid \textbf{Tp}_0 \mid \textbf{Tp}_{i>0}\{\underbrace{\textbf{Config} : \ldots : \textbf{Config}}_{i\ \textbf{Config}\text{'s}}\} \mid [\textbf{Zone}]
\end{array}$$

[1] To anticipate linguistically a little, a hypothetical subtype emitted by a relative pronoun corresponding to a long-distance dependency will enter a stoup, percolate in stoups, maybe contracting to create (parasitic) gaps, and finally permute into a (host) extraction site.

For a type A, its sort $s(A)$ is the i such that $A \in \mathbf{Tp}_i$; for example:

(29) $s((S{\uparrow_1}N){\uparrow_1}N) = s((S{\uparrow_1}N){\uparrow_2}N) = 2$

For a configuration Γ, its sort $s(\Gamma)$ is $|\Gamma|_1$, i.e. the number of points of discontinuity 1 which Γ contains. For a zone Ξ, its sort $s(\Xi)$ is the sort of its configuration since stoup types are of sort is 0; for example:

(30) $s(N; 1, 1, (S{\uparrow_1}N){\uparrow_2}N\{N/CN, CN : 1\}) = s(1, 1, (S{\uparrow_1}N){\uparrow_2}N\{N/CN, CN : 1\}) = 3$

Sequents are of the form:

(31) $\mathbf{Zone} \Rightarrow \mathbf{Tp}$ such that $s(\mathbf{Zone}) = s(\mathbf{Tp})$

A sequent $\Xi \Rightarrow A$ is *valid*, $\models \Xi \Rightarrow A$, iff $[\Xi] \subseteq [A]$ in every interpretation.

The *figure* $\overrightarrow{A}$ of a type A is defined by:

$$(32)\quad \overrightarrow{A} = \begin{cases} A & \text{if } s(A) = 0 \\ A\{\underbrace{1 : \ldots : 1}_{s(A)\ 1's}\} & \text{if } s(A) > 0 \end{cases}$$

Where Γ is a configuration of sort i and $\Delta_1, \ldots, \Delta_i$ are configurations, the *fold* $\Gamma \otimes \langle \Delta_1 : \ldots : \Delta_i \rangle$ is the result of replacing the successive 1's in Γ by $\Delta_1, \ldots, \Delta_i$ respectively; similarly, where Ξ is a zone of sort i and $\Delta_1, \ldots, \Delta_i$ are configurations, the *fold* $\Xi \otimes \langle \Delta_1 : \ldots : \Delta_i \rangle$ is the result of replacing the successive 1's in the configuration of Ξ by $\Delta_1, \ldots, \Delta_i$ respectively.

Where Γ is a configuration of sort i, the hyperoccurrence notation $\Delta\langle\Gamma\rangle$ abbreviates $\Delta_0(\Gamma \otimes \langle \Delta_1 : \ldots : \Delta_i \rangle)$, i.e. a context configuration Δ (which is externally Δ_0 and internally $\Delta_1, \ldots, \Delta_i$) with a potentially discontinuous distinguished subconfiguration Γ; similarly, where Ξ is a zone of sort i, the hyperoccurrence notation $\Xi\langle\Gamma\rangle$ abbreviates $\zeta; \Delta_0(\Gamma \otimes \langle \Delta_1 : \ldots : \Delta_i \rangle)$ where $\Xi = \zeta; \Delta$, i.e. a context zone $\zeta; \Delta$ (which is externally $\zeta; \Delta_0$ and internally $\Delta_1, \ldots, \Delta_i$) with a potentially discontinuous distinguished subconfiguration Γ.

Where Δ is a configuration of sort $i > 0$ and Γ is a configuration, the kth *metalinguistic intercalation* $\Delta|_k \Gamma$, $1 \leq k \leq i$, is given by:

$$(33)\quad \Delta|_k \Gamma =_{df} \Delta \otimes \langle \underbrace{1 : \ldots : 1}_{k-1\ 1\text{'s}} : \Gamma : \underbrace{1 : \ldots : 1}_{i-k\ 1\text{'s}} \rangle$$

i.e. the kth metalinguistic intercalation $\Delta|_k \Gamma$ is the configuration resulting from replacing by Γ the kth separator in Δ.

A semantically labelled sequent is a sequent in which the antecedent syntactic type occurrences A_1, $\ldots$, A_n are labelled by distinct variables $x_1, \ldots, x_n$ of semantic types $T(A_1), \ldots, T(A_n)$ respectively, and the succedent type A is labelled by a term of type $T(A)$ with free variables drawn from $x_1, \ldots, x_n$.

8.2 The identity rules

The identity axiom is (34a); since we adopt the convention that empty stoups can be omitted, we write (34b):

$$(34)\quad \text{a.}\ \frac{}{\emptyset; \overrightarrow{P} : x \Rightarrow P : x}\, id \qquad \text{b.}\ \frac{}{\overrightarrow{P} : x \Rightarrow P : x}\, id$$

This states that a type is derivable from itself, asserting the reflexivity of set inclusion and the derivability relation. The Cut rule is:

$$(35)\quad \frac{\zeta_1; \Gamma \Rightarrow A \qquad \zeta_2; \Delta\langle\overrightarrow{A}\rangle \Rightarrow B}{\zeta_1 \uplus \zeta_2; \Delta\langle\Gamma\rangle \Rightarrow B}\, Cut$$

This states the contextual generalisation of the transitivity of set inclusion and the derivability relation.

8.3 Continuous multiplicatives

The continuous multiplicatives $\{/, \backslash, \bullet, I\}$ of Figure 8.1, Lambek (1958[46]; 1988[45]), are defined in relation to concatenation; they are the basic means of categorial (sub)categorization.

1. $$\frac{\zeta_1; \Gamma \Rightarrow B{:}\, \psi \qquad \zeta_2; \Delta\langle \overrightarrow{C}: z\rangle \Rightarrow D{:}\, \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \overrightarrow{C/B}{:}\, x, \Gamma\rangle \Rightarrow D{:}\, \omega\{(x\ \psi)/z\}} /L \qquad \frac{\zeta; \Gamma, \overrightarrow{B}: y \Rightarrow C{:}\, \chi}{\zeta; \Gamma \Rightarrow C/B{:}\, \lambda y \chi} /R$$

2. $$\frac{\zeta_1; \Gamma \Rightarrow A{:}\, \phi \qquad \zeta_2; \Delta\langle \overrightarrow{C}: z\rangle \Rightarrow D{:}\, \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \Gamma, \overrightarrow{A\backslash C}{:}\, y\rangle \Rightarrow D{:}\, \omega\{(y\ \phi)/z\}} \backslash L \qquad \frac{\zeta; \overrightarrow{A}: x, \Gamma \Rightarrow C{:}\, \chi}{\zeta; \Gamma \Rightarrow A\backslash C{:}\, \lambda x \chi} \backslash R$$

3. $$\frac{\Xi\langle \overrightarrow{A}: x, \overrightarrow{B}: y\rangle \Rightarrow D{:}\, \omega}{\Xi\langle \overrightarrow{A \bullet B}{:}\, z\rangle \Rightarrow D{:}\, \omega\{\pi_1 z/x, \pi_2 z/y\}} \bullet L \qquad \frac{\zeta_1; \Gamma_1 \Rightarrow A{:}\, \phi \qquad \zeta_2; \Gamma_2 \Rightarrow B{:}\, \psi}{\zeta_1 \uplus \zeta_2; \Gamma_1, \Gamma_2 \Rightarrow A \bullet B{:}\, (\phi, \psi)} \bullet R$$

4. $$\frac{\Xi\langle \Lambda\rangle \Rightarrow A{:}\, \phi}{\Xi\langle \overrightarrow{I}: x\rangle \Rightarrow A{:}\, \phi} IL \qquad \frac{}{\Lambda \Rightarrow I{:}\, 0} IR$$

Figure 8.1: Continuous multiplicative rules

The directional divisions over, $/$, and under, $\backslash$, are exemplified by assignments such as **the**: $N/CN : \iota$ for **the man**: $N : (\iota\ man)$ and **sings**: $N\backslash S : sing$ for **John sings**: $S : (sing\ j)$, and **loves**: $(N\backslash S)/N : love$ for **John loves Mary**: $S : ((love\ m)\ j)$. Hence, for **the man**:

(36) $$\dfrac{CN \Rightarrow CN \qquad N \Rightarrow N}{N/CN, CN \Rightarrow N}\ /L$$

And for **John sings** and **John loves Mary**:

(37) $$\dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\ \backslash L \qquad \dfrac{N \Rightarrow N \qquad \dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\ \backslash L}{N, (N\backslash S)/N, N \Rightarrow S}\ /L$$

The continuous product $\bullet$ is exemplified by a ‘small clause’ assignment

$$\textbf{considers}: (N\backslash S)/(N\bullet(CN/CN)) : \lambda x(consider\ ((\pi_2 x\ \lambda y[y = \pi_1 x])\ \pi_1 x))$$

for **John considers Mary socialist**: S: $((consider\ ((socialist\ \lambda y[y = m])\ m))\ j)$.

(38) $$\dfrac{\dfrac{N \Rightarrow N \qquad \dfrac{\dfrac{CN \Rightarrow CN \qquad CN \Rightarrow CN}{CN/CN, CN \Rightarrow CN}\ /L}{CN/CN \Rightarrow CN/CN}\ /R}{N, CN/CN \Rightarrow N\bullet(CN/CN)}\ \bullet R \qquad \dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\ \backslash L}{N, (N\backslash S)/(N\bullet(CN/CN)), N, CN/CN \Rightarrow S}\ /L$$

Of course this use of product is not essential: we could just as well have used $((N\backslash S)/(CN/CN))/N$ since in general we have both $A/(C\bullet B) \Rightarrow (A/B)/C$ (currying) and $(A/B)/C \Rightarrow A/(C\bullet B)$ (uncurrying). For essential use of product, in the antecedent, see the treatment of past participles in Morrill (2011[81]), of which the assignment to **loved** in Chapter 11 here is a discontinuous generalisation.[2]

8.4 Discontinuous multiplicatives

The discontinuous multiplicatives $\{\uparrow, \downarrow, \odot, J\}$ of Figure 8.2, the displacement connectives, Morrill and Valentín (2010[70]), Morrill, Valentín and Fadda (2011[79]), are defined in relation to intercalation; they are the basic means of obtaining displacement effects.

[2] The continuous unit can be used together with additive disjunction to express the optionality of a complement as in **eats**: $(N\backslash S)/(N\oplus I)$ for **John eats fish**: S and **John eats**: S. It can also we used in conjunction with difference to prevent the null string being supplied as argument to an intensifier as in **very**: $(CN/CN)/((CN/CN) - I)$ for **very tall man**: CN but ***very man**: CN.

5. $$\frac{\zeta_1;\Gamma\Rightarrow B\colon\psi \qquad \zeta_2;\Delta\langle\overrightarrow{C}\colon z\rangle\Rightarrow D\colon\omega}{\zeta_1\uplus\zeta_2;\Delta\langle\overrightarrow{C{\uparrow_k}B}\colon x\,|_k\,\Gamma\rangle\Rightarrow D\colon\omega\{(x\ \psi)/z\}}\uparrow_k L \qquad \frac{\zeta;\Gamma\,|_k\,\overrightarrow{B}\colon y\Rightarrow C\colon\chi}{\zeta;\Gamma\Rightarrow C{\uparrow_k}B\colon\lambda y\chi}\uparrow_k R$$

6. $$\frac{\zeta_1;\Gamma\Rightarrow A\colon\phi \qquad \zeta_2;\Delta\langle\overrightarrow{C}\colon z\rangle\Rightarrow D\colon\omega}{\zeta_1\uplus\zeta_2;\Delta\langle\Gamma\,|_k\,\overrightarrow{A{\downarrow_k}C}\colon y\rangle\Rightarrow D\colon\omega\{(y\ \phi)/z\}}\downarrow_k L \qquad \frac{\zeta;\overrightarrow{A}\colon x\,|_k\,\Gamma\Rightarrow C\colon\chi}{\zeta;\Gamma\Rightarrow A{\downarrow_k}C\colon\lambda x\chi}\downarrow_k R$$

7. $$\frac{\Xi\langle\overrightarrow{A}\colon x\,|_k\,\overrightarrow{B}\colon y\rangle\Rightarrow D\colon\omega}{\Xi\langle\overrightarrow{A{\odot_k}B}\colon z\rangle\Rightarrow D\colon\omega\{\pi_1 z/x,\pi_2 z/y\}}\odot_k L \qquad \frac{\zeta_1;\Gamma_1\Rightarrow A\colon\phi \qquad \zeta_2;\Gamma_2\Rightarrow B\colon\psi}{\zeta_1\uplus\zeta_2;\Gamma_1\,|_k\,\Gamma_2\Rightarrow A{\odot_k}B}\odot_k R$$

8. $$\frac{\Xi\langle 1\rangle\Rightarrow A\colon\phi}{\Xi\langle\overrightarrow{J}\colon x\rangle\Rightarrow A\colon\phi}JL \qquad \frac{}{1\Rightarrow J\colon 0}JR$$

Figure 8.2: Discontinuous multiplicative rules

When the value of the k subscript is one it may be omitted, i.e. it defaults to one. Circumfixation, $\uparrow$, is exemplified by a discontinuous idiom assignment **gives**+1+**the**+**cold**+**shoulder**: $(N\backslash S){\uparrow}N\colon$ *shun* for **Mary gives the man the cold shoulder**: $S\colon((\mathit{shun}\ (\iota\ \mathit{man}))\ m)$:

(39) $$\frac{\dfrac{CN\Rightarrow CN \qquad N\Rightarrow N}{N/CN, CN\Rightarrow N}/L \qquad \dfrac{N\Rightarrow N \qquad S\Rightarrow S}{N, N\backslash S\Rightarrow S}\backslash L}{N,(N\backslash S){\uparrow}N\{N/CN, CN\}\Rightarrow S}\uparrow L$$

Infixation, $\downarrow$, and extractation together are exemplified by a quantifier assignment

$$\textbf{everyone}\colon (S{\uparrow}N){\downarrow}S\colon \lambda x\forall y[(\mathit{person}\ y)\rightarrow(x\ y)]$$

simulating Montague's S14 quantifying in:

(40) $$\frac{\dfrac{\ldots,N,\ldots\ \Rightarrow S}{\ldots,1,\ldots\ \Rightarrow S{\uparrow}N}\uparrow R \qquad \dfrac{}{S\Rightarrow S}id}{\ldots,(S{\uparrow}N){\downarrow}S,\ldots\ \Rightarrow S}\downarrow L$$

Circumfixation and discontinuous product, $\odot$, are illustrated in an assignment to a relative pronoun

$$\textbf{that}\colon (CN\backslash CN)/((S{\uparrow}N)\odot I)\colon \lambda x\lambda y\lambda z[(y\ z)\wedge(\pi_1 x\ z)]$$

allowing both peripheral and medial extraction, **that John likes**: $CN\backslash CN\colon \lambda y\lambda z[(y\ z)\wedge((\mathit{like}\ z)\ j)]$ and **that John saw today**: $CN\backslash CN\colon \lambda y\lambda z[(y\ z)\wedge(\mathit{today}\ ((\mathit{see}\ z)\ j))]$:[3]

(41) $$\frac{\dfrac{\dfrac{N,(N\backslash S)/N, N\Rightarrow S}{N,(N\backslash S)/N, 1\Rightarrow S{\uparrow}N}\uparrow R \qquad \dfrac{}{\Rightarrow I}IL}{N,(N\backslash S)/N\Rightarrow (S{\uparrow}N)\odot I}\odot R \qquad CN\backslash CN\Rightarrow CN\backslash CN}{(CN\backslash CN)/((S{\uparrow}N)\odot I), N,(N\backslash S)/N\Rightarrow CN\backslash CN}/L$$

(42) $$\frac{\dfrac{\dfrac{N,(N\backslash S)/N, N, S\backslash S\Rightarrow S}{N,(N\backslash S)/N, 1, S\backslash S\Rightarrow S{\uparrow}N}\uparrow R \qquad \dfrac{}{\Rightarrow I}IL}{N,(N\backslash S)/N, S\backslash S\Rightarrow (S{\uparrow}N)\odot I}\odot R \qquad CN\backslash CN\Rightarrow CN\backslash CN}{(CN\backslash CN)/((S{\uparrow}N)\odot I), N,(N\backslash S)/N, S\backslash S\Rightarrow CN\backslash CN}/L$$

[3]Use of the discontinuous product unit, J, in conjunction with difference is illustrated in a pronoun assignment **him**: $(((S{\uparrow}N){\uparrow_2}N)-(J\bullet((N\backslash S){\uparrow}N)){\downarrow_2}(S{\uparrow}N)$ blocking a subject antecedent (a Principle B effect).

8.5 Additives

The additive conjunction and disjunction {&, ⊕} of Figure 8.3, Lambek (1961[44]), Morrill (1990[57]), and Kanazawa (1992[32]), capture polymorphism.

9.
$$\frac{\Xi\langle\overrightarrow{A}: x\rangle \Rightarrow C: \chi}{\Xi\langle\overrightarrow{A\&B}: z\rangle \Rightarrow C: \chi\{\pi_1 z/x\}}\ \&L_1 \qquad \frac{\Xi\langle\overrightarrow{B}: y\rangle \Rightarrow C: \chi}{\Xi\langle\overrightarrow{A\&B}: z\rangle \Rightarrow C: \chi\{\pi_2 z/y\}}\ \&L_2$$

$$\frac{\Xi \Rightarrow A: \phi \qquad \Xi \Rightarrow B: \psi}{\Xi \Rightarrow A\&B: (\phi, \psi)}\ \&R$$

10.
$$\frac{\Xi\langle\overrightarrow{A}: x\rangle \Rightarrow C: \chi_1 \qquad \Xi\langle\overrightarrow{B}: y\rangle \Rightarrow C: \chi_2}{\Xi\langle\overrightarrow{A\oplus B}: z\rangle \Rightarrow C: z{-}{>}x.\chi_1; y.\chi_2}\ \oplus L$$

$$\frac{\Xi \Rightarrow A: \phi}{\Xi \Rightarrow A\oplus B: \iota_1\phi}\ \oplus R_1 \qquad \frac{\Xi \Rightarrow B: \psi}{\Xi \Rightarrow A\oplus B: \iota_2\psi}\ \oplus R_2$$

Figure 8.3: Additive rules

For example the additive conjunction & can be used for **rice**: $N\&CN$: $((gen\ rice), rice)$ as in **rice grows**: S: $(grow\ (gen\ rice))$ and **the rice grows**: S: $(grow\ (\iota\ rice))$:

(43)
$$\frac{\dfrac{N \Rightarrow N}{N\&CN \Rightarrow N}\ \&L_1 \qquad S \Rightarrow S}{N\&CN, N\backslash S \Rightarrow S}\ \backslash L \qquad \frac{N/CN, CN, N\backslash S \Rightarrow S}{N/CN, N\&CN, N\backslash S \Rightarrow S}\ \&L_2$$

The additive disjunction ⊕ can be used for **is**: $(N\backslash S)/(N\oplus(CN/CN))$: $\lambda x\lambda y x \to z.[z = y]; w.((w\ \lambda u[y = u])\ y)$ as in **Tully is Cicero**: S: $[t = c]$ and **Tully is humanist**: S: $((humanist\ \lambda u[t = u])\ t)$:

(44)
$$\frac{\dfrac{N \Rightarrow N}{N \Rightarrow N\oplus(CN/CN)}\ \oplus R_1 \qquad N\backslash S \Rightarrow N\backslash S}{(N\backslash S)/(N\oplus(CN/CN)), N \Rightarrow N\backslash S}\ /L \qquad \frac{\dfrac{CN/CN \Rightarrow CN/CN}{CN/CN \Rightarrow N\oplus(CN/CN)}\ \oplus R_2 \qquad N\backslash S \Rightarrow N\backslash S}{(N\backslash S)/(N\oplus(CN/CN)), CN/CN \Rightarrow N\backslash S}\ /L$$

8.6 Quantifiers

The 1st order quantifiers $\{\bigwedge, \bigvee\}$ of Figure 8.4, Morrill (1994[80]), have application to features.

$$11. \quad \frac{\Xi\langle\overrightarrow{A[t/v]}: x\rangle \Rightarrow B: \psi}{\Xi\langle\overrightarrow{\bigwedge vA}: z\rangle \Rightarrow B: \psi\{(z\ t)/x\}} \bigwedge L \qquad \frac{\Xi \Rightarrow A[a/v]: \phi}{\Xi \Rightarrow \bigwedge vA: \lambda v\phi} \bigwedge R^{\dagger}$$

$$12. \quad \frac{\Xi\langle\overrightarrow{A[a/v]}: x\rangle \Rightarrow B: \psi}{\Xi\langle\overrightarrow{\bigvee vA}: z\rangle \Rightarrow B: \psi\{\pi_2 z/x\}} \bigvee L^{\dagger} \qquad \frac{\Xi \Rightarrow A[t/v]: \phi}{\Xi \Rightarrow \bigvee vA: (t, \phi)} \bigvee R$$

Figure 8.4: Quantifier rules; † indicates that there is no a in the conclusion

For example, we can generalise over singular and plural number in **sheep**: $\bigwedge nCNn: \lambda n(n\ sheep)$ for **the sheep grazes**: $S: (graze\ (\iota\ (sg\ sheep)))$ and **the sheep graze**: $S: (graze\ (\iota\ (pl\ sheep)))$:

$$(45) \quad \frac{CNsg \Rightarrow CNsg}{\bigwedge nCNn \Rightarrow CNsg} \bigwedge L \qquad \frac{CNpl \Rightarrow CNpl}{\bigwedge nCNn \Rightarrow CNpl} \bigwedge L$$

And we can express a past, present or future tense finite sentence complement:

said: $(N\backslash S)/\bigvee tSf(t): \lambda x(say\ (\pi_1 x\ \pi_2 x))$

in

John said Mary walked: $S: ((say\ (past\ (walk\ m)))\ j)$,

John said Mary walks: $S: ((say\ (pres\ (walk\ m)))\ j)$

and

John said Mary will walk: $S: ((say\ (fut\ (walk\ m)))\ j)$:

$$(46) \quad \frac{Sf(past) \Rightarrow Sf(past)}{Sf(past) \Rightarrow \bigvee tSf(t)} \bigvee R \qquad \frac{Sf(pres) \Rightarrow Sf(pres)}{Sf(pres) \Rightarrow \bigvee tSf(t)} \bigvee R \qquad \frac{Sf(fut) \Rightarrow Sf(fut)}{Sf(fut) \Rightarrow \bigvee tSf(t)} \bigvee R$$

8.7 Normal modalities

With respect to the normal modalities $\{\Box, \Diamond\}$ of Figure 8.5, Morrill (1990[59]) and Moortgat (1997[54]), the universal has application to intensionality. For example, for a propositional attitude verb we can have

$$13. \quad \frac{\Xi\langle\overrightarrow{A}: x\rangle \Rightarrow B: \psi}{\Xi\langle\overrightarrow{\Box A}: z\rangle \Rightarrow B: \psi\{^{\vee}z/x\}} \Box L \qquad \frac{\boxtimes\Xi \Rightarrow A: \phi}{\boxtimes\Xi \Rightarrow \Box A: {}^{\wedge}\phi} \Box R$$

$$14. \quad \frac{\boxtimes\Xi\langle\overrightarrow{A}: x\rangle \Rightarrow \diamond\!\!\!+ B: \psi}{\boxtimes\Xi\langle\overrightarrow{\Diamond A}: z\rangle \Rightarrow \diamond\!\!\!+ B: \psi\{^{\cup}z/x\}} \Diamond L \qquad \frac{\Xi \Rightarrow A: \phi}{\Xi \Rightarrow \Diamond A: {}^{\cap}\phi} \Diamond R$$

Figure 8.5: Normal modality rules; $\boxtimes/\diamond\!\!\!+$ marks a structure all the types of which have principal connective a box/diamond

an assignment **believes**: $\Box((N\backslash S)/\Box S): believe$ with a modality outermost since the word, like all words, has a sense, and its sentential complement is an intensional domain, but its subject is not.[4]

8.8 Bracket modalities

The bracket modalities $\{[\,]^{-1}, \langle\rangle\}$ of Figure 8.6, Morrill (1992[60]) and Moortgat (1995[53]), have application to syntactical domains such as islands. They are semantically transparent. For example, **walks**: $\langle\rangle N\backslash S$:

[4] The existential normal modality has no application known to us, but we think a candidate may be dynamic semantics, when the indices are interpreted like discourse markers.

15. $$\frac{\Xi\langle\overrightarrow{A}: x\rangle \Rightarrow B: \psi}{\Xi\langle[\overrightarrow{[\,]^{-1}A}: x]\rangle \Rightarrow B: \psi}[\,]^{-1}L \qquad \frac{[\Xi] \Rightarrow A: \phi}{\Xi \Rightarrow [\,]^{-1}A: \phi}[\,]^{-1}R$$

16. $$\frac{\Xi\langle[\overrightarrow{A}: x]\rangle \Rightarrow B: \psi}{\Xi\langle\overrightarrow{\langle\rangle A}: x\rangle \Rightarrow B: \psi}\langle\rangle L \qquad \frac{\Xi \Rightarrow A: \phi}{[\Xi] \Rightarrow \langle\rangle A: \phi}\langle\rangle R$$

Figure 8.6: Bracket modality rules

walk for the sentential subject condition, and

$$\textbf{before}: [\,]^{-1}(VP\backslash VP)/VP: \lambda x\lambda y\lambda z((\mathit{before}\ (x\ z))\ (y\ z))$$

for the adverbial island constraint, which are weak islands, and can contain parasitic gaps, see the next section; for a strong island such as a coordinate structure, which cannot contain a parasitic gap, we define doubly bracketed strong islands — **and**: $(S\backslash[\,]^{-1}[\,]^{-1}S)/S: \lambda x\lambda y[y \wedge x]$.

(47) a. $$\frac{\dfrac{N \Rightarrow N}{[N] \Rightarrow \langle\rangle N}\langle\rangle R \qquad S \Rightarrow S}{[N], \langle\rangle N\backslash S}\backslash L$$

b. $$\frac{S \Rightarrow S \qquad \dfrac{S \Rightarrow S \qquad \dfrac{\dfrac{\dfrac{S \Rightarrow S}{[[\,]^{-1}S] \Rightarrow S}[\,]^{-1}L}{[[[\,]^{-1}[\,]^{-1}S]] \Rightarrow S}[\,]^{-1}L}{[[S, S\backslash[\,]^{-1}[\,]^{-1}S]] \Rightarrow S}\backslash S}{[[S, (S\backslash[\,]^{-1}[\,]^{-1}S)/S]] \Rightarrow S}/S$$

Without bracket modalities we do not capture the Coordinate Structure Constraint (CSC) that coordinate structures are islands to left extraction such as relativisation, e.g.

∗**man that John laughs and Mary likes**

would be derived from the theorem

$$CN, (CN\backslash CN)/(S/N), N, N\backslash S, (S\backslash S)/S, N, (N\backslash S)/N \Rightarrow CN.$$

8.9 Subexponentials

The exponentials $\{!, ?\}$ of Figure 8.7, Girard (1987[22]), Barry et al. (1991[9]), Morrill (1994[80], 2002[62]), 2011[81]) and Morrill and Valentín (2015[73], have application to sharing. Using the universal exponential, !, we can assign to a relative pronoun thus:

$$\textbf{that}: (CN\backslash CN)/(S/!N): \lambda x\lambda y\lambda z[(y\ z) \wedge (x\ z)]$$

allowing parasitic extraction, Morrill (2011[81]), Morrill and Valentín (2015[73]), such as

paper that John filed without reading: $CN: \lambda z[(\mathit{paper}\ z) \wedge [((\mathit{file}\ z)\ j) \wedge \neg((\mathit{read}\ z)\ j)]]$,

where parasitic gaps can appear only in (weak) islands, but can be iterated in (weak) subislands (see Chapter 16).[5] Using the existential exponential, ?, we can assign a coordinator type **and**: $(?S\backslash S)/S: (\alpha^+\ \mathit{and})$ allowing iterated coordination as in **I came, I saw and I conquered**: $N: [(\mathit{come}\ i) \wedge [(\mathit{see}\ i) \wedge (\mathit{conquer}\ i)]]$ where α^+ is a map apply function; see Chapter 14.

[5] The universal exponential enjoys the laws $!A \Rightarrow A$ and $!A \Rightarrow !!A$ of **S4** modal logic, which are derived as follows:

(i) a. $$\frac{\dfrac{A \Rightarrow A}{A; \Rightarrow A}!P}{!A \Rightarrow A}!L$$

b. $$\frac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A; \Rightarrow A}!P}{A; \Rightarrow !A}!R}{A; \Rightarrow !!A}!R}{A \Rightarrow !!A}!L$$

17.
$$\frac{\Xi(\zeta \uplus \{A\colon x\}; \Gamma_1, \Gamma_2) \Rightarrow B\colon \psi}{\Xi(\zeta; \Gamma_1, !A\colon x, \Gamma_2) \Rightarrow B\colon \psi}\,!L \qquad \frac{\zeta;\ \Rightarrow A\colon \phi}{\zeta;\ \Rightarrow !A\colon \phi}\,!R, \zeta \neq \emptyset$$

$$\frac{\Xi(\zeta; \Gamma_1, A\colon x, \Gamma_2) \Rightarrow B\colon \psi}{\Xi(\zeta \uplus \{A\colon x\}; \Gamma_1, \Gamma_2) \Rightarrow B\colon \psi}\,!P$$

$$\frac{\Xi(\zeta \uplus \{A_0\colon x_0, \ldots, A_n\colon x_n\}; \Gamma_1, [\{A_0\colon y_0, \ldots, A_n\colon y_n\}; \Gamma_2], \Gamma_3) \Rightarrow B\colon \psi}{\Xi(\zeta \uplus \{A_0\colon x_0, \ldots A_n\colon x_n\}; \Gamma_1, \Gamma_2, \Gamma_3) \Rightarrow B\colon \psi\{x_0/y_0, \ldots, x_n/y_n\}}\,!C$$

18.
$$\frac{\Xi(A\colon x_1) \Rightarrow D\colon \omega([x_1]) \qquad \Xi(A\colon x_1, A\colon x_2) \Rightarrow D\colon \omega([x_1, x_2]) \qquad \cdots}{\Xi(?A\colon x) \Rightarrow D\colon \omega(x)}\,?L$$

$$\frac{\Xi \Rightarrow A\colon \phi}{\Xi \Rightarrow ?A\colon [\phi]}\,?R \qquad \frac{\zeta; \Gamma \Rightarrow A\colon \phi \qquad \zeta'; \Delta \Rightarrow ?A\colon \psi}{\zeta \uplus \zeta'; \Gamma, \Delta \Rightarrow ?A\colon [\phi|\psi]}\,?M$$

Figure 8.7: Exponential rules

8.10 Semantically inactive connectives

The semantically inactive multiplicatives {•−, ⊸, ⟜, −•, ◐, ◑, ↥, ⫰, ⫯, ↧, ◓, ◒} of Figures 8.8 and 8.9, Morrill and Valentín (2014[72]), can be used for the subcategorization without vacuous lambda abstraction of semantically void elements. For example:

(48) a. **rains**: **it**⊸S: *rain* for **it rains**: S: *rain*
b. **give**: $(N\backslash S)/(N$◑**the + cold+shoulder**$)$: *shun*

This is a type logical formulation of the old idea that heads can co-occur with fixed strings to yield non-compositional meanings, and it runs up against the standard problem that some such strings can be compositionally modified, for example:

(49) John gave Mary the same offensively cold shoulder that she gave him.

(Gazdar et al. 1985[21]), i.e. there is *semi*-compositionality, or semi-idiomaticity, depending on whether we want to see the bottle as half empty or as half full. In the face of this dilemma it seems we can only let the grammar decide and take consolation in the fact that all grammars leak. We shall in fact assume a treatment of the form (48b), and we suggest that (49) is interpreted by 'poetic license'.

The semantically inactive additives {⊓, ⊔} of Figure 8.10, Morrill (1994[80]), can be used for polymorphism which is syntactic but not semantically active. For example (assuming as semantics an identity function across types), **to**: $(PP/N)\sqcap(VP/VP)$: λxx for

to Paris: PP and **to run**: VP and **thinks**: $(N\backslash S)/(S\sqcup CP)$: *think*

for

John thinks Suzy sings: S: ((*think* (*sing* *s*)) *j*)

and

John thinks that Suzy sings: S: ((*think* (*sing* *s*)) *j*).

The semantically inactive first-order quantifiers of Figure 8.11, see Morrill (1994[80]), have application to syntactic features. For example, to transmit number agreement features in a definite noun phrase there can be an assignment **the**: $\forall n(Nt(n)/CNn)$: ι indicating that the number and gender on a definite noun phrase come from its head. And in a sentence, **likes**: $((\exists gNt(s(g))\backslash S)/\exists aNa)$: *like* indicating that the object can have any agreement features but the subject must be third person singular, of any gender.

19. $$\frac{\zeta;\Gamma \Rightarrow B\colon\psi \qquad \Xi(\zeta';\Delta_1\langle\overrightarrow{C}\colon z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{C{\bullet}{-}B}\colon x,\Gamma\rangle,\Delta_2) \Rightarrow D\colon\omega\{0/z\}}\ {\bullet}{-}L \qquad \frac{\zeta;\Gamma,\overrightarrow{B}\colon y \Rightarrow C\colon\chi}{\zeta;\Gamma \Rightarrow C{\bullet}{-}B\colon 0}\ {\bullet}{-}R$$

20. $$\frac{\zeta;\Gamma \Rightarrow A\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{C}\colon z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{A{-}{\circ}C}\colon y\rangle,\Delta_2) \Rightarrow D\colon\omega\{y/z\}}\ {-}{\circ}L \qquad \frac{\zeta;\overrightarrow{A}\colon x,\Gamma \Rightarrow C\colon\chi}{\zeta;\Gamma \Rightarrow A{-}{\circ}C\colon\chi}\ {-}{\circ}R$$

21. $$\frac{\Xi\langle\overrightarrow{A}\colon x,\overrightarrow{B}\colon y\rangle \Rightarrow D\colon\omega}{\Xi\langle\overrightarrow{A\LEFTcircle B}\colon z\rangle \Rightarrow D\colon\omega\{z/y\}}\ {\LEFTcircle}L \qquad \frac{\zeta;\Gamma_1 \Rightarrow A\colon\phi \qquad \zeta';\Gamma_2 \Rightarrow B\colon\psi}{\zeta\uplus\zeta';\Gamma_1,\Gamma_2 \Rightarrow A\LEFTcircle B\colon\psi}\ {\LEFTcircle}R$$

22. $$\frac{\zeta;\Gamma \Rightarrow B\colon\psi \qquad \Xi(\zeta';\Delta_1\langle\overrightarrow{C}\colon z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{C{\circ}{-}B}\colon x,\Gamma\rangle,\Delta_2) \Rightarrow D\colon\omega\{x/z\}}\ {\circ}{-}L \qquad \frac{\zeta;\Gamma,\overrightarrow{B}\colon y \Rightarrow C\colon\chi}{\zeta;\Gamma \Rightarrow C{\circ}{-}B\colon\chi}\ {\circ}{-}R$$

23. $$\frac{\zeta;\Gamma \Rightarrow A\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{C}\colon z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{A{-}{\bullet}C}\colon y\rangle,\Delta_2) \Rightarrow D\colon\omega\{0/z\}}\ {-}{\bullet}L \qquad \frac{\zeta;\overrightarrow{A}\colon x,\Gamma \Rightarrow C\colon\chi}{\zeta;\Gamma \Rightarrow A{-}{\bullet}C\colon 0}\ {-}{\bullet}R$$

24. $$\frac{\Xi\langle\overrightarrow{A}\colon x,\overrightarrow{B}\colon y\rangle \Rightarrow D\colon\omega}{\Xi\langle\overrightarrow{A\RIGHTcircle B}\colon z\rangle \Rightarrow D\colon\omega\{z/x\}}\ {\RIGHTcircle}L \qquad \frac{\zeta;\Gamma_1 \Rightarrow A\colon\phi \qquad \zeta';\Gamma_2 \Rightarrow B\colon\psi}{\zeta\uplus\zeta';\Gamma_1,\Gamma_2 \Rightarrow A\RIGHTcircle B\colon\phi}\ {\RIGHTcircle}R$$

Figure 8.8: Semantically inactive continuous multiplicative rules

With respect to the semantically inactive modalities $\{\blacksquare, \blacklozenge\}$ of Figure 8.12, Hepple (1990[27]) and Morrill (1994[80]), the universal can be applied to classify rigid designators such as proper names as in e.g. **John**: $\blacksquare N\colon j$.

The deterministic synthetic multiplicatives of Figure 8.13, left and right projection and injection $\{\triangleleft^{-1}, \triangleright^{-1}, \triangleleft, \triangleright\}$, Morrill et al. (2009[78]), and split and bridge $\{\check{}\,, \hat{}\,\}$, Morrill and Merenciano (1996[69]), are abbreviatory unary operators. By way of example, a relative pronoun assignment allowing both peripheral and medial extraction may now be expressed **that**: $(CN\backslash CN)/\hat{}(S{\uparrow}N)\colon \lambda x\lambda y\lambda z[(y\ z) \wedge (x\ z)]$.

The continuous and discontinuous non-deterministic synthetic multiplicatives $\{\div, \circ, \overset{\wedge}{=}, \overset{\vee}{=}, \diamond\}$ of Figure 8.14, Morrill, Valentín and Fadda (2011[79]), are abbreviatory binary operators. By way of example of the continuous operators, an adsentential preposition can be allowed to appear both sentence initially and sentence finally by an assignment such as **in**: $(S{\div}S)/N\colon in$ for e.g.

in summer John swims: $S\colon ((in\ summer)\ (swim\ j))$

and

John swims in summer: $S\colon ((in\ summer)\ (swim\ j))$,

and complement alternation can be expressed in e.g. **talks**: $(N\backslash S)/(PPto{\circ}PPabout)\colon talk$ for both

John talks to Mary about Bill: $S\colon ((talk\ (m, b))\ j)$

and

John talks about Bill to Mary: $S\colon ((talk\ (m, b))\ j)$.

By way of example of the discontinuous operators, the particle shift of a particle verb can be captured by e.g. **calls**+1+**up**+1: $\check{}(N\backslash S)\overset{\wedge}{=}N\colon phone$ which produces both **John calls Mary up**: $S\colon ((phone\ m)\ j)$ and **John calls up Mary**: $S\colon ((phone\ m)\ j)$.

25. $$\frac{\zeta;\Gamma \Rightarrow B\colon\psi \qquad \Xi(\zeta';\Delta_1\langle\overrightarrow{C}\colon z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{C\overset{\bullet}{|}_k B}\colon x\,|_k\,\Gamma\rangle,\Delta_2) \Rightarrow D\colon\omega\{x/z\}}\ \overset{\bullet}{|}_k L \qquad \frac{\zeta;\Gamma\,|_k\,\overrightarrow{B}\colon y \Rightarrow C\colon\chi}{\zeta;\Gamma \Rightarrow C\overset{\bullet}{|}_k B\colon\chi}\ \overset{\bullet}{|}_k R$$

26. $$\frac{\zeta;\Gamma \Rightarrow A\colon\phi \qquad \Omega(\zeta';\Delta_1,\langle\overrightarrow{C}\colon z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Omega(\zeta\uplus\zeta';\Delta_1,\langle\Gamma\,|_k\,\overrightarrow{A\underset{\circ}{|}_k C}\colon y\rangle,\Delta_2) \Rightarrow D\colon\omega\{0/z\}}\ \underset{\circ}{|}_k L \qquad \frac{\zeta;\overrightarrow{A}\colon x\,|_k\,\Gamma \Rightarrow C\colon\chi}{\zeta;\Gamma \Rightarrow A\underset{\circ}{|}_k C\colon 0}\ \underset{\circ}{|}_k R$$

27. $$\frac{\Omega\langle\overrightarrow{A}\colon x\,|_k\,\overrightarrow{B}\colon y\rangle \Rightarrow D\colon\omega}{\Omega\langle\overrightarrow{A◓_k B}\colon z\rangle \Rightarrow D\colon\omega\{z/y\}}\ ◓_k L \qquad \frac{\zeta;\Gamma_1 \Rightarrow A\colon\phi \qquad \zeta';\Gamma_2 \Rightarrow B\colon\psi}{\zeta\uplus\zeta';\Gamma_1\,|_k\,\Gamma_2 \Rightarrow A◓_k B\colon\psi}\ ◓_k R$$

28. $$\frac{\zeta;\Gamma \Rightarrow B\colon\psi \qquad \Omega(\zeta';\Delta_1\langle\overrightarrow{C}\colon z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Omega(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{C\overset{\circ}{|}_k B}\colon x\,|_k\,\Gamma\rangle,\Delta_2) \Rightarrow D\colon\omega\{0/z\}}\ \overset{\circ}{|}_k L \qquad \frac{\zeta;\Gamma\,|_k\,\overrightarrow{B}\colon y \Rightarrow C\colon\chi}{\zeta;\Gamma \Rightarrow C\overset{\circ}{|}_k B\colon 0}\ \overset{\circ}{|}_k R$$

29. $$\frac{\zeta;\Gamma \Rightarrow A\colon\phi \qquad \Omega(\zeta';\Delta_1,\langle\overrightarrow{C}\colon z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Omega(\zeta\uplus\zeta';\Delta_1,\langle\Gamma\,|_k\,\overrightarrow{A\underset{\bullet}{|}_k C}\colon y\rangle,\Delta_2) \Rightarrow D\colon\omega\{y/z\}}\ \underset{\bullet}{|}_k L \qquad \frac{\zeta;\overrightarrow{A}\colon x\,|_k\,\Gamma \Rightarrow C\colon\chi}{\zeta;\Gamma \Rightarrow A\underset{\bullet}{|}_k C\colon\chi}\ \underset{\bullet}{|}_k R$$

30. $$\frac{\Omega\langle\overrightarrow{A}\colon x\,|_k\,\overrightarrow{B}\colon y\rangle \Rightarrow D\colon\omega}{\Omega\langle\overrightarrow{A◒_k B}\colon z\rangle \Rightarrow D\colon\omega\{z/x\}}\ ◒_k L \qquad \frac{\zeta;\Gamma_1 \Rightarrow A\colon\phi \qquad \zeta';\Gamma_2 \Rightarrow B\colon\psi}{\zeta\uplus\zeta';\Gamma_1\,|_k\,\Gamma_2 \Rightarrow A◒_k B\colon\phi}\ ◒_R$$

Figure 8.9: Semantically inactive discontinuous multiplicative rules

31. $$\frac{\Xi\langle\overrightarrow{A}\colon x\rangle \Rightarrow C\colon\chi}{\Xi\langle\overrightarrow{A\sqcap B}\colon x\rangle \Rightarrow C\colon\chi}\ \sqcap L_1 \qquad \frac{\Xi\langle\overrightarrow{B}\colon y\rangle \Rightarrow C\colon\chi}{\Xi\langle\overrightarrow{A\sqcap B}\colon y\rangle \Rightarrow C\colon\chi}\ \sqcap L_2$$

$$\frac{\Xi \Rightarrow A\colon\chi \qquad \Xi \Rightarrow B\colon\chi}{\Xi \Rightarrow A\sqcap B\colon\chi}\ \sqcap R$$

32. $$\frac{\Xi\langle\overrightarrow{A}\colon z\rangle \Rightarrow C\colon\chi \qquad \Xi\langle\overrightarrow{B}\colon z\rangle \Rightarrow C\colon\chi}{\Xi\langle\overrightarrow{A\sqcup B}\colon z\rangle \Rightarrow C\colon\chi}\ \sqcup L$$

$$\frac{\Xi \Rightarrow A\colon\phi}{\Xi \Rightarrow A\sqcup B\colon\phi}\ \sqcup R_1 \qquad \frac{\Xi \Rightarrow B\colon\psi}{\Xi \Rightarrow A\sqcup B\colon\psi}\ \sqcup R_2$$

Figure 8.10: Semantically inactive additive rules

33. $$\frac{\Xi\langle\overrightarrow{A[t/v]}\colon x\rangle \Rightarrow B\colon\psi}{\Xi\langle\overrightarrow{\forall vA}\colon x\rangle \Rightarrow B\colon\psi}\ \forall L \qquad \frac{\Xi \Rightarrow A[a/v]\colon\phi}{\Xi \Rightarrow \forall vA\colon\phi}\ \forall R^{\dagger}$$

34. $$\frac{\Xi\langle\overrightarrow{A[a/v]}\colon x\rangle \Rightarrow B\colon\psi}{\Xi\langle\overrightarrow{\exists vA}\colon x\rangle \Rightarrow B\colon\psi}\ \exists L^{\dagger} \qquad \frac{\Xi \Rightarrow A[t/v]\colon\phi}{\Xi \Rightarrow \exists vA\colon\phi}\ \exists R$$

Figure 8.11: Semantically inactive quantifier rules; † indicates that there is no *a* in the conclusion

35. $$\frac{\Xi\langle\overrightarrow{A}\colon x\rangle \Rightarrow B\colon\psi}{\Xi\langle\overrightarrow{\blacksquare A}\colon x\rangle \Rightarrow B\colon\psi}\,\blacksquare L \qquad \frac{\boxtimes\Xi \Rightarrow A\colon\phi}{\boxtimes\Xi \Rightarrow \blacksquare A\colon\phi}\,\blacksquare R$$

36. $$\frac{\boxtimes\Xi\langle\overrightarrow{A}\colon x\rangle \Rightarrow \diamondplus B\colon\psi}{\boxtimes\Xi\langle\overrightarrow{\blacklozenge A}\colon x\rangle \Rightarrow \diamondplus B\colon\psi}\,\blacklozenge L \qquad \frac{\Xi \Rightarrow A\colon\phi}{\Xi \Rightarrow \blacklozenge A\colon\phi}\,\blacklozenge R$$

Figure 8.12: Semantically inactive normal modality rules; $\boxtimes/\diamondplus$ marks a structure all the types of which have principal connective a box/diamond

37. $$\frac{\Xi\langle\overrightarrow{A}\colon x\rangle \Rightarrow B\colon\psi}{\Xi\langle\overrightarrow{A}\colon x, 1\rangle \Rightarrow B\colon\psi}\,\triangleleft^{-1}L \qquad \frac{\zeta;\Gamma, 1 \Rightarrow A\colon\phi}{\zeta;\Gamma \Rightarrow \triangleleft^{-1}A\colon\phi}\,\triangleleft^{-1}R$$

38. $$\frac{\Xi\langle\overrightarrow{A}\colon x\rangle \Rightarrow B\colon\psi}{\Xi\langle 1, \overrightarrow{A}\colon x\rangle \Rightarrow B\colon\psi}\,\triangleright^{-1}L \qquad \frac{\zeta; 1, \Gamma \Rightarrow A\colon\phi}{\zeta;\Gamma \Rightarrow \triangleright^{-1}A\colon\phi}\,\triangleright^{-1}R$$

39. $$\frac{\Xi\langle\overrightarrow{A}\colon x, 1\rangle \Rightarrow B\colon\psi}{\Xi\langle\overrightarrow{\triangleleft A}\colon x\rangle \Rightarrow B\colon\psi}\,\triangleleft L \qquad \frac{\zeta;\Gamma \Rightarrow A\colon\phi}{\zeta;\Gamma, 1 \Rightarrow \triangleleft A\colon\phi}\,\triangleleft R$$

40. $$\frac{\Xi\langle 1, \overrightarrow{A}\colon x\rangle \Rightarrow B\colon\psi}{\Xi\langle\overrightarrow{\triangleright A}\colon x\rangle \Rightarrow B\colon\psi}\,\triangleright L \qquad \frac{\zeta;\Gamma \Rightarrow A\colon\phi}{\zeta; 1, \Gamma \Rightarrow \triangleright A\colon\phi}\,\triangleleft R$$

41. $$\frac{\Xi\langle\overrightarrow{B}\colon y\rangle \Rightarrow C\colon\chi}{\Xi\langle\overrightarrow{\check{}_k B}\colon y\,|_k\,\Lambda\rangle \Rightarrow C\colon\chi}\,\check{}_k L \qquad \frac{\zeta;\Delta\,|_k\,\Lambda \Rightarrow A\colon\psi}{\zeta;\Delta \Rightarrow \check{}_k A\colon\psi}\,\check{}_k R$$

42. $$\frac{\Xi\langle\overrightarrow{B}\colon y\,|_k\,\Lambda\rangle \Rightarrow C\colon\chi}{\Xi\langle\overrightarrow{\hat{}_k B}\colon y\rangle \Rightarrow C\colon\chi}\,\hat{}_k L \qquad \frac{\zeta;\Delta \Rightarrow A\colon\psi}{\zeta;\Delta\,|_k\,\Lambda \Rightarrow \hat{}_k A\colon\psi}\,\hat{}_k R$$

Figure 8.13: Deterministic synthetic multiplicative rules

43.
$$\frac{\zeta;\Gamma\Rightarrow A\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{C}\colon z\rangle,\Delta_2)\Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{B\div A}\colon y\rangle,\Delta_2)\Rightarrow D\colon\omega\{(y\ \phi)/z\}}\div L_1 \qquad \frac{\zeta;\Gamma\Rightarrow A\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{C}\colon z\rangle,\Delta_2)\Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{B\div A}\colon y,\Gamma\rangle,\Delta_2)\Rightarrow D\colon\omega\{(y\ \phi)/z\}}\div L_2$$

$$\frac{\zeta;\overrightarrow{A}\colon x,\Gamma\Rightarrow C\colon\chi \qquad \zeta;\Gamma,\overrightarrow{A}\colon x\Rightarrow C\colon\chi}{\zeta;\Gamma\Rightarrow C\div A\colon\lambda x\chi}\div R$$

44.
$$\frac{\Xi\langle\overrightarrow{A}\colon x,\overrightarrow{B}\colon y\rangle\Rightarrow D\colon\omega \qquad \Xi\langle\overrightarrow{B}\colon y,\overrightarrow{A}\colon x\rangle\Rightarrow D\colon\omega}{\Xi\langle\overrightarrow{A\circ B}\colon z\rangle\Rightarrow D\colon\omega\{\pi_1 z/x,\pi_2 z/y\}}\circ L$$

$$\frac{\zeta;\Gamma_1\Rightarrow A\colon\phi \qquad \zeta';\Gamma_2\Rightarrow B\colon\psi}{\zeta\uplus\zeta';\Gamma_1,\Gamma_2\Rightarrow A\circ B\colon(\phi,\psi)}\circ R_1 \qquad \frac{\zeta;\Gamma_1\Rightarrow B\colon\psi \qquad \zeta';\Gamma_2\Rightarrow A\colon\phi}{\zeta\uplus\zeta';\Gamma_1,\Gamma_2\Rightarrow A\circ B\colon(\phi,\psi)}\circ R_2$$

45.
$$\frac{\zeta;\Gamma\Rightarrow A\colon\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{B}\colon z\rangle,\Delta_2)\Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{B\overset{\wedge}{=}A}\colon x\,|_k\,\Gamma\rangle,\Delta_2)\Rightarrow D\colon\omega\{(x\ \psi)/z\}}\overset{\wedge}{=}L \qquad \frac{\zeta;\Gamma\,|_1\,\overrightarrow{B}\colon y\Rightarrow C\colon\chi \qquad \cdots \qquad \zeta;\Gamma\,|_{sC'-sB}\,\overrightarrow{B}\colon y\Rightarrow C\colon\chi}{\zeta;\Gamma\Rightarrow C\overset{\wedge}{=}B\colon\lambda y\chi}\overset{\wedge}{=}R$$

46.
$$\frac{\zeta;\Gamma\Rightarrow A\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{B}\colon z\rangle,\Delta_2)\Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma\,|_k\,\overrightarrow{A\overset{\vee}{=}B}\colon y\rangle,\Delta_2)\Rightarrow D\colon\omega\{(y\ \phi)/z\}}\overset{\vee}{=}L \qquad \frac{\zeta;\overrightarrow{A}\colon x\,|_1\,\Gamma\Rightarrow C\colon\chi \qquad \cdots \qquad \zeta;\overrightarrow{A}\colon x\,|_{sA}\,\Gamma\Rightarrow C\colon\chi}{\zeta;\Gamma\Rightarrow A\overset{\vee}{=}C\colon\lambda x\chi}\overset{\vee}{=}R$$

47.
$$\frac{\Xi\langle\overrightarrow{A}\colon x\,|_1\,\overrightarrow{B}\colon y\rangle\Rightarrow D\colon\omega \qquad \cdots \qquad \Xi\langle\overrightarrow{A}\colon x\,|_{sA}\,\overrightarrow{B}\colon y\rangle\Rightarrow D\colon\omega}{\Xi\langle\overrightarrow{A\diamond B}\colon z\rangle\Rightarrow D\colon\omega\{\pi_1 z/x,\pi_2 z/y\}}\diamond L \qquad \frac{\zeta;\Gamma_1\Rightarrow A\colon\phi \qquad \zeta';\Gamma_2\Rightarrow B\colon\psi}{\zeta\uplus\zeta';\Gamma_1\,|_k\,\Gamma_2\Rightarrow A\diamond B\colon(\phi,\psi)}\diamond R$$

Figure 8.14: Non-deterministic synthetic multiplicative rules

8.11 Limited contraction and limited expansion

The limited contraction and limited expansion of Figure 8.15 can be used for anaphora and for words as types respectively.

48. $$\frac{\zeta; \Gamma \Rightarrow A{:}\,\phi \qquad \zeta'; \Delta\langle\overrightarrow{A}{:}\,x; \overrightarrow{B}{:}\,y\rangle \Rightarrow D{:}\,\omega}{\zeta \uplus \zeta'; \Delta\langle\Gamma; \overrightarrow{B|A}{:}\,z\rangle \Rightarrow D{:}\,\omega\{\phi/x, (z\ \phi)/y\}}\,|L$$

$$\frac{\zeta; \Gamma\langle\overrightarrow{B_0}{:}\,y_0 : \ldots : \overrightarrow{B_n}{:}\,y_n\rangle \Rightarrow D{:}\,\omega}{\zeta; \Gamma\langle\overrightarrow{B_0|A}{:}\,z_0 : \ldots : \overrightarrow{B_n|A}{:}\,z_n\rangle \Rightarrow D|A{:}\,\lambda x\omega\{(z_0\ x)/y_0, \ldots, (z_n\ x)/y_n\}}\,|R$$

49. $$\frac{\Xi\langle\Lambda\rangle \Rightarrow A{:}\,\phi}{\Xi\langle\overrightarrow{0}{:}\,x\rangle \Rightarrow A{:}\,\phi}\,+L \qquad \frac{\Xi\langle\overrightarrow{v{:}\,x, \overrightarrow{w}{:}\,y}\rangle \Rightarrow D{:}\,\omega}{\Xi\langle\overrightarrow{v + w}{:}\,z\rangle \Rightarrow D{:}\,\omega\{z/x, z/y\}}\,+L$$

$$\frac{}{\Lambda \Rightarrow 0{:}\,0}\,+R \qquad \frac{\zeta_1; \Gamma_1 \Rightarrow v{:}\,\phi \qquad \zeta_2; \Gamma_2 \Rightarrow w{:}\,\phi}{\zeta_1 \uplus \zeta_2; \Gamma_1, \Gamma_2 \Rightarrow v + w{:}\,0}\,+R$$

Figure 8.15: Limited contraction and limited expansion rules

The limited contraction, |, of Jäger (2005[29]), can be used for anaphora in an assignment like **it**: $(S{\uparrow}N){\downarrow}(S|N){:}\,\lambda xx$ for, e.g., **the company**$_i$ **said it**$_i$ **flourished**: $S{:}\,((say\ (flourish\ (\iota\ company)))\ (\iota\ company))$, and it can be used for **such that** relativisation in an assignment **such that**: $(CN\backslash CN)/(S|N){:}\,\lambda x\lambda y\lambda z[(y\ z) \wedge (x\ z)]$ for, say, **man such that**$_i$ **he**$_i$ **thinks Mary loves him**$_i$: $CN{:}\,\lambda z[(man\ z) \wedge ((think\ ((love\ z)\ m))\ z)]$; see Chapter 13. The limited expansion can be used for words as types as in Morrill and Valentín (2014[72]) for semantically void words. For example **rains**: $it\backslash S{:}\,\lambda x rain$ for **it rains**.

8.12 Difference

(50) $$\frac{\zeta_1; \Gamma \Rightarrow A{:}\,\phi \qquad \zeta_2; \Delta\langle\overrightarrow{A}{:}\,x\rangle \Rightarrow B{:}\,\psi}{\zeta_1 \uplus \zeta_2; \Delta\langle\Gamma\rangle \Rightarrow B{:}\,\psi\{\phi/x\}}\,Cut$$

Negation as failure:[6]

(51) 50. $$\frac{}{\Delta \Rightarrow \neg A}\,\neg R, \nvdash \Delta \Rightarrow A$$

Difference: $A - B = A \& \neg B$:

(52) $$\frac{\Delta \Rightarrow A}{\Delta \Rightarrow A - B}\,-R, \nvdash \Delta \Rightarrow B$$

[6] For difference − (connective number 50), however, Cut is not appropriate. We consider this a *metalogical* connective which is added to the system for which Cut-elimination has been proved.

Part III

PARSING

Even in Cut-free sequent proof search there are in general many sequent proofs for each theorem, differing in inessential rule reorderings. And in particular, in Cut-free hedge sequent proof search there are in general many semantically equivalent hedge sequent proofs for each theorem, giving rise to great redundancy in hedge sequent calculus parsing/theorem proving. In this part we present Andreoli's method of 'focusing' for greatly reducing the redundancy of hedge sequent calculus. In addition, theorems are subject to van Benthem's 'count' invariance properties necessary (though not sufficient) for theoremhood. In this part we also present infinitary count invariance for categorial logic including subexponentials, which can be used in hedge sequent calculus proof search to filter hedge sequents and subsequents by rapidly establishing non-count invariance and hence non-provability.

Chapter 9

Focalised Sequent Calculus

Categorial grammar operates under the slogans 'grammar as logic' and 'parsing as deduction'. The grammar is a substructural, indeed sublinear, logic such as displacement logic, and the parsing paradigm is typically backward-chaining sequent proof-search. Spurious ambiguity is the phenomenon whereby distinct derivations in grammar may assign the same structural reading, resulting in redundancy in the parse search space and inefficiency in parsing. Understanding the problem depends on identifying the essential mathematical structure of derivations. This is trivial in the case of CFG, where the parse structures are ordered trees; in the case of categorial grammar, the parse structures are proof nets. However, with respect to multiplicatives intrinsic proof nets have not yet been given for displacement calculus **D**, and proof nets for additives, for example, which have applications to polymorphism, are complex. In this chapter we approach parsing as deduction for Full Displacement Logic **FDL** by means of the proof-theoretic technique of focalisation, and we prove completeness for the case of displacement calculus with additives **DA**.

9.1 Spurious ambiguity

9.1.1 Introduction

In CFG sequential rewriting derivations exhibit spurious ambiguity: distinct rewriting derivations may correspond to the same parse structure (tree) and the same structural reading. In this case it is transparent to develop parsing algorithms avoiding spurious ambiguity by reference to parse trees. In categorial grammar (CG) the problem is more subtle. The Cut-free Lambek sequent proof search space is finite, but involves a combinatorial explosion of spuriously ambiguous sequential proofs: sequent proofs of the same endsequent with the same Curry-Howard term reading. This can be understood, analogously to CFG, as inessential rule reorderings, which we can parallelise in underlying geometric parse structures which are (planar) proof nets. The planarity of Lambek proof nets reflects that the formalism is continuous or concatenative. But the challenge of natural grammar is discontinuity or apparent movement, whereby there is syntactic/semantic mismatch, or elements appearing out of place. Hence the subsumption of Lambek calculus, as a logic of strings with appending, by displacement calculus **D**, as a logic of strings with holes including plugging as well as appending (Morrill et al. 2011[79]).

Proof nets for **D** must be partially nonplanar; steps towards intrinsic correctness criteria for displacement proof nets are made in Fadda (2010[19]) and Moot (2014[55]). However, even in the case of Lambek calculus, in our experience parsing by reference to intrinsic criteria (Morrill 2011[81] appendix B, Moot and Retoré 2012[56]) is not more efficient than parsing by reference to extrinsic criteria of normalised sequent calculus (Hendriks 1993[25], Morrill 2011[63], 2012[64]). In its turn, on the other hand, normalisation does not extend to product left rules and product unit left rules, nor to additives. The focalisation of Andreoli (1992[5]) represents a methodology midway between proof nets and normalisation. Here we apply the focusing discipline to the parsing as deduction of **FDL**.

$$\dfrac{CN \Rightarrow CN \qquad \dfrac{\dfrac{\dfrac{N \Rightarrow N \qquad \dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\backslash L}{N, (N\backslash S)/N, N \Rightarrow S}/L}{(N\backslash S)/N, N \Rightarrow N\backslash S}\backslash R \qquad S \Rightarrow S}{S/(N\backslash S), (N\backslash S)/N, N \Rightarrow S}/L}{(S/(N\backslash S))/CN, CN, (N\backslash S)/N, N \Rightarrow S}/L$$

$$\dfrac{N \Rightarrow N \qquad \dfrac{CN \Rightarrow CN \qquad \dfrac{\dfrac{\dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\backslash L}{N\backslash S \Rightarrow N\backslash S}\backslash R \qquad S \Rightarrow S}{S/(N\backslash S), N\backslash S \Rightarrow S}/L}{(S/(N\backslash S))/CN, CN, N\backslash S \Rightarrow S}/L}{(S/(N\backslash S))/CN, CN, (N\backslash S)/N, N \Rightarrow S}/L$$

Figure 9.1: Spurious ambiguity

9.1.2 Spurious ambiguity in CFG and CG

Consider the following production rules:

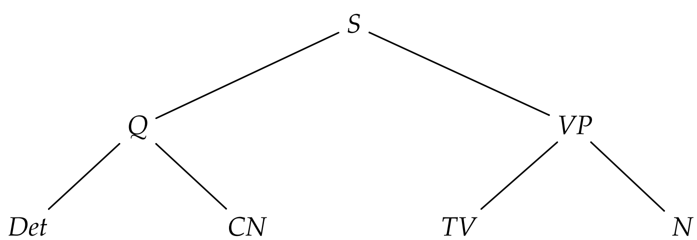
(53) $S \to Q\ VP$
$Q \to Det\ CN$
$VP \to TV\ N$

These generate the following sequential rewriting derivations:

(54) $S \to Q\ VP \to Det\ CN\ VP \to Det\ CN\ TV\ N$
$S \to Q\ VP \to Q\ TV\ N \to Det\ CN\ TV\ N$

These sequential rewriting derivations correspond to the same parellelised parse structure:

(55)



And they correspond to the same structural reading; sequential rewriting has *spurious ambiguity*.

Recall the definitions of types, configurations and sequents in the Lambek calculus **L** (Lambek 1958[46]), in terms of a set $\mathcal{P}$ of primitive types, where Λ is the metalinguistic empty string:

(56) Types $\mathcal{F} ::= \mathcal{P} \mid \mathcal{F}\backslash\mathcal{F} \mid \mathcal{F}/\mathcal{F} \mid \mathcal{F}\bullet\mathcal{F}$
Configurations $\mathcal{O} ::= \Lambda \mid \mathcal{F} \mid \mathcal{O}, \mathcal{O}$
Sequents $\Sigma ::= \mathcal{O} \Rightarrow \mathcal{F}$

$\Delta(\Gamma)$ indicates a configuration Δ and a distinguished subconfiguration Γ; the logical rules of **L** are:

(57)
$$\frac{\Gamma \Rightarrow A \qquad \Delta(C) \Rightarrow D}{\Delta(A\backslash C) \Rightarrow D}\backslash L \qquad \frac{A, \Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C}\backslash R$$

$$\frac{\Gamma \Rightarrow B \qquad \Delta(C) \Rightarrow D}{\Delta(C/B, \Gamma) \Rightarrow D}/L \qquad \frac{\Gamma, B \Rightarrow C}{\Gamma \Rightarrow C/B}/R$$

$$\frac{\Delta(A, B) \Rightarrow D}{\Delta(A\bullet B) \Rightarrow D}\bullet L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A\bullet B}\bullet R$$

Even amongst Cut-free proofs there is spurious ambiguity; consider for example the sequential derivations of Figure 9.1. These have the same parallelised parse structure (proof net), given in Figure 9.2.

Lambek proof structures are planar graphs which must satisfy certain global and local properties to be correct as proofs (proof nets). Proof nets provide a geometric perspective on derivational equivalence. Alternatively we may identify the same algebraic parse structure (Curry-Howard term): $((x_{Det}\, x_{CN})\ \lambda x((x_{TV}\, x_N)\, x))$ for the categorial examples here.

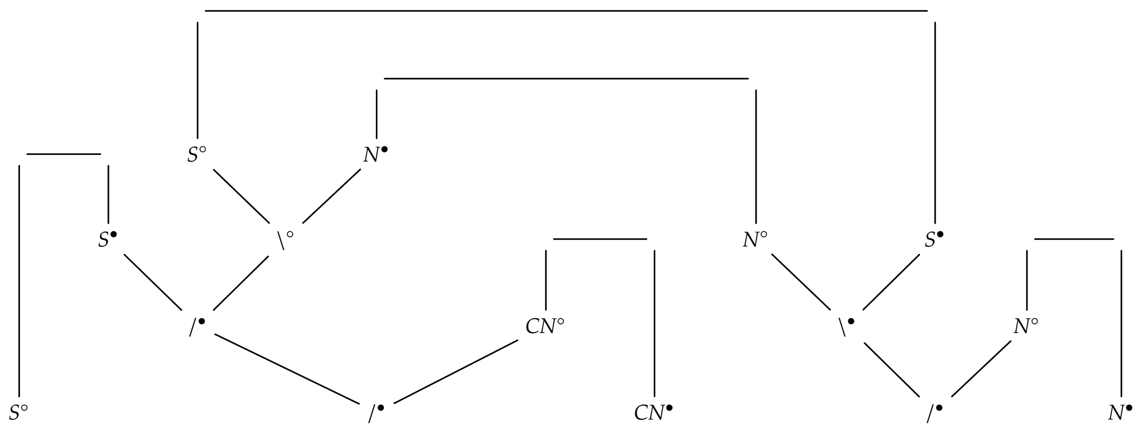


Figure 9.2: Proof net

9.2 Focalisation for FDL

The discipline of focalisation depends fundamentally on the distinction between invertible and noninvertible rules. A rule is invertible (or reversable) if its premises are derivable from its conclusion — for example $\backslash R$ and $\oplus L$ — otherwise it is noninvertible. In focalisation situated (antecedent, input, $^\bullet$ / succedent, output, $^\circ$) connectives are classified as of negative (asynchronous) or positive (synchronous) *polarity* according as their rule is invertible or not respectively. There are alternating phases of don't-care non-deterministic negative rule application, and positive rule application locking on to *focalised* formulas. Given a sequent, invertible rules are applied in a don't care non-determinitic fashion until no longer possible. Once given a subgoal with no occurrences of negative formulas, one chooses a positive formula as principal formula; we say it is focalised — in our presentation this formula is boxed — and we apply proof search to its subformulas while these remain positive. When one finds a negative formula or a literal, invertible rules are applied in a don't care non-determinitic fashion again until no longer possible, when another positive formula is chosen, and so on.

We define input and output synchronous and asynchronous types of **FDL**, P, Q, N, M, as follows:

(58) Synch. output P ::= $A{\bullet}B^\circ \mid I^\circ \mid A{\odot_k}B^\circ \mid J^\circ \mid A{\oplus}B^\circ \mid \bigvee vA^\circ \mid \Diamond A^\circ \mid \langle\rangle A^\circ \mid ?A^\circ$
$A◐B^\circ \mid A◑B^\circ \mid A◓_kB^\circ \mid A◒_kB^\circ \mid A{\sqcup}B^\circ \mid \exists vA^\circ \mid ♦A^\circ \mid \triangleleft A^\circ \mid \triangleright A^\circ \mid {}^{\wedge k}A^\circ \mid$
$A{\circ}B^\circ \mid A{\diamond}B^\circ$

Synch. input Q ::= $A\backslash C^\bullet \mid C/B^\bullet \mid A{\downarrow_k}C^\bullet \mid C{\uparrow_k}B^\bullet \mid A\&B^\bullet \mid \bigwedge vA^\bullet \mid \Box A^\bullet \mid [\,]^{-1}A^\bullet \mid !A^\bullet \mid$
$A{\multimap}C^\bullet \mid C{\bullet\!\!-}B^\bullet \mid A{-\!\!\bullet}C^\bullet \mid C{\circ\!\!-}B^\bullet \mid A{\downarrow^\circ_k}C^\bullet \mid C{\uparrow^\bullet_k}B^\bullet \mid A{\downarrow^\bullet_k}C^\bullet \mid C{\uparrow^\circ_k}B^\bullet \mid$
$A{\sqcap}B^\bullet \mid \forall vA^\bullet \mid \blacksquare A^\bullet \mid \triangleleft^{-1}A^\bullet \mid \triangleright^{-1}A^\bullet \mid {}^{\vee k}A^\bullet \mid C{\div}A^\bullet \mid A\overset{\vee}{=}C^\bullet \mid C\overset{\wedge}{=}B^\bullet$

Asynch. output N ::= $A\backslash C^\circ \mid C/B^\circ \mid A{\downarrow_k}C^\circ \mid C{\uparrow_k}B^\circ \mid A\&B^\circ \mid \bigwedge vA^\circ \mid \Box A^\circ \mid [\,]^{-1}A^\circ \mid !A^\circ \mid$
$A{\multimap}C^\circ \mid C{\bullet\!\!-}B^\circ \mid A{-\!\!\bullet}C^\circ \mid C{\circ\!\!-}B^\circ \mid A{\downarrow^\circ_k}C^\circ \mid C{\uparrow^\bullet_k}B^\circ \mid A{\downarrow^\bullet_k}C^\circ \mid C{\uparrow^\circ_k}B^\circ \mid$
$A{\sqcap}B^\circ \mid \forall vA^\circ \mid \blacksquare A^\circ \mid \triangleleft^{-1}A^\circ \mid \triangleright^{-1}A^\circ \mid {}^{\vee k}A^\circ \mid C{\div}A^\circ \mid A\overset{\vee}{=}C^\circ \mid C\overset{\wedge}{=}B^\circ$

Asynch. input M ::= $A{\bullet}B^\bullet \mid I^\bullet \mid A{\odot_k}B^\bullet \mid J^\bullet \mid A{\oplus}B^\bullet \mid \bigvee vA^\bullet \mid \Diamond A^\bullet \mid \langle\rangle A^\bullet \mid ?A^\bullet \mid$
$A◐B^\bullet \mid A◑B^\bullet \mid A◓_kB^\bullet \mid A◒_kB^\bullet \mid A{\sqcup}B^\bullet \mid \exists vA^\bullet \mid ♦A^\bullet \mid \triangleleft A^\bullet \mid \triangleright A^\bullet \mid {}^{\wedge k}A^\bullet \mid$
$A{\circ}B^\bullet \mid A{\diamond}B^\bullet$

Sequents now include *stoups* $\mathcal{Z}$ which are lists of sort 0 types to which structural rules can apply. Sequents with such stoups can be defined:

(59) $$\begin{array}{lll}\textbf{Stoup} & ::= & \emptyset \mid \textbf{Tp}^{\bullet}_{0}, \textbf{Stoup}\\ \textbf{Config} & ::= & \Lambda \mid 1 \mid \textbf{Tp}^{\bullet}_{0} \mid \textbf{Tp}^{\bullet}_{i>0}\{\underbrace{\textbf{Config} : \ldots : \textbf{Config}}_{i\ \textbf{Config}\text{'s}}\} \mid \textbf{Config}, \textbf{Config} \mid [\textbf{Stoup}; \textbf{Config}]\end{array}$$

The definition of sequents Σ is:

(60) $\Sigma ::= \textbf{Stoup}; \textbf{Config} \Rightarrow \textbf{Tp}^{\circ}$ such that $s(\textbf{Config}) = s(\textbf{Tp}^{\circ})$

The focused sequent calculus for this categorial logic is defined in the following sections.

9.3 Asynchronous rules

(61) $$\frac{}{\overrightarrow{P}: x \Rightarrow P: x}\ id, P \text{ atomic}$$

9.3.1 Primary connectives

1. $$\frac{\zeta; \Gamma, \overrightarrow{B}: y \Rightarrow C: \chi}{\zeta; \Gamma \Rightarrow C/B: \lambda y \chi}\ /R$$

2. $$\frac{\zeta; \overrightarrow{A}: x, \Gamma \Rightarrow C: \chi}{\zeta; \Gamma \Rightarrow A\backslash C: \lambda x \chi}\ \backslash R$$

3. $$\frac{\Xi\langle \overrightarrow{A}: x, \overrightarrow{B}: y\rangle \Rightarrow D: \omega}{\Xi\langle \overrightarrow{A \bullet B}: z\rangle \Rightarrow D: \omega\{\pi_1 z/x, \pi_2 z/y\}}\ \bullet L$$

4. $$\frac{\Xi\langle \Lambda\rangle \Rightarrow A: \phi}{\Xi\langle \overrightarrow{I}: x\rangle \Rightarrow A: \phi}\ IL$$

5. $$\frac{\zeta; \Gamma |_k \overrightarrow{B}: y \Rightarrow C: \chi}{\zeta; \Gamma \Rightarrow C\uparrow_k B: \lambda y \chi}\ \uparrow_k R$$

6. $$\frac{\zeta; \overrightarrow{A}: x |_k \Gamma \Rightarrow C: \chi}{\zeta; \Gamma \Rightarrow A\downarrow_k C: \lambda x \chi}\ \downarrow_k R$$

7. $$\frac{\Xi\langle \overrightarrow{A}: x |_k \overrightarrow{B}: y\rangle \Rightarrow D: \omega}{\Xi\langle \overrightarrow{A \odot_k B}: z\rangle \Rightarrow D: \omega\{\pi_1 z/x, \pi_2 z/y\}}\ \odot_k L$$

8. $$\frac{\Xi\langle 1\rangle \Rightarrow A: \phi}{\Xi\langle \overrightarrow{J}: x\rangle \Rightarrow A: \phi}\ JL$$

Figure 9.3: Asynchronous multiplicative rules

9. $$\frac{\Xi \Rightarrow A: \phi \qquad \Xi \Rightarrow B: \psi}{\Xi \Rightarrow A\&B: (\phi, \psi)}\ \&R$$

10. $$\frac{\Xi\langle \overrightarrow{A}: x\rangle \Rightarrow C: \chi_1 \qquad \Xi\langle \overrightarrow{B}: y\rangle \Rightarrow C: \chi_2}{\Xi\langle \overrightarrow{A \oplus B}: z\rangle \Rightarrow C: z \to x.\chi_1; y.\chi_2}\ \oplus L$$

Figure 9.4: Asynchronous additive rules

11. $$\frac{\Xi \Rightarrow A[a/v]: \phi}{\Xi \Rightarrow \bigwedge vA: \lambda v \phi}\ \bigwedge R^{\dagger}$$

12. $$\frac{\Xi\langle \overrightarrow{A[a/v]}: x\rangle \Rightarrow B: \psi}{\Xi\langle \overrightarrow{\bigvee vA}: z\rangle \Rightarrow B: \psi\{\pi_2 z/x\}}\ \bigvee L^{\dagger}$$

Figure 9.5: Asynchronous quantifier rules, where † indicates that there is no a in the conclusion

$$13.\quad \frac{\boxtimes\Xi \Rightarrow A\colon \phi}{\boxtimes\Xi \Rightarrow \Box A\colon {}^{\wedge}\phi}\,\Box R \qquad 14.\quad \frac{\boxtimes\Xi\langle \overrightarrow{A\colon x}\rangle \Rightarrow \diamondplus B\colon \psi}{\boxtimes\Xi\langle \overrightarrow{\Diamond A\colon z}\rangle \Rightarrow \diamondplus B\colon \psi\{{}^{\cup}z/x\}}\,\Diamond L$$

Figure 9.6: Asynchronous normal modality rules; $\boxtimes/\diamondplus$ marks a structure all the types of which have principal connective a box/diamond

$$15.\quad \frac{[\Xi] \Rightarrow A\colon \phi}{\Xi \Rightarrow [\,]^{-1}A\colon \phi}\,[\,]^{-1}R \qquad 16.\quad \frac{\Xi\langle [\overrightarrow{A\colon x}]\rangle \Rightarrow B\colon \psi}{\Xi\langle \overrightarrow{\langle\rangle A\colon x}\rangle \Rightarrow B\colon \psi}\,\langle\rangle L$$

Figure 9.7: Asynchronous bracket modality rules

$$17.\qquad \frac{\zeta; \Lambda \Rightarrow A\colon \phi}{\zeta; \Lambda \Rightarrow\, !A\colon \phi}\,!R$$

$$\frac{\Xi(\zeta \uplus \{A\colon x\}; \Gamma_1, \Gamma_2) \Rightarrow B\colon \psi}{\Xi(\zeta; \Gamma_1, !A\colon x, \Gamma_2) \Rightarrow B\colon \psi}\,!P$$

Figure 9.8: Asynchronous subexponential rules

9.3.2 Semantically inactive variants

19. $$\frac{\zeta;\Gamma,\overrightarrow{B}\colon y \Rightarrow C\colon \chi}{\zeta;\Gamma \Rightarrow C{\bullet\!\!-}B\colon 0}\ {\bullet\!\!-}R$$

20. $$\frac{\zeta;\overrightarrow{A}\colon x,\Gamma \Rightarrow C\colon \chi}{\zeta;\Gamma \Rightarrow A{\multimap}C\colon \chi}\ {\multimap}R$$

21. $$\frac{\zeta;\Gamma,\overrightarrow{B}\colon y \Rightarrow C\colon \chi}{\zeta;\Gamma \Rightarrow C{\circ\!\!-}B\colon \chi}\ {\circ\!\!-}R$$

22. $$\frac{\zeta;\overrightarrow{A}\colon x,\Gamma \Rightarrow C\colon \chi}{\zeta;\Gamma \Rightarrow A{-\!\!\bullet}C\colon 0}\ {-\!\!\bullet}R$$

23. $$\frac{\Xi\langle\overrightarrow{A}\colon x,\overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega}{\Xi\langle\overrightarrow{A◐B}\colon z\rangle \Rightarrow D\colon \omega\{z/y\}}\ ◐L$$

24. $$\frac{\Xi\langle\overrightarrow{A}\colon x,\overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega}{\Xi\langle\overrightarrow{A◑B}\colon z\rangle \Rightarrow D\colon \omega\{z/x\}}\ ◑L$$

Figure 9.9: Asynchronous semantically inactive continuous multiplicative rules

25. $$\frac{\zeta;\Gamma|_k\overrightarrow{B}\colon y \Rightarrow C\colon \chi}{\zeta;\Gamma \Rightarrow C\overset{\bullet}{|}_k B\colon \chi}\ \overset{\bullet}{|}_k R$$

26. $$\frac{\zeta;\overrightarrow{A}\colon x|_k\Gamma \Rightarrow C\colon \chi}{\zeta;\Gamma \Rightarrow A\underset{\circ}{|}_k C\colon 0}\ \underset{\circ}{|}_k R$$

27. $$\frac{\zeta;\Gamma|_k\overrightarrow{B}\colon y \Rightarrow C\colon \chi}{\zeta;\Gamma \Rightarrow C\overset{\circ}{|}_k B\colon 0}\ \overset{\circ}{|}_k R$$

28. $$\frac{\zeta;\overrightarrow{A}\colon x|_k\Gamma \Rightarrow C\colon \chi}{\zeta;\Gamma \Rightarrow A\underset{\bullet}{|}_k C\colon \chi}\ \underset{\bullet}{|}_k R$$

29. $$\frac{\Xi\langle\overrightarrow{A}\colon x|_k\overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega}{\Xi\langle\overrightarrow{A◓_k B}\colon z\rangle \Rightarrow D\colon \omega\{z/y\}}\ ◓_k L$$

30. $$\frac{\Xi\langle\overrightarrow{A}\colon x|_k\overrightarrow{B}\colon y\rangle \Rightarrow D\colon \omega}{\Xi\langle\overrightarrow{A◒_k B}\colon z\rangle \Rightarrow D\colon \omega\{z/x\}}\ ◒_k L$$

Figure 9.10: Asynchronous semantically inactive discontinuous multiplicative rules

$$31. \quad \frac{\Xi \Rightarrow A\colon \chi \qquad \Xi \Rightarrow B\colon \chi}{\Xi \Rightarrow A \sqcap B\colon \chi}\sqcap R$$

$$32. \quad \frac{\Xi\langle \overrightarrow{A}\colon z\rangle \Rightarrow C\colon \chi \qquad \Xi\langle \overrightarrow{B}\colon z\rangle \Rightarrow C\colon \chi}{\Xi\langle \overrightarrow{A \sqcup B}\colon z\rangle \Rightarrow C\colon \chi}\sqcup L$$

Figure 9.11: Asynchronous semantically inactive additive rules

$$33. \quad \frac{\Xi \Rightarrow A[a/v]\colon \phi}{\Xi \Rightarrow \forall v A\colon \phi}\forall R^{\dagger} \qquad 34. \quad \frac{\Xi\langle \overrightarrow{A[a/v]}\colon x\rangle \Rightarrow B\colon \psi}{\Xi\langle \overrightarrow{\exists v A}\colon x\rangle \Rightarrow B\colon \psi}\exists L^{\dagger}$$

Figure 9.12: Asynchronous semantically inactive quantifier rules, where † indicates that there is no a in the conclusion

$$35. \quad \frac{\boxtimes\Xi \Rightarrow A\colon \phi}{\boxtimes\Xi \Rightarrow \blacksquare A\colon \phi}\blacksquare R \qquad 36. \quad \frac{\boxtimes\Xi\langle \overrightarrow{A}\colon x\rangle \Rightarrow \diamondplus B\colon \psi}{\boxtimes\Xi\langle \overrightarrow{\blacklozenge A}\colon x\rangle \Rightarrow \diamondplus B\colon \psi}\blacklozenge L$$

Figure 9.13: Asynchronous semantically inactive normal modality rules; $\boxtimes/\diamondplus$ marks a structure all the types of which have principal connective a box/diamond

9.3.3 Unary synthetic multiplicatives

37. $\dfrac{\zeta;\Gamma,1 \Rightarrow A\colon\phi}{\zeta;\Gamma \Rightarrow \triangleleft^{-1}A\colon\phi}\triangleleft^{-1}R$ 38. $\dfrac{\Xi\langle\overrightarrow{A}\colon x,1\rangle \Rightarrow B\colon\psi}{\Xi\langle\overrightarrow{\triangleleft A}\colon x\rangle \Rightarrow B\colon\psi}\triangleleft L$

39. $\dfrac{\zeta;1,\Gamma \Rightarrow A\colon\phi}{\zeta;\Gamma \Rightarrow \triangleright^{-1}A\colon\phi}\triangleright^{-1}R$

40. $\dfrac{\Xi\langle 1,\overrightarrow{A}\colon x\rangle \Rightarrow B\colon\psi}{\Xi\langle\overrightarrow{\triangleright A}\colon x\rangle \Rightarrow B\colon\psi}\triangleright L$

41. $\dfrac{\zeta;\Delta\,|_k\,\Lambda \Rightarrow B\colon\psi}{\zeta;\Delta \Rightarrow \check{}^k B\colon\psi}\check{}^k R$ 42. $\dfrac{\Xi\langle\overrightarrow{B}\colon y\,|_k\,\Lambda\rangle \Rightarrow C\colon\chi}{\Xi\langle\overrightarrow{\hat{}^k B}\colon y\rangle \Rightarrow C\colon\chi}\hat{}^k L$

Figure 9.14: Asynchronous deterministic synthetic multiplicative rules

9.3.4 Binary synthetic multiplicatives

43. $\dfrac{\zeta;\overrightarrow{A}\colon x,\Gamma \Rightarrow C\colon\chi \qquad \zeta;\Gamma,\overrightarrow{A}\colon x \Rightarrow C\colon\chi}{\zeta;\Gamma \Rightarrow C\div A\colon\lambda x\chi}\div R$

44. $\dfrac{\Xi\langle\overrightarrow{A}\colon x,\overrightarrow{B}\colon y\rangle \Rightarrow D\colon\omega \qquad \Xi\langle\overrightarrow{B}\colon y,\overrightarrow{A}\colon x\rangle \Rightarrow D\colon\omega}{\Xi\langle\overrightarrow{A\circ B}\colon z\rangle \Rightarrow D\colon\omega\{\pi_1 z/x,\pi_2 z/y\}}\circ L$

45. $\dfrac{\zeta;\Gamma\,|_1\,\overrightarrow{B}\colon y \Rightarrow C\colon\chi \qquad \cdots \qquad \zeta;\Gamma\,|_{sC'-sB}\,\overrightarrow{B}\colon y \Rightarrow C\colon\chi}{\zeta;\Gamma \Rightarrow C\overset{\wedge}{=}B\colon\lambda y\chi}\overset{\wedge}{=}R$

46. $\dfrac{\zeta;\overrightarrow{A}\colon x\,|_1\,\Gamma \Rightarrow C\colon\chi \qquad \cdots \qquad \zeta;\overrightarrow{A}\colon x\,|_{sA}\,\Gamma \Rightarrow C\colon\chi}{\zeta;\Gamma \Rightarrow A\overset{\vee}{=}C\colon\lambda x\chi}\overset{\vee}{=}R$

47. $\dfrac{\Xi\langle\overrightarrow{A}\colon x\,|_1\,\overrightarrow{B}\colon y\rangle \Rightarrow D\colon\omega \qquad \cdots \qquad \Xi\langle\overrightarrow{A}\colon x\,|_{sA}\,\overrightarrow{B}\colon y\rangle \Rightarrow D\colon\omega}{\Xi\langle\overrightarrow{A\diamond B}\colon z\rangle \Rightarrow D\colon\omega\{\pi_1 z/x,\pi_2 z/y\}}\diamond L$

Figure 9.15: Asynchronous non-deterministic synthetic multiplicative rules

9.4 Synchronous left rules

(62) $$\frac{}{\overrightarrow{\boxed{P}}: x \Rightarrow P: x}\ id, \text{ if atomic\_focus(inp, } P)$$

9.4.1 Primary connectives

1. $$\frac{\zeta_1; \Gamma \Rightarrow \boxed{P}: \psi \qquad \zeta_2; \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \overrightarrow{\boxed{Q/P}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ /L \qquad \frac{\zeta_1; \Gamma \Rightarrow N: \psi \qquad \zeta_2; \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \overrightarrow{\boxed{Q/N}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ /L$$

$$\frac{\zeta_1; \Gamma \Rightarrow \boxed{P}: \psi \qquad \zeta_2; \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \overrightarrow{\boxed{M/P}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ /L \qquad \frac{\zeta_1; \Gamma \Rightarrow M: \psi \qquad \zeta_2; \Delta\langle \overrightarrow{N}: z\rangle, \Delta \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \overrightarrow{\boxed{N/M}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ /L$$

2. $$\frac{\zeta_1; \Gamma \Rightarrow \boxed{P}: \phi \qquad \zeta_2; \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta_1, \langle \Gamma, \overrightarrow{\boxed{P\backslash Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ \backslash L \qquad \frac{\zeta_1; \Gamma \Rightarrow \boxed{P}: \phi \qquad \zeta_2; \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta_1, \langle \Gamma, \overrightarrow{\boxed{P\backslash M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ \backslash L$$

$$\frac{\zeta_1; \Gamma \Rightarrow N: \phi \qquad \zeta_2; \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \Gamma, \overrightarrow{\boxed{N\backslash Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ \backslash L \qquad \frac{\zeta_1; \Gamma \Rightarrow N: \phi \qquad \zeta_2; \Delta_1, \langle \overrightarrow{M}: z\rangle, \Delta_2) \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \Gamma, \overrightarrow{\boxed{N\backslash M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ \backslash L$$

Figure 9.16: Left synchronous continuous multiplicative rules

5. $$\frac{\zeta_1; \Gamma \Rightarrow \boxed{P}: \psi \qquad \zeta_2; \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \overrightarrow{\boxed{Q{\uparrow_k}P}}: x\,|_k\,\Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ {\uparrow_k}L \qquad \frac{\zeta; \Gamma \Rightarrow N: \psi \qquad \zeta_1; \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \overrightarrow{\boxed{Q{\uparrow_k}N}}: x\,|_k\,\Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ {\uparrow_k}L$$

$$\frac{\zeta; \Gamma \Rightarrow \boxed{P}: \psi \qquad \zeta_1; \Delta\langle \overrightarrow{M}: z\rangle, \Delta \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \overrightarrow{\boxed{M{\uparrow_k}P}}: x\,|_k\,\Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ {\uparrow_k}L \qquad \frac{\zeta_1; \Gamma \Rightarrow N: \psi \qquad \zeta_2; \Delta\langle \overrightarrow{M}: z\rangle, \Delta_2) \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \overrightarrow{\boxed{M{\uparrow_k}N}}: x\,|_k\,\Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}}\ {\uparrow_k}L$$

6. $$\frac{\zeta_1; \Gamma \Rightarrow \boxed{P}: \phi \qquad \zeta_2; \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle, \Delta \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \Gamma\,|_k\, \overrightarrow{\boxed{P{\downarrow_k}Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ {\downarrow_k}L \qquad \frac{\zeta_1; \Gamma \Rightarrow \boxed{P}: \phi \qquad \zeta_2; \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \Gamma\,|_k\, \overrightarrow{\boxed{P{\downarrow_k}M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ {\downarrow_k}L$$

$$\frac{\zeta_1; \Gamma \Rightarrow N: \phi \qquad \zeta_2; \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \Gamma\,|_k\, \overrightarrow{\boxed{N{\downarrow_k}Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ {\downarrow_k}L \qquad \frac{\zeta_1; \Gamma \Rightarrow N: \phi \qquad \zeta_2; \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\zeta_1 \uplus \zeta_2; \Delta\langle \Gamma\,|_k\, \overrightarrow{\boxed{N{\downarrow_k}M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}}\ {\downarrow_k}L$$

Figure 9.17: Left synchronous discontinuous multiplicative rules

9.
$$\frac{\Xi\langle\overrightarrow{\boxed{Q}}: x\rangle \Rightarrow C\colon \chi}{\Xi\langle\overrightarrow{\boxed{Q\&B}}: z\rangle \Rightarrow C\colon \chi\{\pi_1 z/x\}}\,\&L_1 \qquad \frac{\Xi\langle\overrightarrow{M}: x\rangle \Rightarrow C\colon \chi}{\Xi\langle\overrightarrow{\boxed{M\&B}}: z\rangle \Rightarrow C\colon \chi\{\pi_1 z/x\}}\,\&L_1$$

$$\frac{\Xi\langle\overrightarrow{\boxed{Q}}: y\rangle \Rightarrow C\colon \chi}{\Xi\langle\overrightarrow{\boxed{A\&Q}}: z\rangle \Rightarrow C\colon \chi\{\pi_2 z/y\}}\,\&L_2 \qquad \frac{\Xi\langle\overrightarrow{M}: y\rangle \Rightarrow C\colon \chi}{\Xi\langle\overrightarrow{\boxed{A\&M}}: z\rangle \Rightarrow C\colon \chi\{\pi_2 z/y\}}\,\&L_2$$

Figure 9.18: Left synchronous additive rules

11.
$$\frac{\Xi\langle\overrightarrow{\boxed{Q[t/v]}}: x\rangle \Rightarrow B\colon \psi}{\Xi\langle\overrightarrow{\boxed{\bigwedge vQ}}: z\rangle \Rightarrow B\colon \psi\{(z\ t)/x\}}\,\bigwedge L \qquad \frac{\Xi\langle\overrightarrow{M[t/v]}: x\rangle \Rightarrow B\colon \psi}{\Xi\langle\overrightarrow{\boxed{\bigwedge vM}}: z\rangle \Rightarrow B\colon \psi\{(z\ t)/x\}}\,\bigwedge L$$

Figure 9.19: Left synchronous quantifier rules

13.
$$\frac{\Xi\langle\overrightarrow{\boxed{Q}}: x\rangle \Rightarrow B\colon \psi}{\Xi\langle\overrightarrow{\boxed{\Box Q}}: z\rangle \Rightarrow B\colon \psi\{^{\vee}z/x\}}\,\Box L \qquad \frac{\Xi\langle\overrightarrow{M}: x\rangle \Rightarrow B\colon \psi}{\Xi\langle\overrightarrow{\boxed{\Box M}}: z\rangle \Rightarrow B\colon \psi\{^{\vee}z/x\}}\,\Box L$$

Figure 9.20: Left synchronous normal modality rules

15.
$$\frac{\Xi\langle\overrightarrow{\boxed{Q}}: x\rangle \Rightarrow B\colon \psi}{\Xi\langle[\overrightarrow{\boxed{[\,]^{-1}Q}}: x]\rangle \Rightarrow B\colon \psi}\,[\,]^{-1}L \qquad \frac{\Xi\langle\overrightarrow{M}: x\rangle \Rightarrow B\colon \psi}{\Xi\langle[\overrightarrow{\boxed{[\,]^{-1}M}}: x]\rangle \Rightarrow B\colon \psi}\,[\,]^{-1}L$$

Figure 9.21: Left synchronous bracket modality rules

9.4.2 Semantically inactive variants

19.
$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}:\psi \qquad \Xi(\zeta';\Delta_1\langle\overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q\bullet\!-P}}:x,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{0/z\}}\ \bullet\!-L \qquad \frac{\zeta;\Gamma\Rightarrow N:\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q\bullet\!-N}}:x,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{0/z\}}\ \bullet\!-L$$

$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}:\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{M}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{M\bullet\!-P}}:x,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{0/z\}}\ \bullet\!-L \qquad \frac{\zeta;\Gamma\Rightarrow M:\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{N}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta;\Delta_1,\langle\overrightarrow{\boxed{N\bullet\!-M}}:x,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{0/z\}}\ \bullet\!-L$$

20.
$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}:\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{\boxed{P-\!\circ Q}}:y\rangle,\Delta_2)\Rightarrow D:\omega\{y/z\}}\ -\!\circ L \qquad \frac{\zeta;\Gamma\Rightarrow\boxed{P}:\phi \qquad \Xi(\zeta;\Delta_1,\langle\overrightarrow{M}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{\boxed{P-\!\circ M}}:y\rangle,\Delta_2)\Rightarrow D:\omega\{y/z\}}\ -\!\circ L$$

$$\frac{\zeta;\Gamma\Rightarrow N:\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{\boxed{N-\!\circ Q}}:y\rangle,\Delta_2)\Rightarrow D:\omega\{y/z\}}\ -\!\circ L \qquad \frac{\zeta;\Gamma\Rightarrow N:\phi \qquad \Xi(\zeta;\Delta_1,\langle\overrightarrow{M}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{\boxed{N-\!\circ M}}:y\rangle,\Delta_2)\Rightarrow D:\omega\{y/z\}}\ -\!\circ L$$

21.
$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}:\psi \qquad \Xi(\zeta';\Delta_1\langle\overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q\circ\!-P}}:x,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{x/z\}}\ \circ\!-L \qquad \frac{\zeta;\Gamma\Rightarrow N:\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q\circ\!-N}}:x,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{x/z\}}\ \circ\!-L$$

$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}:\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{M}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{M\circ\!-P}}:x,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{x/z\}}\ \circ\!-L \qquad \frac{\zeta;\Gamma\Rightarrow M:\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{N}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta;\Delta_1,\langle\overrightarrow{\boxed{N\circ\!-M}}:x,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{x/z\}}\ \circ\!-L$$

22.
$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}:\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{\boxed{P-\!\bullet Q}}:y\rangle,\Delta_2)\Rightarrow D:\omega\{0/z\}}\ -\!\bullet L \qquad \frac{\zeta;\Gamma\Rightarrow\boxed{P}:\phi \qquad \Xi(\zeta;\Delta_1,\langle\overrightarrow{M}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{\boxed{P-\!\bullet M}}:y\rangle,\Delta_2)\Rightarrow D:\omega\{0/z\}}\ -\!\bullet L$$

$$\frac{\zeta;\Gamma\Rightarrow N:\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{\boxed{N-\!\bullet Q}}:y\rangle,\Delta_2)\Rightarrow D:\omega\{0/z\}}\ -\!\bullet L \qquad \frac{\zeta;\Gamma\Rightarrow N:\phi \qquad \Xi(\zeta;\Delta_1,\langle\overrightarrow{M}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{\boxed{N-\!\bullet M}}:y\rangle,\Delta_2)\Rightarrow D:\omega\{0/z\}}\ -\!\bullet L$$

Figure 9.22: Left synchronous semantically inactive continuous multiplicative rules

25.
$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}{:}\,\psi \qquad \Xi(\zeta';\Delta_1\langle\overrightarrow{\boxed{Q}}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q\overset{\bullet}{\mid}_k P}}{:}\,x|_k\Gamma\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{x/z\}}\,\overset{\bullet}{\mid}_k L \qquad \frac{\zeta;\Gamma\Rightarrow N{:}\,\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q\overset{\bullet}{\mid}_k N}}{:}\,x|_k\Gamma\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{x/z\}}\,\overset{\bullet}{\mid}_k L$$

$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}{:}\,\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{M}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{M\overset{\bullet}{\mid}_k P}}{:}\,x|_k\Gamma\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{x/z\}}\,\overset{\bullet}{\mid}_k L \qquad \frac{\zeta;\Gamma\Rightarrow M{:}\,\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{N}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta;\Delta_1,\langle\overrightarrow{\boxed{N\overset{\bullet}{\mid}_k M}}{:}\,x|_k\Gamma\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{x/z\}}\,\overset{\bullet}{\mid}_k L$$

26.
$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}{:}\,\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma|_k\overrightarrow{\boxed{P\underset{\circ}{\mid}_k Q}}{:}\,y\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{0/z\}}\,\underset{\circ}{\mid}_k L \qquad \frac{\zeta;\Gamma\Rightarrow\boxed{P}{:}\,\phi \qquad \Xi(\zeta;\Delta_1,\langle\overrightarrow{M}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma|_k\overrightarrow{\boxed{P\underset{\circ}{\mid}_k M}}{:}\,y\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{0/z\}}\,\underset{\circ}{\mid}_k L$$

$$\frac{\zeta;\Gamma\Rightarrow N{:}\,\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma|_k\overrightarrow{\boxed{N\underset{\circ}{\mid}_k Q}}{:}\,y\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{0/z\}}\,\underset{\circ}{\mid}_k L \qquad \frac{\zeta;\Gamma\Rightarrow N{:}\,\phi \qquad \Xi(\zeta;\Delta_1,\langle\overrightarrow{M}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma|_k\overrightarrow{\boxed{N\underset{\circ}{\mid}_k M}}{:}\,y\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{0/z\}}\,\underset{\circ}{\mid}_k L$$

27.
$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}{:}\,\psi \qquad \Xi(\zeta';\Delta_1\langle\overrightarrow{\boxed{Q}}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q\overset{\circ}{\mid}_k P}}{:}\,x|_k\Gamma\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{0/z\}}\,\overset{\circ}{\mid}_k L \qquad \frac{\zeta;\Gamma\Rightarrow N{:}\,\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q\overset{\circ}{\mid}_k N}}{:}\,x|_k\Gamma\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{0/z\}}\,\overset{\circ}{\mid}_k L$$

$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}{:}\,\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{M}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{M\overset{\circ}{\mid}_k P}}{:}\,x|_k\Gamma\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{0/z\}}\,\overset{\circ}{\mid}_k L \qquad \frac{\zeta;\Gamma\Rightarrow M{:}\,\psi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{N}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta;\Delta_1,\langle\overrightarrow{\boxed{N\overset{\circ}{\mid}_k M}}{:}\,x|_k\Gamma\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{0/z\}}\,\overset{\circ}{\mid}_k L$$

28.
$$\frac{\zeta;\Gamma\Rightarrow\boxed{P}{:}\,\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma|_k\overrightarrow{\boxed{P\underset{\bullet}{\mid}_k Q}}{:}\,y\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{y/z\}}\,\underset{\bullet}{\mid}_k L \qquad \frac{\zeta;\Gamma\Rightarrow\boxed{P}{:}\,\phi \qquad \Xi(\zeta;\Delta_1,\langle\overrightarrow{M}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma|_k\overrightarrow{\boxed{P\underset{\bullet}{\mid}_k M}}{:}\,y\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{y/z\}}\,\underset{\bullet}{\mid}_k L$$

$$\frac{\zeta;\Gamma\Rightarrow N{:}\,\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma|_k\overrightarrow{\boxed{N\underset{\bullet}{\mid}_k Q}}{:}\,y\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{y/z\}}\,\underset{\bullet}{\mid}_k L \qquad \frac{\zeta;\Gamma\Rightarrow N{:}\,\phi \qquad \Xi(\zeta;\Delta_1,\langle\overrightarrow{M}{:}\,z\rangle,\Delta_2)\Rightarrow D{:}\,\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma|_\omega\overrightarrow{\boxed{N\underset{\bullet}{\mid}_k M}}{:}\,y\rangle,\Delta_2)\Rightarrow D{:}\,\omega\{y/z\}}\,\underset{\bullet}{\mid}_k L$$

Figure 9.23: Left synchronous semantically inactive discontinuous multiplicative rules

31.
$$\frac{\Xi\langle\overrightarrow{\boxed{Q}}{:}\,x\rangle\Rightarrow C{:}\,\chi}{\Xi\langle\overrightarrow{\boxed{Q\sqcap B}}{:}\,x\rangle\Rightarrow C{:}\,\chi}\,\sqcap L_1 \qquad \frac{\Xi\langle\overrightarrow{M}{:}\,x\rangle\Rightarrow C{:}\,\chi}{\Xi\langle\overrightarrow{\boxed{M\sqcap B}}{:}\,x\rangle\Rightarrow C{:}\,\chi}\,\sqcap L_1$$

$$\frac{\Xi\langle\overrightarrow{\boxed{Q}}{:}\,y\rangle\Rightarrow C{:}\,\chi}{\Xi\langle\overrightarrow{\boxed{A\sqcap Q}}{:}\,y\rangle\Rightarrow C{:}\,\chi}\,\sqcap L_2 \qquad \frac{\Xi\langle\overrightarrow{M}{:}\,y\rangle\Rightarrow C{:}\,\chi}{\Xi\langle\overrightarrow{\boxed{A\sqcap M}}{:}\,y\rangle\Rightarrow C{:}\,\chi}\,\sqcap L_2$$

Figure 9.24: Semantically inactive additives

$$33. \quad \frac{\Xi\langle \overrightarrow{\boxed{Q[t/v]}} : x\rangle \Rightarrow B : \psi}{\Xi\langle \overrightarrow{\boxed{\forall v Q}} : x\rangle \Rightarrow B : \psi}\,\forall L \qquad \frac{\Xi\langle \overrightarrow{M[t/v]} : x\rangle \Rightarrow B : \psi}{\Xi\langle \overrightarrow{\boxed{\forall v M}} : x\rangle \Rightarrow B : \psi}\,\forall L$$

Figure 9.25: Left synchronous semantically inactive quantifier rules

$$35. \quad \frac{\Xi\langle \overrightarrow{\boxed{Q}} : x\rangle \Rightarrow B : \psi}{\Xi\langle \overrightarrow{\boxed{\blacksquare Q}} : x\rangle \Rightarrow B : \psi}\,\blacksquare L \qquad \frac{\Xi\langle \overrightarrow{M} : x\rangle \Rightarrow B : \psi}{\Xi\langle \overrightarrow{\boxed{\blacksquare M}} : x\rangle \Rightarrow B : \psi}\,\blacksquare L$$

Figure 9.26: Left synchronous semantically inactive normal modality rules

9.4.3 Unary synthetic multiplicatives

37. $$\frac{\Xi\langle\overrightarrow{\boxed{Q}}: x\rangle \Rightarrow B\colon\psi}{\Xi\langle\overrightarrow{\boxed{\triangleleft^{-1}Q}}: x, 1\rangle \Rightarrow B\colon\psi}\,\triangleleft^{-1}L \qquad \frac{\Xi\langle\overrightarrow{M}: x\rangle \Rightarrow B\colon\psi}{\Xi\langle\overrightarrow{\boxed{\triangleleft^{-1}M}}: x, 1\rangle \Rightarrow B\colon\psi}\,\triangleleft^{-1}L$$

38. $$\frac{\Xi\langle\overrightarrow{\boxed{Q}}: x\rangle \Rightarrow B\colon\psi}{\Xi\langle 1, \overrightarrow{\boxed{\triangleright^{-1}Q}}: x\rangle \Rightarrow B\colon\psi}\,\triangleright^{-1}L \qquad \frac{\Xi\langle\overrightarrow{M}: x\rangle \Rightarrow B\colon\psi}{\Xi\langle 1, \overrightarrow{\boxed{\triangleright^{-1}M}}: x\rangle \Rightarrow B\colon\psi}\,\triangleright^{-1}L$$

41. $$\frac{\Xi\langle\overrightarrow{\boxed{Q}}: y\rangle \Rightarrow C\colon\chi}{\Xi\langle\overrightarrow{\boxed{\check{}_k Q}}: y\,|_k\,\Lambda\rangle \Rightarrow C\colon\chi}\,\check{}_k L \qquad \frac{\Xi\langle\overrightarrow{M}: y\rangle \Rightarrow C\colon\chi}{\Xi\langle\overrightarrow{\boxed{\check{}_k M}}: y\,|_k\,\Lambda\rangle \Rightarrow C\colon\chi}\,\check{}_k L$$

Figure 9.27: Left synchronous deterministic synthetic multiplicative rules

9.4.4 Binary synthetic multiplicatives

43. $$\frac{\zeta;\Gamma \Rightarrow \boxed{P}\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}: z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{\boxed{Q{\div}P}}: y\rangle,\Delta_2) \Rightarrow D\colon\omega\{(y\ \phi)/z\}}\,\div L_1 \qquad \frac{\zeta;\Gamma \Rightarrow N\colon\phi \qquad \Xi(\zeta';\Delta_1,\overrightarrow{\boxed{Q}}: z,\Delta_2) \Rightarrow D\colon\omega}{\Delta\langle\zeta\uplus\zeta';\Delta_1,\Gamma,\overrightarrow{\boxed{Q{\div}N}}: y,\Delta_2\rangle \Rightarrow D\colon\omega\{(y\ \phi)/z\}}\,\div L_1$$

$$\frac{\zeta;\Gamma \Rightarrow \boxed{P}\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{M}: z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{\boxed{M{\div}P}}: y\rangle,\Delta_2) \Rightarrow D\colon\omega\{(y\ \phi)/z\}}\,\div L_1 \qquad \frac{\zeta;\Gamma \Rightarrow N\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{M}: z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma,\overrightarrow{\boxed{M{\div}N}}: y\rangle,\Delta_2) \Rightarrow D\colon\omega\{(y\ \phi)/z\}}\,\div L_1$$

$$\frac{\zeta;\Gamma \Rightarrow \boxed{P}\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}: z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q{\div}P}}: y,\Gamma\rangle,\Delta_2) \Rightarrow D\colon\omega\{(y\ \phi)/z\}}\,\div L_2 \qquad \frac{\zeta;\Gamma \Rightarrow N\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q}}: z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{Q{\div}N}}: y,\Gamma\rangle,\Delta_2) \Rightarrow D\colon\omega\{(y\ \phi)/z\}}\,\div L_2$$

$$\frac{\zeta;\Gamma \Rightarrow \boxed{P}\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{M}: z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{M{\div}P}}: y,\Gamma\rangle,\Delta_2) \Rightarrow D\colon\omega\{(y\ \phi)/z\}}\,\div L_2 \qquad \frac{\zeta;\Gamma \Rightarrow N\colon\phi \qquad \Xi(\zeta';\Delta_1,\langle\overrightarrow{M}: z\rangle,\Delta_2) \Rightarrow D\colon\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\overrightarrow{\boxed{M{\div}N}}: y,\Gamma\rangle,\Delta_2) \Rightarrow D\colon\omega\{(y\ \phi)/z\}}\,\div L_2$$

Figure 9.28: Left synchronous continuous non-deterministic synthetic multiplicative rules

45.

$$\frac{\zeta;\Gamma \Rightarrow \boxed{P}:\psi \qquad \Xi(\zeta';\Delta_1,\langle \overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle \overrightarrow{\boxed{Q\overset{\wedge}{=}P}}:x\,|_k\,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{(x\ \psi)/z\}}\overset{\wedge}{=}L \qquad \frac{\zeta;\Gamma \Rightarrow N:\psi \qquad \Xi(\zeta';\Delta_1,\langle \overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle \overrightarrow{\boxed{Q\overset{\wedge}{=}N}}:x\,|_k\,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{(x\ \psi)/z\}}\overset{\wedge}{=}L$$

$$\frac{\zeta;\Gamma \Rightarrow \boxed{P}:\psi \qquad \Xi(\zeta';\Delta_1,\langle \overrightarrow{M}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi\langle\zeta\uplus\zeta';\Delta_1,\langle \overrightarrow{\boxed{M\overset{\wedge}{=}P}}:x\,|_k\,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{(x\ \psi)/z\}}\overset{\wedge}{=}L \qquad \frac{\zeta;\Gamma \Rightarrow N:\psi \qquad \Xi(\zeta';\Delta_1,\langle \overrightarrow{M}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle \overrightarrow{\boxed{M\overset{\wedge}{=}N}}:x\,|_k\,\Gamma\rangle,\Delta_2)\Rightarrow D:\omega\{(x\ \psi)/z\}}\overset{\wedge}{=}L$$

46.

$$\frac{\zeta;\Gamma \Rightarrow \boxed{P}:\phi \qquad \Xi(\zeta';\Delta_1,\langle \overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma\,|_k\,\overrightarrow{\boxed{P\overset{\vee}{=}Q}}:y\rangle,\Delta_2)\Rightarrow D:\omega\{(y\ \phi)/z\}}\overset{\vee}{=}L \qquad \frac{\zeta;\Gamma \Rightarrow \boxed{P}:\phi \qquad \Xi(\zeta';\Delta_1,\langle \overrightarrow{M}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma\,|_k\,\overrightarrow{\boxed{P\overset{\vee}{=}M}}:y\rangle,\Delta_2)\Rightarrow D:\omega\{(y\ \phi)/z\}}\overset{\vee}{=}L$$

$$\frac{\zeta;\Gamma \Rightarrow N:\phi \qquad \Xi(\zeta';\Delta_1,\langle \overrightarrow{\boxed{Q}}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Delta\langle\zeta\uplus\zeta';\Delta_1,\Gamma\,|_k\,\overrightarrow{\boxed{N\overset{\vee}{=}Q}}:y,\Delta_2\rangle\Rightarrow D:\omega\{(y\ \phi)/z\}}\overset{\vee}{=}L \qquad \frac{\zeta;\Gamma \Rightarrow N:\phi \qquad \Xi(\zeta';\Delta_1,\langle \overrightarrow{M}:z\rangle,\Delta_2)\Rightarrow D:\omega}{\Xi(\zeta\uplus\zeta';\Delta_1,\langle\Gamma\,|_k\,\overrightarrow{\boxed{N\overset{\vee}{=}M}}:y\rangle,\Delta_2)\Rightarrow D:\omega\{(y\ \phi)/z\}}\overset{\vee}{=}L$$

Figure 9.29: Left synchronous discontinuous non-deterministic synthetic multiplicative rules

9.5 Synchronous stoup rules

$$17. \quad \frac{\Xi(\zeta; \Gamma_1, A\colon x, \Gamma_2) \Rightarrow B\colon \psi}{\Xi(\zeta \uplus \{\boxed{A}\colon x\}; \Gamma_1, \Gamma_2) \Rightarrow B\colon \psi}\ !L$$

$$\frac{\Xi(\zeta \uplus \{\boxed{\mathrm{A}}\colon x\}; \Gamma_1, [\{A\colon y\}; \Gamma_2], \Gamma_3) \Rightarrow B\colon \psi}{\Xi(\zeta \uplus \{\boxed{\mathrm{A}}\colon x\}; \Gamma_1, \Gamma_2, \Gamma_3) \Rightarrow B\colon \psi\{x/y\}}\ C$$

Figure 9.30: Synchronous exponential stoup rules

9.6 Synchronous right rules

$$(63)\quad \frac{}{\overrightarrow{P}\colon x \Rightarrow \boxed{P}\colon x}\ id, \text{ if atomic\_focus(out, } P)$$

9.6.1 Primary connectives

$$3. \quad \frac{\zeta; \Gamma_1 \Rightarrow \boxed{P_1}\colon \phi \qquad \zeta'; \Gamma_2 \Rightarrow \boxed{P_2}\colon \psi}{\zeta \uplus \zeta'; \Gamma_1, \Gamma_2 \Rightarrow \boxed{P_1 \bullet P_2}\colon (\phi, \psi)}\ \bullet R \qquad \frac{\zeta; \Gamma_1 \Rightarrow \boxed{P}\colon \phi \qquad \zeta'; \Gamma_2 \Rightarrow N\colon \psi}{\zeta \uplus \zeta'; \Gamma_1, \Gamma_2 \Rightarrow \boxed{P \bullet N}\colon (\phi, \psi)}\ \bullet R$$

$$\frac{\zeta; \Gamma_1 \Rightarrow N\colon \phi \qquad \zeta'; \Gamma_2 \Rightarrow \boxed{P}\colon \psi}{\zeta \uplus \zeta'; \Gamma_1, \Gamma_2 \Rightarrow \boxed{N \bullet P}\colon (\phi, \psi)}\ \bullet R \qquad \frac{\zeta; \Gamma_1 \Rightarrow N_1\colon \phi \qquad \zeta'; \Gamma_2 \Rightarrow N_2\colon \psi}{\zeta \uplus \zeta'; \Gamma_1, \Gamma_2 \Rightarrow \boxed{N_1 \bullet N_2}\colon (\phi, \psi)}\ \bullet R$$

$$4. \quad \frac{}{\Lambda \Rightarrow \boxed{I}\colon 0}\ IR$$

Figure 9.31: Right synchronous continuous multiplicative rules

$$7. \quad \frac{\zeta; \Gamma_1 \Rightarrow \boxed{P_1}\colon \phi \qquad \zeta'; \Gamma_2 \Rightarrow \boxed{P_2}\colon \psi}{\zeta \uplus \zeta'; \Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{P_1 \odot_k P_2}\colon (\phi, \psi)}\ \odot_k R \qquad \frac{\zeta; \Gamma_1 \Rightarrow \boxed{P}\colon \phi \qquad \zeta'; \Gamma_2 \Rightarrow N\colon \psi}{\zeta \uplus \zeta'; \Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{P \odot_k N}\colon (\phi, \psi)}\ \odot_k R$$

$$\frac{\zeta; \Gamma_1 \Rightarrow N\colon \phi \qquad \zeta'; \Gamma_2 \Rightarrow \boxed{P}\colon \psi}{\zeta \uplus \zeta'; \Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{N \odot_k P}\colon (\phi, \psi)}\ \odot_k R \qquad \frac{\zeta; \Gamma_1 \Rightarrow N_1\colon \phi \qquad \zeta'; \Gamma_2 \Rightarrow N_2\colon \psi}{\zeta \uplus \zeta'; \Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{N_1 \odot_k N_2}\colon (\phi, \psi)}\ \odot_k R$$

$$8. \quad \frac{}{1 \Rightarrow \boxed{J}\colon 0}\ JR$$

Figure 9.32: Right synchronous discontinuous multiplicative rules

10. $\dfrac{\Xi \Rightarrow \boxed{P}\colon \phi}{\Xi \Rightarrow \boxed{P{\oplus}B}\colon \iota_1\phi}\oplus R_1 \qquad \dfrac{\Xi \Rightarrow N\colon \phi}{\Xi \Rightarrow \boxed{N{\oplus}B}\colon \iota_1\phi}\oplus R_1$

$\dfrac{\Xi \Rightarrow \boxed{P}\colon \psi}{\Xi \Rightarrow \boxed{A{\oplus}P}\colon \iota_2\psi}\oplus R_2 \qquad \dfrac{\Xi \Rightarrow N\colon \psi}{\Xi \Rightarrow \boxed{A{\oplus}N}\colon \iota_2\psi}\oplus R_2$

Figure 9.33: Right synchronous additive rules

12. $\dfrac{\Xi \Rightarrow \boxed{P[t/v]}\colon \phi}{\Xi \Rightarrow \boxed{\bigvee vP}\colon (t,\phi)}\bigvee R \qquad \dfrac{\Xi \Rightarrow N[t/v]\colon \phi}{\Xi \Rightarrow \boxed{\bigvee vN}\colon (t,\phi)}\bigvee R$

Figure 9.34: Right synchronous quantifier rules

14. $\dfrac{\Xi \Rightarrow \boxed{P}\colon \phi}{\Xi \Rightarrow \boxed{\Diamond P}\colon {}^{\cap}\phi}\Diamond R \qquad \dfrac{\Xi \Rightarrow N\colon \phi}{\Xi \Rightarrow \boxed{\Diamond N}\colon {}^{\cap}\phi}\Diamond R$

Figure 9.35: Right synchronous normal modality rules

16. $\dfrac{\Xi \Rightarrow \boxed{P}\colon \phi}{[\Xi] \Rightarrow \boxed{\langle\rangle P}\colon \phi}\langle\rangle R \qquad \dfrac{\Xi \Rightarrow N\colon \phi}{[\Xi] \Rightarrow \boxed{\langle\rangle N}\colon \phi}\langle\rangle R$

Figure 9.36: Right synchronous bracket modality rules

18. $\dfrac{\Xi \Rightarrow \boxed{P}\colon \phi}{\Xi \Rightarrow \boxed{?P}\colon [\phi]}?R \qquad \dfrac{\Xi \Rightarrow N\colon \phi}{\Xi \Rightarrow \boxed{?N}\colon [\phi]}?R$

$\dfrac{\zeta;\Gamma \Rightarrow \boxed{P}\colon \phi \qquad \zeta';\Delta \Rightarrow \boxed{?P}\colon \psi}{\zeta \uplus \zeta';\Gamma,\Delta \Rightarrow \boxed{?P}\colon [\phi|\psi]}?M \qquad \dfrac{\zeta;\Gamma \Rightarrow N\colon \phi \qquad \zeta';\Delta \Rightarrow ?N\colon \psi}{\zeta \uplus \zeta';\Gamma,\Delta \Rightarrow \boxed{?N}\colon [\phi|\psi]}?M$

Figure 9.37: Right synchronous subexponential rules

9.6.2 Semantically inactive variants

23.
$$\frac{\zeta;\Gamma_1 \Rightarrow \boxed{P_1}:\phi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P_2}:\psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{P_1 \mathbin{◐} P_2}:\psi}\ ◐R \qquad \frac{\zeta;\Gamma_1 \Rightarrow \boxed{P}:\phi \qquad \zeta';\Gamma_2 \Rightarrow N:\psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{P \mathbin{◐} N}:\psi}\ ◐R$$

$$\frac{\zeta;\Gamma_1 \Rightarrow N:\phi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P}:\psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{N \mathbin{◐} P}:\psi}\ ◐R \qquad \frac{\zeta;\Gamma_1 \Rightarrow N_1:\phi \qquad \zeta';\Gamma_2 \Rightarrow N_2:\psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{N_1 \mathbin{◐} N_2}:\psi}\ ◐R$$

24.
$$\frac{\zeta;\Gamma_1 \Rightarrow \boxed{P_1}:\phi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P_2}:\psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{P_1 \mathbin{◑} P_2}:\phi}\ ◑R \qquad \frac{\zeta;\Gamma_1 \Rightarrow \boxed{P}:\phi \qquad \zeta';\Gamma_2 \Rightarrow N:\psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{P \mathbin{◑} N}:\phi}\ ◑R$$

$$\frac{\zeta;\Gamma_1 \Rightarrow N:\phi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P}:\psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{N \mathbin{◑} P}:\phi}\ ◑R \qquad \frac{\zeta;\Gamma_1 \Rightarrow N_1:\phi \qquad \zeta';\Gamma_2 \Rightarrow N_2:\psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{N_1 \mathbin{◑} N_2}:\phi}\ ◑R$$

Figure 9.38: Right synchronous rules for semantically inactive continuous multiplicatives

29.
$$\frac{\zeta;\Gamma_1 \Rightarrow \boxed{P_1}:\phi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P_2}:\psi}{\zeta \uplus \zeta';\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{P_1 ◓_k P_2}:\psi}\ ◓_k R \qquad \frac{\zeta;\Gamma_1 \Rightarrow \boxed{P}:\phi \qquad \zeta';\Gamma_2 \Rightarrow N:\psi}{\zeta \uplus \zeta';\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{P ◓_k N}:\psi}\ ◓_k R$$

$$\frac{\zeta;\Gamma_1 \Rightarrow N:\phi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P}:\psi}{\zeta \uplus \zeta';\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{N ◓_k P}:\psi}\ ◓_k R \qquad \frac{\zeta;\Gamma_1 \Rightarrow N_1:\phi \qquad \zeta';\Gamma_2 \Rightarrow N_2:\psi}{\zeta \uplus \zeta';\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{N_1 ◓_k N_2}:\psi}\ ◓_k R$$

30.
$$\frac{\zeta;\Gamma_1 \Rightarrow \boxed{P_1}:\phi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P_2}:\psi}{\zeta \uplus \zeta';\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{P_1 ◒_k P_2}:\phi}\ ◒_R \qquad \frac{\zeta;\Gamma_1 \Rightarrow \boxed{P}:\phi \qquad \zeta';\Gamma_2 \Rightarrow N:\psi}{\zeta \uplus \zeta';\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{P ◒_k N}:\phi}\ ◒_k R$$

$$\frac{\zeta;\Gamma_1 \Rightarrow N:\phi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P}:\psi}{\zeta \uplus \zeta';\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{N ◒_k P}:\phi}\ ◒_k R \qquad \frac{\zeta;\Gamma_1 \Rightarrow N_1:\phi \qquad \zeta';\Gamma_2 \Rightarrow N_2:\psi}{\zeta \uplus \zeta';\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{N_1 ◒_k N_2}:\phi}\ ◒_k R$$

Figure 9.39: Right synchronous rules for semantically inactive discontinuous multiplicatives

32. $$\frac{\Xi \Rightarrow \boxed{P}\colon \phi}{\Xi \Rightarrow \boxed{P \sqcup B}\colon \phi}\sqcup R_1 \qquad \frac{\Xi \Rightarrow N\colon \phi}{\Xi \Rightarrow \boxed{N \sqcup B}\colon \phi}\sqcup R_1$$

$$\frac{\Xi \Rightarrow P\colon \psi}{\Xi \Rightarrow \boxed{A \sqcup P}\colon \psi}\sqcup R_2 \qquad \frac{\Xi \Rightarrow N\colon \psi}{\Xi \Rightarrow \boxed{A \sqcup N}\colon \psi}\sqcup R_2$$

Figure 9.40: Right synchronous semantically inactive additive rules

34. $$\frac{\Xi \Rightarrow \boxed{P[t/v]}\colon \phi}{\Xi \Rightarrow \boxed{\exists v P}\colon \phi}\exists R \qquad \frac{\Xi \Rightarrow N[t/v]\colon \phi}{\Xi \Rightarrow \boxed{\exists v N}\colon \phi}\exists R$$

Figure 9.41: Right synchronous semantically inactive quantifier rules

36. $$\frac{\Xi \Rightarrow \boxed{P}\colon \phi}{\Xi \Rightarrow \boxed{\blacklozenge P}\colon \phi}\blacklozenge R \qquad \frac{\Xi \Rightarrow N\colon \phi}{\Xi \Rightarrow \boxed{\blacklozenge N}\colon \phi}\blacklozenge R$$

Figure 9.42: Right synchronous semantically inactive normal modality rules

9.6.3 Unary synthetic multiplicatives

39.
$$\frac{\zeta;\Gamma \Rightarrow \boxed{P}\colon \phi}{\zeta;\Gamma,1 \Rightarrow \boxed{\triangleleft P}\colon \phi}\triangleleft R \qquad \frac{\zeta;\Gamma \Rightarrow N\colon \phi}{\zeta;\Gamma,1 \Rightarrow \boxed{\triangleleft N}\colon \phi}\triangleleft R$$

40.
$$\frac{\zeta;\Gamma \Rightarrow \boxed{P}\colon \phi}{\zeta;1,\Gamma \Rightarrow \boxed{\triangleright P}\colon \phi}\triangleright R \qquad \frac{\zeta;\Gamma \Rightarrow N\colon \phi}{\zeta;1,\Gamma \Rightarrow \boxed{\triangleright N}\colon \phi}\triangleleft R$$

42.
$$\frac{\zeta;\Delta \Rightarrow \boxed{P}\colon \psi}{\zeta;\Delta|_k\Lambda \Rightarrow \boxed{\hat{}_k P}\colon \psi}\hat{}_k R \qquad \frac{\zeta;\Delta \Rightarrow N\colon \psi}{\zeta;\Delta|_k\Lambda \Rightarrow \boxed{\hat{}_k N}\colon \psi}\hat{}_k R$$

Figure 9.43: Right synchronous deterministic synthetic multiplicative rules

9.6.4 Binary synthetic multiplicatives

44.
$$\frac{\zeta;\Gamma_1 \Rightarrow \boxed{P_1}\colon \phi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P_2}\colon \psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{P_1 \circ P_2}\colon (\phi,\psi)}\circ R_1 \qquad \frac{\zeta;\Gamma_1 \Rightarrow \boxed{P}\colon \phi \qquad \zeta';\Gamma_2 \Rightarrow N\colon \psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{P \circ N}\colon (\phi,\psi)}\circ R_1$$

$$\frac{\zeta;\Gamma_1 \Rightarrow \boxed{N}\colon \phi \qquad \zeta';\Gamma_2 \Rightarrow P\colon \psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{N \circ P}\colon (\phi,\psi)}\circ R_1 \qquad \frac{\zeta;\Gamma_1 \Rightarrow N_1\colon \phi \qquad \zeta';\Gamma_2 \Rightarrow N_2\colon \psi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{N_1 \circ N_2}\colon (\phi,\psi)}\circ R_1$$

$$\frac{\zeta;\Gamma_1 \Rightarrow \boxed{P_2}\colon \psi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P_1}\colon \phi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{P_1 \circ P_2}\colon (\phi,\psi)}\circ R_2 \qquad \frac{\zeta;\Gamma_1 \Rightarrow N\colon \psi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P}\colon \phi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{P \circ N}\colon (\phi,\psi)}\circ R_2$$

$$\frac{\zeta;\Gamma_1 \Rightarrow \boxed{P}\colon \psi \qquad \zeta';\Gamma_2 \Rightarrow N\colon \phi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{N \circ P}\colon (\phi,\psi)}\circ R_2 \qquad \frac{\zeta;\Gamma_1 \Rightarrow N_2\colon \psi \qquad \zeta';\Gamma_2 \Rightarrow N_1\colon \phi}{\zeta \uplus \zeta';\Gamma_1,\Gamma_2 \Rightarrow \boxed{N_1 \circ N_2}\colon (\phi,\psi)}\circ R_2$$

Figure 9.44: Right synchronous continuous non-deterministic derived multiplicative rules

47.
$$\frac{\zeta;\Gamma_1 \Rightarrow \boxed{P_1}\colon \phi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P_2}\colon \psi}{\zeta \uplus \zeta';\Gamma_1|_k\Gamma_2 \Rightarrow \boxed{P_1 \diamond P_2}\colon (\phi,\psi)}\diamond R \qquad \frac{\zeta;\Gamma_1 \Rightarrow \boxed{P}\colon \phi \qquad \zeta';\Gamma_2 \Rightarrow B\colon \psi}{\zeta \uplus \zeta';\Gamma_1|_k\Gamma_2 \Rightarrow \boxed{P \diamond N}\colon (\phi,\psi)}\diamond R$$

$$\frac{\zeta;\Gamma_1 \Rightarrow N\colon \phi \qquad \zeta';\Gamma_2 \Rightarrow \boxed{P}\colon \psi}{\zeta \uplus \zeta';\Gamma_1|_k\Gamma_2 \Rightarrow \boxed{N \diamond P}\colon (\phi,\psi)}\diamond R \qquad \frac{\zeta;\Gamma_1 \Rightarrow N_1\colon \phi \qquad \zeta';\Gamma_2 \Rightarrow N_2\colon \psi}{\zeta \uplus \zeta';\Gamma_1|_k\Gamma_2 \Rightarrow \boxed{N_1 \diamond N_2}\colon (\phi,\psi)}\diamond R$$

Figure 9.45: Right synchronous discontinuous non-deterministic synthetic multiplicative rules

48.

$$\frac{\zeta;\Gamma\langle\overrightarrow{B_0}\colon y_0;\ldots;\boxed{\overrightarrow{Q}};\ldots;\overrightarrow{B_n}\colon y_n\rangle \Rightarrow D\colon\omega}{\zeta;\Gamma\langle\overrightarrow{B_0|A}\colon z_0;\ldots;\boxed{\overrightarrow{Q|A}};\ldots;\overrightarrow{B_n|A}\colon z_n\rangle \Rightarrow D|A\colon \lambda x\omega\{(z_0\ x)/y_0,\ldots,(z_n\ x)/y_n\}}\,|R$$

$$\frac{\zeta;\Gamma\langle\overrightarrow{B_0}\colon y_0;\ldots;\overrightarrow{M};\ldots;\overrightarrow{B_n}\colon y_n\rangle \Rightarrow D\colon\omega}{\zeta;\Gamma\langle\overrightarrow{B_0|A}\colon z_0;\ldots;\boxed{\overrightarrow{M|A}};\ldots;\overrightarrow{B_n|A}\colon z_n\rangle \Rightarrow D|A\colon \lambda x\omega\{(z_0\ x)/y_0,\ldots,(z_n\ x)/y_n\}}\,|R$$

Figure 9.46: Right synchronous rules for limited contraction

49.

$$\frac{}{\emptyset;\Lambda \Rightarrow \boxed{W(0)}\colon 0}\,WR \qquad \frac{\zeta_1;\Gamma_1 \Rightarrow \boxed{W(v)}\colon\phi \qquad \zeta_2;\Gamma_2 \Rightarrow \boxed{W(w)}\colon\phi}{\zeta_1 \uplus \zeta_2;\Gamma_1,\Gamma_2 \Rightarrow \boxed{W(v+w)}\colon 0}\,WR$$

Figure 9.47: Right synchronous rules for limited expansion

50. $$\frac{\Xi \Rightarrow \boxed{P_1}: \psi \qquad \nvdash \Xi \Rightarrow \boxed{P_2}: \_}{\Xi \Rightarrow \boxed{P_1 - P_2}: \psi} -R \qquad \frac{\Xi \Rightarrow \boxed{P}: \psi \qquad \nvdash \Xi \Rightarrow N: \_}{\Xi \Rightarrow \boxed{P - N}: \psi} -R$$

$$\frac{\Xi \Rightarrow N: \psi \qquad \nvdash \Xi \Rightarrow \boxed{P}: \_}{\Xi \Rightarrow \boxed{N - P}: \psi} -R \qquad \frac{\Xi \Rightarrow N_1: \psi \qquad \nvdash \Xi \Rightarrow N_2: \_}{\Xi \Rightarrow \boxed{N_1 - N_2}: \psi} -R$$

Figure 9.48: Right synchronous rules for difference

9.7 Completeness of focalisation for DA

We shall be dealing with three systems: the displacement calculus **DA** with sequents notated $\Delta \Rightarrow A$, the *weakly focalised* displacement calculus with additives $\mathbf{DA_{foc}}$ with sequents notated $\Delta \Longrightarrow_w A$, and the *strongly focalised* displacement calculus with additives $\mathbf{DA_{Foc}}$ with sequents notated $\Delta \Longrightarrow A$. Sequents of both $\mathbf{DA_{foc}}$ and $\mathbf{DA_{Foc}}$ may contain at most one focalised formula, possibly A. When a $\mathbf{DA_{foc}}$ sequent is notated $\Delta \Longrightarrow_w A \diamond \text{foc}$, it means that the sequent possibly contains a (unique) focalised formula. Otherwise, $\Delta \Longrightarrow_w A$ means that the sequent does not contain a focus.

In this section we prove the strong focalisation property for the displacement calculus with additives **DA**. The focalisation property for Linear Logic was discovered by Andreoli (1992[5]). In this paper we follow the proof idea from [47], which we adapt to the intuitionistic non-commutative case **DA** with twin multiplicative modes of combination, the continuous (concatenation) and the discontinuous (intercalation) products. The proof relies heavily on the Cut-elimination property for weakly focalised **DA** which is proved in the appendix. In our presentation of focalisation we have avoided the *react* rules of [5] and [12], and use instead a simpler, box, notation suitable for non-commutativity.

$\mathbf{DA_{Foc}}$ is a subsystem of $\mathbf{DA_{foc}}$. $\mathbf{DA_{foc}}$ has the focusing rules *foc* and Cut rules *p-Cut*$_1$, *p-Cut*$_2$, *n-Cut*$_1$ and *n-Cut*$_2$[1] shown in (64), and the synchronous and asynchronous rules displayed before, which are read as allowing in synchronous rules the occurrence of asynchronous formulas, and in asynchronous rules as allowing arbitrary sequents with possibly one focalised formula. $\mathbf{DA_{Foc}}$ has the focusing rules but not the Cut rules, and the synchronous and asynchronous rules displayed before, which are such that focalised sequents cannot contain any complex asynchronous formulas, whereas sequents with at least one complex asynchronous formula cannot contain a focalised formula. Hence, strongly focalised proof search operates in alternating asynchronous and synchronous phases. The weakly focalised calculus $\mathbf{DA_{foc}}$ is an intermediate logic which we use to prove the completeness of $\mathbf{DA_{Foc}}$ for **DA**.

(64) $$\frac{\Delta\langle \overrightarrow{\boxed{Q}} \rangle \Longrightarrow_w A}{\Delta\langle \overrightarrow{Q} \rangle \Longrightarrow_w A} foc \qquad \frac{\Delta \Longrightarrow_w \boxed{P}}{\Delta \Longrightarrow_w P} foc$$

$$\frac{\Gamma \Longrightarrow_w \boxed{P} \qquad \Delta\langle \overrightarrow{P} \rangle \Longrightarrow_w C \diamond \text{foc}}{\Delta\langle \Gamma \rangle \Longrightarrow_w C \diamond \text{foc}} p\text{-}Cut_1 \qquad \frac{\Gamma \Longrightarrow_w N \diamond \text{foc} \qquad \Delta\langle \overrightarrow{\boxed{N}} \rangle \Longrightarrow_w C}{\Delta\langle \Gamma \rangle \Longrightarrow_w C \diamond \text{foc}} p\text{-}Cut_2$$

$$\frac{\Gamma \Longrightarrow_w P \diamond \text{foc} \qquad \Delta\langle \overrightarrow{P} \rangle \Longrightarrow_w C}{\Delta\langle \Gamma \rangle \Longrightarrow_w C \diamond \text{foc}} n\text{-}Cut_1 \qquad \frac{\Gamma \Longrightarrow_w N \qquad \Delta\langle \overrightarrow{N} \rangle \Longrightarrow_w C \diamond \text{foc}}{\Delta\langle \Gamma \rangle \Longrightarrow_w C \diamond \text{foc}} n\text{-}Cut_2$$

As before, Q and P denote synchronous formulas in input and output position respectively, whereas M and N denote asynchronous formulas in input and output position respectively; we will abbreviate thus: *left sync., right synch., left async., and right async.*. Occurrences of P, Q, M and N are supposed not to be focalised, which means that their focalised occurrence *must* be signalled with a box. By contrast, occurrences of A, B, C may be focalised (if the sequent they occur in is judged $\diamond$ foc) or not. The table below summarizes the notational convention on formulas P, Q, M and N:

[1] If it is convenient, we may drop the subscripts.

	input	output
mboxsync.	**Q**	**P**
async. hline	**M**	**N**

2

9.7.1 Embedding of DA into $\mathbf{DA_{foc}}$

The identity axiom we consider for **DA** and for both $\mathbf{DA_{foc}}$ and $\mathbf{DA_{Foc}}$ is restricted to atomic types; recalling that atomic types are classified into positive bias At^+ and negative bias At^-:

(65) If $P \in At^+$, $P \Longrightarrow_w \boxed{P}$ and $P \Longrightarrow \boxed{P}$
If $Q \in At^-$, $\boxed{Q} \Longrightarrow_w Q$ and $\boxed{Q} \Longrightarrow Q$

In fact, the Identity rule holds of any type A. It has the following formulation in the sequent calculi considered here:

(66)
$$\begin{cases} \vec{A} \Rightarrow A & & \text{in } \mathbf{DA} \\ \vec{P} \Longrightarrow_w \boxed{P} & \overrightarrow{\boxed{N}} \Longrightarrow_w N & \text{in } \mathbf{DA_{foc}} \\ \vec{P} \Longrightarrow P & \vec{N} \Longrightarrow N & \text{in } \mathbf{DA_{Foc}} \end{cases}$$

The Identity axiom for arbitrary types is also known as *Eta-expansion*. Eta-expansion is easy to prove in both **DA** and $\mathbf{DA_{foc}}$, but the same is not the case for $\mathbf{DA_{Foc}}$. This is the reason to consider what we have called weak focalisation, which helps us to prove smoothly this crucial property for the proof of strong focalisation.

Theorem 9.7.1 (Embedding of **DA** into $\mathbf{DA_{foc}}$) *For any configuration Δ and type A, we have that if $\Delta \Rightarrow A$ then $\Delta \Longrightarrow_w A$.*

Proof. We proceed by induction on the length of the derivation of **DA** proofs. In the following lines, we apply the induction hypothesis (i.h.) for each premise of **DA** rules (with the exception of the Identity rule and the right rules of units):

- Identity axiom:

(67)
$$\frac{\vec{P} \Longrightarrow_w \boxed{\mathrm{P}}}{\vec{P} \Longrightarrow_w P}\,foc \qquad \frac{\overrightarrow{\boxed{\mathrm{N}}} \Longrightarrow_w N}{\vec{N} \Longrightarrow_w N}\,foc$$

- Cut rule: just apply *n-Cut*.
- Units

(68)
$$\frac{}{\Lambda \Rightarrow I}\,IR \quad \rightsquigarrow \quad \frac{\dfrac{}{\Lambda \Longrightarrow_w \boxed{I}}\,IR}{\Lambda \Longrightarrow_w I}\,foc$$

(69)
$$\frac{}{1 \Rightarrow J}\,JR \quad \rightsquigarrow \quad \frac{\dfrac{}{1 \Longrightarrow_w \boxed{J}}\,JR}{1 \Longrightarrow_w J}\,foc$$

Left unit rules apply as in the case of **DA**.

- Left discontinuous product: directly translates.
- Right discontinuous product. There are cases $P_1 \odot_k P_2$, $N_1 \odot_k N_2$, $N \odot_k P$ and $P \odot_k N$. We show one representative example:

$$\frac{\Delta \Rightarrow P \qquad \Gamma \Rightarrow N}{\Delta |_k \Gamma \Rightarrow P \odot_k N}\odot_k R \quad\rightsquigarrow\quad \frac{\dfrac{\Gamma \Longrightarrow_w N \qquad \dfrac{\Delta \Longrightarrow_w P \qquad \dfrac{\vec{P} \Longrightarrow_w \boxed{P} \qquad \dfrac{\boxed{\vec{N}} \Longrightarrow_w N}{\vec{N} \Longrightarrow_w N}\,foc}{\vec{P} |_k \vec{N} \Rightarrow \boxed{P \odot_k N}}\odot_k R}{\Delta |_k \vec{N} \Longrightarrow_w \boxed{P \odot_k N}}\,n\text{-}Cut}{\Delta |_k \Gamma \Longrightarrow_w \boxed{P \odot_k N}}\,n\text{-}Cut}{\Delta |_k \Gamma \Longrightarrow_w P \odot_k N}\,foc$$

- Left discontinuous $\uparrow_k$ rule (the left rule for $\downarrow_k$ is entirely similar). Like in the case for the right discontinuous product $\odot_k$ rule, we only show one representative example:

$$\frac{\Gamma \Rightarrow P \qquad \Delta\langle \vec{N} \rangle \Rightarrow A}{\Delta\langle \overrightarrow{N\uparrow_k P} |_k \Gamma \rangle \Rightarrow A}\uparrow_k L \quad\rightsquigarrow\quad \frac{\dfrac{\Gamma \Longrightarrow_w P \qquad \dfrac{\dfrac{\vec{P} \Longrightarrow_w \boxed{\mathrm{P}} \qquad \overrightarrow{\boxed{N}} \Longrightarrow_w N}{\overrightarrow{\boxed{N\uparrow_k P}} |_k \vec{P} \Longrightarrow_w N}\uparrow_k L \qquad \Delta\langle \vec{N} \rangle \Longrightarrow_w A}{\Delta\langle \overrightarrow{\boxed{N\uparrow_k P}} |_k \vec{P} \rangle \Longrightarrow_w A}\,n\text{-}Cut}{\Delta\langle \overrightarrow{\boxed{N\uparrow_k P}} |_k \Gamma \rangle \Longrightarrow_w A}\,n\text{-}Cut}{\Delta\langle \overrightarrow{N\uparrow_k P} |_k \Gamma \rangle \Longrightarrow_w A}\,foc$$

- Right discontinuous $\uparrow_k$ rule (the right discontinuous rule for $\downarrow_k$ is entirely similar):

(70) $$\frac{\Delta |_k \vec{A} \Rightarrow B}{\Delta \Rightarrow B\uparrow_k A}\uparrow_k R \quad\rightsquigarrow\quad \frac{\Delta |_k \vec{A} \Longrightarrow_w B}{\Delta \Longrightarrow_w B\uparrow_k A}\uparrow_k R$$

- Product and implicative continuous rules. These follow the same pattern as the discontinuous case. We interchange the metalinguistic k-th intercalation $|_k$ with the metalinguistic concatenation ',', and we interchange $\odot_k$, $\uparrow_k$ and $\downarrow_k$ with $\bullet$, $/$, and $\backslash$ respectively. Concerning additives, conjunction Right translates directly and we consider then conjunction Left (disjunction is symmetric):

$$\frac{\Delta\langle \vec{P} \rangle \Rightarrow C}{\Delta\langle \overrightarrow{P\&M} \rangle \Rightarrow C}\&L \quad\rightsquigarrow\quad \frac{\overrightarrow{P\&M} \Longrightarrow_w P \qquad \Delta\langle \vec{P} \rangle \Longrightarrow_w C}{\Delta\langle \overrightarrow{P\&M} \rangle \Longrightarrow_w C}\,n\text{-}Cut$$

where by Eta expansion and application of the *foc* rule, we have $\overrightarrow{P\&M} \Longrightarrow_w P$. This completes the proof.

9.7.2 Embedding of $\mathbf{DA_{foc}}$ into $\mathbf{DA_{Foc}}$

Theorem 9.7.2 (Embedding of $\mathbf{DA_{foc}}$ into $\mathbf{DA_{Foc}}$) *For any configuration Δ and type A, we have that if $\Delta \Longrightarrow_w A$ with one focalised formula and no asynchronous formula occurrence, then $\Delta \Longrightarrow A$ with the same formula focalised. If $\Delta \Longrightarrow_w A$ with no focalised formula and with at least one asynchronous formula, then $\Delta \Longrightarrow A$.*

Proof. We proceed by induction on the size of $\mathbf{DA_{foc}}$ sequents.[2] We consider Cut-free $\mathbf{DA_{foc}}$ proofs which match the sequents of this theorem. If the last rule is logical (i.e., it is not an instance of the *foc* rule) the i.h. applies directly and we get $\mathbf{DA_{Foc}}$ proofs of the same end-sequent. Now, let us suppose that the last rule is not logical, i.e. it is an instance of the *foc* rule. Let us suppose that the end sequent $\Delta \Longrightarrow_w A$ is a synchronous sequent. Suppose for example that the focalised formula is in the succedent:

[2] For a given type A, the *size* of A, $|A|$, is the number of connectives in A. By recursion on configurations we have:

$$\begin{array}{rcl} |\Lambda| & ::= & 0 \\ |\vec{A}, \Delta| & ::= & |A| + |\Delta|, \text{for } sA = 0 \\ |1, \Delta| & ::= & |\Delta| \\ |A\{\Delta_1 : \cdots : \Delta_{sA}\}| & ::= & |A| + \sum_{i=1}^{sA} |\Delta_i| \end{array}$$

Moreover, we have:

$|\Delta\langle \overrightarrow{\boxed{Q}} \rangle \Longrightarrow_w A| = |\Delta\langle \vec{Q} \rangle \Longrightarrow_w A|$

$|\Delta \Longrightarrow_w \boxed{P}| = |\Delta \Longrightarrow_w P|$

(71) $$\frac{\Delta \Longrightarrow_w \boxed{P}}{\Delta \Longrightarrow_w P}\, foc$$

The sequent $\Delta \Longrightarrow_w \boxed{P}$ arises from a synchronous rule to which we can apply i.h.. Let us suppose now that the end-sequent contains at least one asynchronous formula. We see three cases which are illustrative:

(72)
a. $\Delta\langle\overrightarrow{A\odot_k B}\rangle \Longrightarrow_w \boxed{P}$
b. $\Delta\langle\overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w B\uparrow_k A$
c. $\Delta\langle\overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w A\&B$

We have by Eta expansion that $\overrightarrow{A\odot_k B} \Longrightarrow_w \boxed{\overrightarrow{A\odot_k B}}$. We apply to this sequent the invertible $\odot_k$ left rule, whence $\overrightarrow{A}|_k\overrightarrow{B} \Longrightarrow_w \boxed{\overrightarrow{A\odot_k B}}$. In case (72a), we have the following proof in $\mathbf{DA_{foc}}$:

(73) $$\dfrac{\dfrac{\overrightarrow{A}\,|_k\,\overrightarrow{B} \Longrightarrow_w \boxed{\overrightarrow{A\odot_k B}} \qquad \Delta\langle\overrightarrow{A\odot_k B}\rangle \Longrightarrow_w \boxed{P}}{\Delta\langle\overrightarrow{A}\,|_k\,\overrightarrow{B}\rangle \Longrightarrow_w \boxed{P}}\, p\text{-}Cut_1}{\Delta\langle\overrightarrow{A}\,|_k\,\overrightarrow{B}\rangle \Longrightarrow_w P}\, foc$$

To the above $\mathbf{DA_{foc}}$ proof we apply Cut-elimination and we get the Cut-free $\mathbf{DA_{foc}}$ end-sequent $\Delta\langle\overrightarrow{A}\,|_k\overrightarrow{B}\rangle \Longrightarrow_w P$. We have $|\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle \Longrightarrow_w P| < |\Delta\langle\overrightarrow{A\odot_k B}\rangle \Longrightarrow_w P|$. We can apply then i.h. and we derive the provable $\mathbf{DA_{Foc}}$ sequent $\Delta\langle\overrightarrow{A}\,|_k\,\overrightarrow{B}\rangle \Longrightarrow P$ to which we can apply the left $\odot_k$ rule. We have obtained $\Delta\langle\overrightarrow{A\odot_k B}\rangle \Longrightarrow P$. In the same way, we have that $\boxed{\overrightarrow{B\uparrow_k A}}|_k\overrightarrow{A} \Longrightarrow_w B$. Thus, in case (72b), we have the following proof in $\mathbf{DA_{foc}}$:

(74) $$\dfrac{\dfrac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w B\uparrow_k A \qquad \boxed{\overrightarrow{B\uparrow_k A}}\,|_k\,\overrightarrow{A} \Longrightarrow_w B}{\Delta\langle\overrightarrow{\boxed{Q}}\rangle\,|_k\,\overrightarrow{A} \Longrightarrow_w B}\, p\text{-}Cut_2}{\Delta\langle\overrightarrow{Q}\rangle\,|_k\,\overrightarrow{A} \Longrightarrow_w B}\, foc$$

As before, we apply Cut-elimination to the above proof. We get the Cut-free $\mathbf{DA_{foc}}$ end-sequent $\Delta\langle\overrightarrow{Q}\rangle|_k\overrightarrow{A} \Longrightarrow_w B$. It has size less than $|\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow_w B\uparrow_k A|$. We can apply i.h. and we get the $\mathbf{DA_{Foc}}$ provable sequent $\Delta\langle\overrightarrow{Q}\rangle|_k\overrightarrow{A} \Longrightarrow B$ to which we apply the $\uparrow_k$ right rule. In case (72c):

(75) $$\frac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w A\&B}{\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow_w A\&B}\, foc$$

by applying the *foc* rule we get the provable $\mathbf{DA_{foc}}$ sequents $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow_w A$ and $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow_w B$. These sequents have smaller size than $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow_w A\&B$. The aforementioned sequents have a Cut-free proof in $\mathbf{DA_{foc}}$. We apply i.h. and we get $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow A$ and $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow B$. We apply the & right rule in $\mathbf{DA_{Foc}}$, and we get $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow A\&B$. This completes the proof.

9.8 Cut-elimination for displacement calculus with additives

We prove this by induction on the complexity (d,h) of top-most instances of Cut, where d is the size[3] of the cut formula and h is the length of the derivation the last rule of which is the Cut rule. There are four cases to consider: Cut with axiom in the minor premise, Cut with axiom in the major premise, principal Cuts, and permutation conversions. In each case, the complexity of the Cut is reduced. In order to save space, we will not be exhaustive showing all the cases because many follow the same pattern. In particular, for any synchronous logical rule there are always four cases to consider corresponding to the polarity of

[3]The size of $|A|$ is the number of connectives appearing in A.

the subformulas. Here, and in the following, we will show only one representative example. Concerning continuous and discontinuous formulas, we will show only the discontinuous cases (discontinuous connectives are less known than the continuous ones of the plain Lambek Calculus). For the continuous instances, the reader has only to interchange the meta-linguistic wrap $|_k$ with the meta-linguistic concatenation $','$, $\odot_k$ with $\bullet$, $\uparrow_k$ with $/$ and $\downarrow_k$ with $\backslash$. The units cases (principal case and permutation conversion cases) are completely trivial.

Proof. - *Id* cases:

(76) $$\dfrac{\overrightarrow{P}\Longrightarrow_w \boxed{P} \qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w B\diamond \text{foc}}{\Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w B\diamond \text{foc}}\,p\text{-}Cut_1 \qquad\rightsquigarrow\qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w B\diamond \text{foc}$$

$$\dfrac{\Delta\Longrightarrow_w N\diamond \text{foc} \qquad \overrightarrow{\boxed{N}}\Longrightarrow_w N}{\Delta\langle\overrightarrow{N}\rangle\Longrightarrow_w B\diamond \text{foc}}\,p\text{-}Cut_2 \qquad\rightsquigarrow\qquad \Delta\langle\overrightarrow{N}\rangle\Longrightarrow_w B\diamond \text{foc}$$

The attentive reader may have wondered whether the following *Id* case could arise:

(77) $$\dfrac{\boxed{\overrightarrow{Q}}\Rightarrow Q \qquad \Gamma\langle\overrightarrow{Q}\rangle\Rightarrow A}{\Gamma\langle\boxed{Q}\rangle\Rightarrow A}\,n\text{-}Cut_i$$

If Q were a primitive type q, and Γ were not the empty context, we would have then a Cut-free underivable sequent. For example, if the right premise of the Cut rule in (77) were the derivable sequent $q, q\backslash s\Rightarrow s$, we would have then as conclusion:

(78) $\boxed{q}, q\backslash s\Rightarrow s$

Since the primitive type q in the antecedent is focalised, there is no possibility of applying the \ left rule, which is a synchronous rule that needs that its active formula to be focalised. Principal cases:

- *foc* cases:

(79) $$\dfrac{\dfrac{\Delta\Longrightarrow_w\boxed{P}}{\Delta\Longrightarrow_w P}\,foc \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w A\diamond \text{foc}}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\diamond \text{foc}}\,n\text{-}Cut_1 \qquad\rightsquigarrow\qquad \dfrac{\Delta\Longrightarrow_w\boxed{P} \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w A\diamond \text{foc}}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\diamond \text{foc}}\,p\text{-}Cut_1$$

(80) $$\dfrac{\Delta\Longrightarrow_w N \qquad \dfrac{\Delta\langle\boxed{N}\rangle\Longrightarrow_w A}{\Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w A}\,foc}{\Gamma\langle\Delta\rangle\Longrightarrow_w A}\,n\text{-}Cut_2 \qquad\rightsquigarrow\qquad \dfrac{\Delta\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{\boxed{N}}\rangle\Longrightarrow_w A}{\Gamma\langle\Delta\rangle\Longrightarrow_w A}\,p\text{-}Cut_2$$

- logical connectives:

(81) $$\dfrac{\dfrac{\Delta|_k\overrightarrow{P_1}\Longrightarrow_w P_2\diamond \text{foc}}{\Delta\Longrightarrow_w P_2\uparrow_k P_1\diamond \text{foc}}\,\uparrow_k R \qquad \dfrac{\Gamma_1\Longrightarrow_w\boxed{P_1} \qquad \Gamma_2\langle\overrightarrow{P_2}\rangle\Longrightarrow_w A}{\Gamma_2\langle\overrightarrow{\boxed{P_2\uparrow_k P_1}}|_k\Gamma_1\rangle\Longrightarrow_w A}\,\uparrow_k L}{\Gamma_2\langle\Delta|_k\Gamma_1\rangle\Longrightarrow_w A\diamond \text{foc}}\,p\text{-}Cut_2 \qquad\rightsquigarrow$$

$$\dfrac{\Gamma_1\Longrightarrow_w\boxed{P_1} \qquad \dfrac{\Delta|_k\overrightarrow{P_1}\Longrightarrow P_2\diamond \text{foc} \qquad \Gamma_2\langle\overrightarrow{P_2}\rangle\Longrightarrow_w A}{\Gamma_2\langle\Delta|_k\overrightarrow{P_1}\rangle\Longrightarrow_w A\diamond \text{foc}}\,n\text{-}Cut_1}{\Gamma_2\langle\Delta|_k\Gamma_1\rangle\Longrightarrow_w A\diamond \text{foc}}\,p\text{-}Cut_1$$

The case of $\downarrow_k$ is entirely similar to the $\uparrow_k$ case and the case of $\downarrow_k$ is entirely similar to the $\uparrow_k$ case.

$$(82)\quad \dfrac{\dfrac{\Delta_1\Longrightarrow_w \boxed{P} \qquad \Delta_2\Longrightarrow_w N}{\Delta_1|_k\Delta_2\Longrightarrow_w \boxed{P\odot_k N}}\odot_k R \qquad \dfrac{\Gamma\langle\overrightarrow{P}|_k\overrightarrow{N}\rangle\Longrightarrow_w A\diamond\text{foc}}{\Gamma\langle\overrightarrow{P\odot_k N}\rangle\Longrightarrow_w A\diamond\text{foc}}\odot_k L}{\Gamma\langle\Delta_1|_k\Delta_2\rangle\Longrightarrow_w A\diamond\text{foc}}\,p\text{-}Cut_1 \qquad \rightsquigarrow$$

$$\dfrac{\Delta_2\Longrightarrow_w N \qquad \dfrac{\Delta_1\Longrightarrow \boxed{P} \qquad \Gamma\langle\overrightarrow{P}|_k\overrightarrow{N}\rangle\Longrightarrow_w A\diamond\text{foc}}{\Gamma\langle\Delta_1|_k\overrightarrow{N}\rangle\Longrightarrow_w A\diamond\text{foc}}\,p\text{-}Cut_1}{\Gamma\langle\Delta_1|_k\Delta_2\rangle\Longrightarrow_w A\diamond\text{foc}}\,n\text{-}Cut_2$$

$$(83)\quad \dfrac{\dfrac{\Delta\Longrightarrow_w Q\diamond\text{foc} \qquad \Delta\Longrightarrow_w A\diamond\text{foc}}{\Delta\Longrightarrow_w Q\&A\diamond\text{foc}}\&R \qquad \dfrac{\Gamma\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w B}{\Gamma\langle\overrightarrow{\boxed{Q\&A}}\rangle\Longrightarrow_w B}\&L}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond\text{foc}}\,p\text{-}Cut_2 \qquad \rightsquigarrow$$

$$\dfrac{\Delta\Longrightarrow Q\diamond\text{foc} \qquad \Gamma\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w B}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond\text{foc}}\,p\text{-}Cut_2$$

$$(84)\quad \dfrac{\dfrac{\Delta\Longrightarrow_w M\diamond\text{foc} \qquad \Delta\Longrightarrow_w A\diamond\text{foc}}{\Delta\Longrightarrow_w M\&A\diamond\text{foc}}\&R \qquad \dfrac{\Gamma\langle\overrightarrow{M}\rangle\Longrightarrow_w B}{\Gamma\langle\overrightarrow{\boxed{M\&A}}\rangle\Longrightarrow_w B}\&L}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond\text{foc}}\,p\text{-}Cut_2 \qquad \rightsquigarrow$$

$$\dfrac{\Delta\Longrightarrow M\diamond\text{foc} \qquad \Gamma\langle\overrightarrow{M}\rangle\Longrightarrow_w B}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond\text{foc}}\,n\text{-}Cut_1$$

- Left commutative *p-Cut* conversions:

$$(85)\quad \dfrac{\dfrac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w N}{\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow_w N}foc \qquad \Gamma\langle\overrightarrow{\boxed{N}}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{Q}\rangle\rangle\Longrightarrow_w C}\,p\text{-}Cut_2 \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{\boxed{N}}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{\boxed{Q}}\rangle\rangle\Longrightarrow_w C}\,p\text{-}Cut_2}{\Gamma\langle\Delta\langle\overrightarrow{Q}\rangle\rangle\Longrightarrow_w C}foc$$

$$(86)\quad \dfrac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w \boxed{P}}{\Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w \boxed{P}}\odot_k L \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_1 \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w \boxed{P} \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Gamma\langle\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_1}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\odot_k L$$

$$(87)\quad \dfrac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w N\diamond\text{foc}}{\Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w N\diamond\text{foc}}\odot_k L \qquad \Gamma\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_2 \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w N\diamond\text{foc} \qquad \Gamma\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,p\text{-}Cut_2}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\odot_k L$$

(88)
$$\dfrac{\dfrac{\Gamma_1 \Longrightarrow_w \boxed{P_1} \qquad \Gamma_2\langle \overrightarrow{\boxed{N_1}} \rangle \Longrightarrow_w N}{\Gamma_2\langle \overrightarrow{\boxed{N_1 \uparrow_k P_1}} |_k \Gamma_1 \rangle \Longrightarrow_w N}\uparrow_k L \qquad \Theta\langle \overrightarrow{\boxed{N}} \rangle \Longrightarrow_w C}{\Theta\langle \Gamma_2\langle \overrightarrow{\boxed{N_1 \uparrow_k P_1}} |_k \Gamma_1 \rangle\rangle \Longrightarrow_w C}\, p\text{-}Cut_2 \quad \rightsquigarrow$$

$$\dfrac{\Gamma_1 \Longrightarrow_w \boxed{P_1} \qquad \dfrac{\Gamma_1\langle \overrightarrow{\boxed{N_1}} \rangle \Longrightarrow_w N \qquad \Theta\langle \overrightarrow{\boxed{N}} \rangle \Longrightarrow_w C}{\Theta\langle \Gamma_2\langle \overrightarrow{\boxed{N_1}} \rangle\rangle \Longrightarrow_w C}\, p\text{-}Cut_2}{\Theta\langle \Gamma_2\langle \overrightarrow{\boxed{N_1 \uparrow_k P_1}} |_k \Gamma_1 \rangle\rangle \Longrightarrow_w C}\uparrow_k L$$

(89)
$$\dfrac{\dfrac{\Gamma\langle \overrightarrow{A} \rangle \Longrightarrow_w \boxed{P} \qquad \Gamma\langle \overrightarrow{B} \rangle \Longrightarrow_w \boxed{P}}{\Gamma\langle \overrightarrow{A \oplus B} \rangle \Longrightarrow_w \boxed{P}}\oplus L \qquad \Delta\langle \overrightarrow{P} \rangle \Longrightarrow_w C \diamond \text{foc}}{\Delta\langle \Gamma\langle \overrightarrow{A \oplus B} \rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\, p\text{-}Cut_1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma\langle \overrightarrow{A} \rangle \Longrightarrow_w \boxed{P} \qquad \Delta\langle \overrightarrow{P} \rangle \Longrightarrow_w C \diamond \text{foc}}{\Delta\langle \Gamma\langle \overrightarrow{A} \rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\, p\text{-}Cut_1 \qquad \dfrac{\Gamma\langle \overrightarrow{B} \rangle \Longrightarrow_w \boxed{P} \qquad \Delta\langle \overrightarrow{P} \rangle \Longrightarrow_w C \diamond \text{foc}}{\Delta\langle \Gamma\langle \overrightarrow{B} \rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\, p\text{-}Cut_1}{\Delta\langle \Gamma\langle \overrightarrow{A \oplus B} \rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\oplus L$$

(90)
$$\dfrac{\dfrac{\Gamma\langle \overrightarrow{A} \rangle \Longrightarrow_w N \diamond \text{foc} \qquad \Gamma\langle \overrightarrow{B} \rangle \Longrightarrow_w N \diamond \text{foc}}{\Gamma\langle \overrightarrow{A \oplus B} \rangle \Longrightarrow_w N \diamond \text{foc}}\oplus L \qquad \Delta\langle \boxed{\overrightarrow{N}} \rangle \Longrightarrow_w C}{\Delta\langle \Gamma\langle \overrightarrow{A \oplus B} \rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\, p\text{-}Cut_2 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma\langle \overrightarrow{A} \rangle \Longrightarrow_w N \diamond \text{foc} \qquad \Delta\langle \boxed{\overrightarrow{N}} \rangle \Longrightarrow_w C}{\Delta\langle \Gamma\langle \overrightarrow{A} \rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\, p\text{-}Cut_2 \qquad \dfrac{\Gamma\langle \overrightarrow{B} \rangle \Longrightarrow_w N \diamond \text{foc} \qquad \Delta\langle \boxed{\overrightarrow{N}} \rangle \Longrightarrow_w C}{\Delta\langle \Gamma\langle \overrightarrow{B} \rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\, p\text{-}Cut_2}{\Delta\langle \Gamma\langle \overrightarrow{A \oplus B} \rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\oplus L$$

- Right commutative *p-Cut* conversions (unordered multiple distinguished occurrences are separated by semicolons):

(91)
$$\dfrac{\Delta \Longrightarrow_w \boxed{P} \qquad \dfrac{\Gamma\langle \overrightarrow{P}; \overrightarrow{\boxed{Q}} \rangle \Longrightarrow_w C}{\Gamma\langle \overrightarrow{P}; \overrightarrow{Q} \rangle \Longrightarrow_w C}\, foc}{\Gamma\langle \Delta; \overrightarrow{Q} \rangle \Longrightarrow_w C}\, p\text{-}Cut_1 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \Longrightarrow_w \boxed{P} \qquad \Gamma\langle \overrightarrow{P}; \overrightarrow{\boxed{Q}} \rangle \Longrightarrow_w C}{\Gamma\langle \Delta; \overrightarrow{\boxed{Q}} \rangle \Longrightarrow_w C}\, p\text{-}Cut_1}{\Gamma\langle \Delta; \overrightarrow{Q} \rangle \Longrightarrow_w C}\, foc$$

(92)
$$\dfrac{\Delta \Longrightarrow_w \boxed{P_1} \qquad \dfrac{\Gamma\langle \overrightarrow{P_1} \rangle \Longrightarrow_w \boxed{P_2}}{\Gamma\langle \overrightarrow{P_1} \rangle \Longrightarrow_w P_2}\, foc}{\Gamma\langle \Delta \rangle \Longrightarrow_w P_2}\, p\text{-}Cut_1 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \Longrightarrow_w \boxed{P_1} \qquad \Gamma\langle \overrightarrow{P_1} \rangle \Longrightarrow_w \boxed{P_2}}{\Gamma\langle \Delta \rangle \Longrightarrow_w \boxed{P_2}}\, p\text{-}Cut_1}{\Gamma\langle \Delta \rangle \Longrightarrow_w P_2}\, foc$$

(93)
$$\dfrac{\Delta \Longrightarrow_w \boxed{P} \qquad \dfrac{\Gamma\langle \overrightarrow{P} \rangle |_k \overrightarrow{A} \Longrightarrow_w B \diamond \text{foc}}{\Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w B \uparrow_k A \diamond \text{foc}}\uparrow_k R}{\Gamma\langle \Delta \rangle \Longrightarrow_w B \uparrow_k A \diamond \text{foc}}\, p\text{-}Cut_1 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \Longrightarrow_w \boxed{P} \qquad \Gamma\langle \overrightarrow{P} \rangle |_k \overrightarrow{A} \Longrightarrow_w B \diamond \text{foc}}{\Gamma\langle \Delta \rangle |_k \overrightarrow{A} \Longrightarrow_w B \diamond \text{foc}}\, p\text{-}Cut_1}{\Gamma\langle \Delta \rangle \Longrightarrow_w B \uparrow_k A \diamond \text{foc}}\uparrow_k R$$

(94)
$$\dfrac{\Delta \Longrightarrow_w N \diamond \text{foc} \qquad \dfrac{\Gamma\langle \boxed{\overrightarrow{N}} \rangle |_k \overrightarrow{A} \Longrightarrow_w B}{\Gamma\langle \boxed{\overrightarrow{N}} \rangle \Longrightarrow_w B \uparrow_k A}\uparrow_k R}{\Gamma\langle \Delta \rangle \Longrightarrow_w B \uparrow_k A \diamond \text{foc}}\, p\text{-}Cut_2 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \Longrightarrow_w N \diamond \text{foc} \qquad \Gamma\langle \boxed{\overrightarrow{N}} \rangle |_k \overrightarrow{A} \Longrightarrow_w B}{\Gamma\langle \Delta \rangle |_k \overrightarrow{A} \Longrightarrow_w B \diamond \text{foc}}\, p\text{-}Cut_2}{\Gamma\langle \Delta \rangle \Longrightarrow_w B \uparrow_k A \diamond \text{foc}}\uparrow_k R$$

(95)
$$\dfrac{\Delta\Longrightarrow_w \boxed{P} \qquad \dfrac{\Gamma\langle\overrightarrow{P};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond \mathrm{foc}}{\Gamma\langle\overrightarrow{P};\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond \mathrm{foc}}\,\odot_k L}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond \mathrm{foc}}\,p\text{-}Cut_1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\Delta\Longrightarrow_w \boxed{P} \qquad \Gamma\langle\overrightarrow{P};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond \mathrm{foc}}{\Gamma\langle\Delta;\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond \mathrm{foc}}\,p\text{-}Cut_1}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond \mathrm{foc}}\,\odot_k L$$

(96)
$$\dfrac{\Delta\Longrightarrow_w N\diamond \mathrm{foc} \qquad \dfrac{\Gamma\langle\overrightarrow{\boxed{N}};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C}{\Gamma\langle\overrightarrow{\boxed{N}};\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C}\,\odot_k L}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond \mathrm{foc}}\,p\text{-}Cut_2 \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\Delta\Longrightarrow_w N\diamond \mathrm{foc} \qquad \Gamma\langle\overrightarrow{\boxed{N}};\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta;\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w C\diamond \mathrm{foc}}\,p\text{-}Cut_2}{\Gamma\langle\Delta;\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w C\diamond \mathrm{foc}}\,\odot_k L$$

(97)
$$\dfrac{\Delta\Longrightarrow_w \boxed{P} \qquad \dfrac{\Gamma\Longrightarrow_w \boxed{P_1} \qquad \Theta\langle\overrightarrow{P_2};\overrightarrow{P}\rangle\Longrightarrow_w C}{\Theta\langle\overrightarrow{\boxed{P_2\uparrow_k P_1}}|_k\Gamma;\overrightarrow{P}\rangle\Longrightarrow_w C}\,\uparrow_k L}{\Theta\langle\overrightarrow{\boxed{P_2\uparrow_k P_1}}|_k\Gamma;\Delta\rangle\Longrightarrow_w C}\,p\text{-}Cut_1 \quad\rightsquigarrow$$

$$\dfrac{\Gamma\Longrightarrow_w \boxed{P_1} \qquad \dfrac{\Delta\Longrightarrow_w \boxed{P} \qquad \Theta\langle\overrightarrow{P_2};\overrightarrow{P}\rangle\Longrightarrow_w C}{\Theta\langle\overrightarrow{P_2};\Delta\rangle\Longrightarrow_w C}\,p\text{-}Cut_1}{\Theta\langle\overrightarrow{\boxed{P_2\uparrow_k P_1}}|_k\Gamma;\Delta\rangle\Longrightarrow_w C}\,\uparrow_k L$$

(98)
$$\dfrac{\Delta\Longrightarrow_w \boxed{P} \qquad \dfrac{\Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w A\diamond \mathrm{foc} \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w B\diamond \mathrm{foc}}{\Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w A\& B\diamond \mathrm{foc}}\,\& R}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\& B\diamond \mathrm{foc}}\,p\text{-}Cut_1 \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\Delta\Longrightarrow_w \boxed{P} \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w A\diamond \mathrm{foc}}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\diamond \mathrm{foc}}\,p\text{-}Cut_1 \qquad \dfrac{\Delta\Longrightarrow_w \boxed{P} \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w B\diamond \mathrm{foc}}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond \mathrm{foc}}\,p\text{-}Cut_1}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\& B\diamond \mathrm{foc}}\,\& R$$

(99)
$$\dfrac{\Delta\Longrightarrow_w N\diamond \mathrm{foc} \qquad \dfrac{\Gamma\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w A \qquad \Gamma\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w B}{\Gamma\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w A\& B}\,\& R}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\& B\diamond \mathrm{foc}}\,p\text{-}Cut_2 \quad\rightsquigarrow$$

$$\dfrac{\dfrac{\Delta\Longrightarrow_w N\diamond \mathrm{foc} \qquad \Gamma\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w A}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\diamond \mathrm{foc}}\,p\text{-}Cut_2 \qquad \dfrac{\Delta\Longrightarrow_w N\diamond \mathrm{foc} \qquad \Gamma\langle\boxed{\overrightarrow{N}}\rangle\Longrightarrow_w B}{\Gamma\langle\Delta\rangle\Longrightarrow_w B\diamond \mathrm{foc}}\,p\text{-}Cut_2}{\Gamma\langle\Delta\rangle\Longrightarrow_w A\& B\diamond \mathrm{foc}}\,\& R$$

- Left commutative *n-Cut* conversions:

(100)
$$\dfrac{\dfrac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w P}{\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow_w P}\,foc \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{Q}\rangle\rangle\Longrightarrow_w C}\,n\text{-}Cut_1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\Delta\overrightarrow{\boxed{Q}}\Longrightarrow_w P \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{\boxed{Q}}\rangle\rangle\Longrightarrow_w C}\,n\text{-}Cut_1}{\Gamma\langle\Delta\langle\overrightarrow{Q}\rangle\rangle\Longrightarrow_w C}\,foc$$

(101)
$$\frac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w P\diamond\text{foc}}{\Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w P\diamond\text{foc}}\odot_k L \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_1 \qquad\leadsto$$

$$\dfrac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w P\diamond\text{foc} \qquad \Gamma\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_1}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\odot_k L$$

(102)
$$\frac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w N}{\Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w N}\odot_k L \qquad \Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_2 \qquad\leadsto$$

$$\dfrac{\dfrac{\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{N}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Gamma\langle\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_2}{\Gamma\langle\Delta\langle\overrightarrow{A\odot_k B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\odot_k L$$

(103)
$$\frac{\dfrac{\Gamma_1\Longrightarrow_w\boxed{P_1} \qquad \Gamma_2\langle\overrightarrow{\boxed{N_1}}\rangle\Longrightarrow_w P}{\Gamma_2\langle\overrightarrow{\boxed{N_1\uparrow_k P_1}}|_k\Gamma_1\rangle\Longrightarrow_w P}\uparrow_k L \qquad \Theta\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Theta\langle\Gamma_2\langle\overrightarrow{\boxed{N_1\uparrow_k P_1}}|_k\Gamma_1\rangle\rangle\Longrightarrow_w C}\,n\text{-}Cut_1 \qquad\leadsto$$

$$\frac{\Gamma_1\Longrightarrow_w\boxed{P_1} \qquad \dfrac{\Gamma_1\langle\overrightarrow{\boxed{N_1}}\rangle\Longrightarrow_w P \qquad \Theta\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Theta\langle\Gamma_2\langle\overrightarrow{\boxed{N_1}}\rangle\rangle\Longrightarrow_w C}\,n\text{-}Cut_1}{\Theta\langle\Gamma_2\langle\overrightarrow{\boxed{N_1\uparrow_k P_1}}|_k\Gamma_1\rangle\rangle\Longrightarrow_w C}\uparrow_k L$$

(104)
$$\frac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w P\diamond\text{foc} \qquad \Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w P\diamond\text{foc}}{\Gamma\langle\overrightarrow{A\oplus B}\rangle\Longrightarrow_w P\diamond\text{foc}}\oplus L \qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_1 \qquad\leadsto$$

$$\frac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w P\diamond\text{foc} \qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Delta\langle\Gamma\langle\overrightarrow{A}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_1 \qquad \dfrac{\Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w P\diamond\text{foc} \qquad \Delta\langle\overrightarrow{P}\rangle\Longrightarrow_w C}{\Delta\langle\Gamma\langle\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_1}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\oplus L$$

(105)
$$\frac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w N}{\Gamma\langle\overrightarrow{A\oplus B}\rangle\Longrightarrow_w N}\oplus L \qquad \Delta\langle\overrightarrow{N}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_2 \qquad\leadsto$$

$$\frac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle\Longrightarrow_w N \qquad \Delta\langle\overrightarrow{N}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{A}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_2 \qquad \dfrac{\Gamma\langle\overrightarrow{B}\rangle\Longrightarrow_w N \qquad \Delta\langle\overrightarrow{N}\rangle\Longrightarrow_w C\diamond\text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\,n\text{-}Cut_2}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle\Longrightarrow_w C\diamond\text{foc}}\oplus L$$

- Right commutative *n-Cut* conversions:

(106)
$$\dfrac{\Delta \Longrightarrow_w N \qquad \dfrac{\Gamma\langle \overrightarrow{N}; \overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w C}{\Gamma\langle \overrightarrow{N}; \overrightarrow{Q}\rangle \Longrightarrow_w C}\,\mathit{foc}}{\Gamma\langle \Delta; \overrightarrow{Q}\rangle \Longrightarrow_w C}\,n\text{-}Cut_2 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \Longrightarrow_w N \qquad \Gamma\langle \overrightarrow{N}; \overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w C}{\Gamma\langle \Delta; \overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w C}\,n\text{-}Cut_2}{\Gamma\langle \Delta; \overrightarrow{Q}\rangle \Longrightarrow_w C}\,\mathit{foc}$$

(107)
$$\dfrac{\Delta \Longrightarrow_w N \qquad \dfrac{\Gamma\langle \overrightarrow{N}\rangle \Longrightarrow_w \boxed{P}}{\Gamma\langle \overrightarrow{N}\rangle \Longrightarrow_w P}\,\mathit{foc}}{\Gamma\langle \Delta\rangle \Longrightarrow_w P}\,n\text{-}Cut_2 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \Longrightarrow_w N \qquad \Gamma\langle \overrightarrow{N}\rangle \Longrightarrow_w \boxed{P}}{\Gamma\langle \Delta\rangle \Longrightarrow_w \boxed{P}}\,n\text{-}Cut_2}{\Gamma\langle \Delta\rangle \Longrightarrow_w P}\,\mathit{foc}$$

(108)
$$\dfrac{\Delta \Longrightarrow_w P \diamond \text{foc} \qquad \dfrac{\Gamma\langle \overrightarrow{P}\rangle|_k \overrightarrow{A} \Longrightarrow_w B}{\Gamma\langle \overrightarrow{P}\rangle \Longrightarrow_w B\uparrow_k A}\,\uparrow_k R}{\Gamma\langle \Delta\rangle \Longrightarrow_w B\uparrow_k A \diamond \text{foc}}\,n\text{-}Cut_1 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \Longrightarrow_w P \diamond \text{foc} \qquad \Gamma\langle \overrightarrow{P}\rangle|_k \overrightarrow{A} \Longrightarrow_w B}{\Gamma\langle \Delta\rangle|_k \overrightarrow{A} \Longrightarrow_w B \diamond \text{foc}}\,n\text{-}Cut_1}{\Gamma\langle \Delta\rangle \Longrightarrow_w B\uparrow_k A \diamond \text{foc}}\,\uparrow_k R$$

(109)
$$\dfrac{\Delta \Longrightarrow_w N \qquad \dfrac{\Gamma\langle \overrightarrow{P}\rangle|_k \overrightarrow{A} \Longrightarrow_w B \diamond \text{foc}}{\Gamma\langle \overrightarrow{P}\rangle \Longrightarrow_w B\uparrow_k A \diamond \text{foc}}\,\uparrow_k R}{\Gamma\langle \Delta\rangle \Longrightarrow_w B\uparrow_k A \diamond \text{foc}}\,n\text{-}Cut_2 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \Longrightarrow_w N \qquad \Gamma\langle \overrightarrow{P}\rangle|_k \overrightarrow{A} \Longrightarrow_w B \diamond \text{foc}}{\Gamma\langle \Delta\rangle|_k \overrightarrow{A} \Longrightarrow_w B \diamond \text{foc}}\,n\text{-}Cut_2}{\Gamma\langle \Delta\rangle \Longrightarrow_w B\uparrow_k A \diamond \text{foc}}\,\uparrow_k R$$

(110)
$$\dfrac{\Delta \Longrightarrow_w P \diamond \text{foc} \qquad \dfrac{\Gamma\langle \overrightarrow{P}; \overrightarrow{A}|_k \overrightarrow{B}\rangle \Longrightarrow_w C}{\Gamma\langle \overrightarrow{P}; \overrightarrow{A\odot_k B}\rangle \Longrightarrow_w C}\,\odot_k L}{\Gamma\langle \Delta; \overrightarrow{A\odot_k B}\rangle \Longrightarrow_w C \diamond \text{foc}}\,n\text{-}Cut_1 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \Longrightarrow_w P \diamond \text{foc} \qquad \Gamma\langle \overrightarrow{P}; \overrightarrow{A}|_k \overrightarrow{B}\rangle \Longrightarrow_w C}{\Gamma\langle \Delta; \overrightarrow{A}|_k \overrightarrow{B}\rangle \Longrightarrow_w C \diamond \text{foc}}\,n\text{-}Cut_1}{\Gamma\langle \Delta; \overrightarrow{A\odot_k B}\rangle \Longrightarrow_w C \diamond \text{foc}}\,\odot_k L$$

(111)
$$\dfrac{\Delta \Longrightarrow_w N \qquad \dfrac{\Gamma\langle \overrightarrow{N}; \overrightarrow{A}|_k \overrightarrow{B}\rangle \Longrightarrow_w C \diamond \text{foc}}{\Gamma\langle \overrightarrow{N}; \overrightarrow{A\odot_k B}\rangle \Longrightarrow_w C \diamond \text{foc}}\,\odot_k L}{\Gamma\langle \Delta; \overrightarrow{A\odot_k B}\rangle \Longrightarrow_w C \diamond \text{foc}}\,n\text{-}Cut_2 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Delta \Longrightarrow_w N \qquad \Gamma\langle \overrightarrow{N}; \overrightarrow{A}|_k \overrightarrow{B}\rangle \Longrightarrow_w C \diamond \text{foc}}{\Gamma\langle \Delta; \overrightarrow{A}|_k \overrightarrow{B}\rangle \Longrightarrow_w C \diamond \text{foc}}\,n\text{-}Cut_2}{\Gamma\langle \Delta; \overrightarrow{A\odot_k B}\rangle \Longrightarrow_w C \diamond \text{foc}}\,\odot_k L$$

(112)
$$\dfrac{\Delta \Longrightarrow_w N \qquad \dfrac{\Gamma \Longrightarrow_w \boxed{P_1} \qquad \Theta\langle \overrightarrow{P_2}; \overrightarrow{N}\rangle \Longrightarrow_w C}{\Theta\langle \overrightarrow{\boxed{P_2\uparrow_k P_1}}|_k \Gamma; \overrightarrow{N}\rangle \Longrightarrow_w C}\,\uparrow_k L}{\Theta\langle \overrightarrow{\boxed{P_2\uparrow_k P_1}}|_k \Gamma; \Delta\rangle \Longrightarrow_w C}\,n\text{-}Cut_2 \quad \rightsquigarrow$$

$$\dfrac{\Gamma \Longrightarrow_w \boxed{P_1} \qquad \dfrac{\Delta \Longrightarrow_w N \qquad \Theta\langle \overrightarrow{P_2}; \overrightarrow{N}\rangle \Longrightarrow_w C}{\Theta\langle \overrightarrow{P_2}; \Delta\rangle \Longrightarrow_w C}\,n\text{-}Cut_2}{\Theta\langle \overrightarrow{\boxed{P_2\uparrow_k P_1}}|_k \Gamma; \Delta\rangle \Longrightarrow_w C}\,\uparrow_k L$$

(113)
$$\dfrac{\Delta \Longrightarrow_w P \diamond \text{foc} \qquad \dfrac{\Gamma\langle \overrightarrow{P}\rangle \Longrightarrow_w A \qquad \Gamma\langle \overrightarrow{P}\rangle \Longrightarrow_w B}{\Gamma\langle \overrightarrow{P}\rangle \Longrightarrow_w A\&B}\,\&R}{\Gamma\langle \Delta\rangle \Longrightarrow_w A\&B \diamond \text{foc}}\,n\text{-}Cut_1 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\Delta \Longrightarrow_w P \qquad \Gamma\langle \overrightarrow{P}\rangle \Longrightarrow_w A}{\Gamma\langle \Delta\rangle \Longrightarrow_w A \diamond \text{foc}}\,n\text{-}Cut_1 \qquad \dfrac{\Delta \Longrightarrow_w P \diamond \text{foc} \qquad \Gamma\langle \overrightarrow{P}\rangle \Longrightarrow_w B}{\Gamma\langle \Delta\rangle \Longrightarrow_w B \diamond \text{foc}}\,n\text{-}Cut_1}{\Gamma\langle \Delta\rangle \Longrightarrow_w A\&B \diamond \text{foc}}\,\&R$$

(114)
$$\dfrac{\Delta \Longrightarrow_w N \qquad \dfrac{\Gamma\langle \overrightarrow{N}\rangle \Longrightarrow_w A \diamond \text{foc} \qquad \Gamma\langle \overrightarrow{N}\rangle \Longrightarrow_w B \diamond \text{foc}}{\Gamma\langle \overrightarrow{N}\rangle \Longrightarrow_w A\&B \diamond \text{foc}}\,\&R}{\Gamma\langle \Delta\rangle \Longrightarrow_w A\&B \diamond \text{foc}}\,n\text{-}Cut_2 \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\Delta \Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{N}\rangle \Longrightarrow_w A \diamond \text{foc}}{\Gamma\langle\Delta\rangle \Longrightarrow_w A \diamond \text{foc}}\, n\text{-}Cut_2 \qquad \dfrac{\Delta \Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{N}\rangle \Longrightarrow_w B \diamond \text{foc}}{\Gamma\langle\Delta\rangle \Longrightarrow_w B \diamond \text{foc}}\, n\text{-}Cut_2}{\Gamma\langle\Delta\rangle \Longrightarrow_w A\&B \diamond \text{foc}}\, \&R$$

This completes the proof.

9.9 Non-admissibility of Cut

Chapter 10

Count-invariance

We define infinitary count invariance for categorial logic extending count invariance for multiplicatives (van Benthem 1991[91]) and additives and bracket modalities (Valentín et al. 2013[90]) to include subexponentials (Kuznetsov et al. 2017[42]). The count invariance provides a powerful tool for pruning proof search in categorial parsing/theorem-proving.

10.1 Introduction

We make some introductory remarks on non-linearity since the main feature here is infinitary count invariance for subexponentials.

10.1.1 Sharing

In standard logic information does not have multiplicity. Thus where + is the notion of addition of information and $\leq$ is the notion of inclusion of information we have $x+x \leq x$ and $x \leq x+x$; together these two properties amount to idempotency: $x+x = x$. These properties are expressed by the rules of inference of Contraction and Expansion:

(115) $$\frac{\Delta(A, A) \Rightarrow B}{\Delta(A) \Rightarrow B}\text{Contraction}$$

$$\frac{\Delta(A) \Rightarrow B}{\Delta(A, A) \Rightarrow B}\text{Expansion}$$

Linguistic resources do not have these properties: grammaticality is not generally preserved under addition or deletion of copies of words or expressions. However, there are some constructions manifesting something similiar. Parasitic gaps involve a kind of Contraction. Parasitic gaps cannot occur anywhere, for example

(116)∗the slave that$_i$ John sold e_i to e_i

Rather, we assume here that as the term 'parasitic' suggests, a parasitic gap must fall within an island. Extraction from weak islands can become fully acceptable when accompanied by a cobound non-island extraction:

(117) a. the man that$_i$ [the friends of e_i] admire e_i
b. the paper that$_i$ John filed e_i [without reading e_i]
c. the paper that$_i$ [the editor of e_i] filed e_i [without reading e_i]

And iterated coordination allows a kind of Expansion:

(118) John likes, Mary dislikes and Bill loves London.

That is, in logical grammar a *controlled* use of idempotency, or sharing, is motivated. Girard (1987[22]) introduced exponentials for such control. Morrill and Valentín (2015[73]) uses versions of the exponentials, dubbed 'subexponentials' by Kanovich et al. (2018[33]) to treat (parasitic) gaps and iterated coordination in categorial grammar.

10.1.2 Count invariance

Van Benthem (1991[91]) introduces count invariance for multiplicatives in (sub)linear logic, which involves simply checking the equality of the number of positive and negative occurrences of each atom in a sequent. Thus where $\#_P(\Sigma)$ is the count of the atom P in the sequent Σ we have:

(119) $\vdash \Sigma \Longrightarrow \forall P, \#_P(\Sigma) = 0$

I.e. there must be an exact balance between the number of positive and the number of negative occurrences of each atom. This provides a necessary, although of course not sufficient, criterion for theoremhood, and can be checked very quickly. It can thus be used as a filter in proof search: if backward chaining proof search generates an initial endsequent goal, or a subgoal, which does not satisfy the count invariant, then the (sub)goal can be safely made to fail immediately. This notion of count for multiplicatives was included in the categorial parser/theorem-prover CatLog (Morrill 2012[64]).

In Valentín et al. (2013[90]) the idea is extended to additives (and bracket modalities). To treat additives, instead of a single count for each atom P of a sequent Σ we have a minimum count $\#_{min,P}(\Sigma)$ and a maximum count $\#_{max,P}(\Sigma)$ and for a sequent to be a theorem it must satisfy two inequations:

(120) $\vdash \Sigma \Longrightarrow \forall P, \#_{min,P}(\Sigma) \leq 0 \leq \#_{max,P}(\Sigma)$

I.e. the count functions $\#_{min,P}$ and $\#_{max,P}$ define an interval which must include the point of equilibrium 0; in the case of the multiplicatives, $\#_{min,P} = \#_{max,P} = \#_P$ and (120) reduces to the special case (119). This generalised notion of count for additives and bracket modalities is included in the categorial parser/theorem-prover CatLog2.[1]

The structure of the rest of the chapter is as follows. In Section 10.2 we present the infinitary count algebra which we employ, define the fragment of categorial logic for which we illustrate count invariance, and define the (infinitary) count functions for this fragment. In Section 10.3 we state and prove our count invariance theorem.

10.2 Infinitary count algebra

We consider terms built over constants 0, 1, $\bot$, $\top$ and $*$ by binary operations of plus (+), minus (−), minimum (*min*) and maximum (*max*), and unary operations of positive extrapolation (X^+) and negative extrapolation (X^-) as follows where i and j are integers and n is a positive integer:[2]

$+$	j	$\bot$	$\top$	$*$
i	$i+j$	$\bot$	$\top$	$*$
$\bot$	$\bot$	$\bot$	$*$	$*$
$\top$	$\top$	$*$	$\top$	$*$
$*$	$*$	$*$	$*$	$*$

$-$	j	$\bot$	$\top$	$*$
i	$i-j$	$\top$	$\bot$	$*$
$\bot$	$\bot$	$*$	$\bot$	$*$
$\top$	$\top$	$\top$	$*$	$*$
$*$	$*$	$*$	$*$	$*$

min	j	$\bot$	$\top$	$*$
i	$\frac{\lvert i+j\rvert-\lvert i-j\rvert}{2}$	$\bot$	i	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
$\top$	j	$\bot$	$\top$	$\bot$
$*$	$\bot$	$\bot$	$\bot$	$\bot$

max	j	$\bot$	$\top$	$*$
i	$\frac{\lvert i+j\rvert+\lvert i-j\rvert}{2}$	i	$\top$	$\top$
$\bot$	j	$\bot$	$\top$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$
$*$	$\top$	$\top$	$\top$	$\top$

[1] `http://www.cs.upc.edu/~droman/index.php`

[2] Positive and negative extrapolation can be defined by $X^+(x) = \frac{2x^2}{x-|x|}$ and $X^-(x) = \frac{-2x^2}{-x-|x|}$ where division by zero is infinite and obeys the sign rule.

	X^+	X^-
$-n$	$-n$	$\bot$
0	0	0
$+n$	$\top$	$+n$
$\bot$	$\bot$	$\bot$
$\top$	$\top$	$\top$
$*$	$*$	$*$

Where for primitive types $\mathcal{P}$, $Q \in \mathcal{P} \cup \{[]\}$, $m \in \{min, max\}$ and $\overline{min} = max$ and $\overline{max} = min$.

$$\#_{m,Q}(\Gamma \Rightarrow A) = \#^{\circ}_{m,Q}(A) - \#^{\bullet}_{\overline{m},Q}(\Gamma)$$

where for types **Tp**:

$$\begin{array}{l}\mathbf{Tp} ::= \mathcal{P} \mid \\ \mathbf{Tp}\backslash\mathbf{Tp} \mid \mathbf{Tp}/\mathbf{Tp} \mid \mathbf{Tp}\bullet\mathbf{Tp} \mid \\ \mathbf{Tp}\&\mathbf{Tp} \mid \mathbf{Tp}\oplus\mathbf{Tp} \mid \\ [\,]^{-1}\mathbf{Tp} \mid \langle\rangle\mathbf{Tp} \mid \\ !\mathbf{Tp} \mid ?\mathbf{Tp}\end{array}$$

for $P \in \mathcal{P}$, $p \in \{^{\bullet}, ^{\circ}\}$, and $\overline{^{\bullet}} = {}^{\circ}$ and $\overline{^{\circ}} = {}^{\bullet}$

$$\begin{array}{rcl}
\#^{p}_{m,Q}(P) & = & \begin{array}{ll} 1 & \text{if } Q = P \\ 0 & \text{if } Q \neq P \end{array} \\
\#^{p}_{m,Q}(A\backslash C) & = & \#^{p}_{m,Q}(C) - \#^{\overline{p}}_{\overline{m},Q}(A) \\
\#^{p}_{m,Q}(C/B) & = & \#^{p}_{m,Q}(C) - \#^{\overline{p}}_{\overline{m},Q}(B) \\
\#^{p}_{m,Q}(A\bullet B) & = & \#^{p}_{m,Q}(A) + \#^{p}_{m,Q}(B) \\
\#^{\circ}_{m,Q}(A\&B) & = & \overline{m}(\#^{\circ}_{m,Q}(A), \#^{\circ}_{m,Q}(B)) \\
\#^{\bullet}_{m,Q}(A\&B) & = & m(\#^{\bullet}_{m,Q}(A), \#^{\bullet}_{m,Q}(B)) \\
\#^{\circ}_{m,Q}(A\oplus B) & = & m(\#^{\circ}_{m,Q}(A), \#^{\circ}_{m,Q}(B)) \\
\#^{\bullet}_{m,Q}(A\oplus B) & = & \overline{m}(\#^{\bullet}_{m,Q}(A), \#^{\bullet}_{m,Q}(B)) \\
\#^{p}_{m,P}([\,]^{-1}A) & = & \#^{p}_{m,P}(A) \\
\#^{p}_{m,[]}([\,]^{-1}A) & = & \#^{p}_{m,[]}(A) - 1 \\
\#^{p}_{m,P}(\langle\rangle A) & = & \#^{p}_{m,P}(A) \\
\#^{p}_{m,[]}(\langle\rangle A) & = & \#^{p}_{m,[]}(A) + 1 \\
\#^{p}_{min,Q}(!A) & = & X^-(\#^{p}_{min,Q}(A)) \\
\#^{\circ}_{max,Q}(!A) & = & X^+(\#^{\circ}_{max,Q}(A)) \\
\#^{\bullet}_{max,P}(!A) & = & X^+(\#^{\bullet}_{max,P}(A)) \\
\#^{\bullet}_{max,[]}(!A) & = & \top \\
\#^{p}_{min,Q}(?A) & = & X^-(\#^{p}_{min,Q}(A)) \\
\#^{p}_{max,Q}(?A) & = & X^+(\#^{p}_{max,Q}(A))
\end{array}$$

To present sequents we define *configurations* **Config** and *tree terms* **TreeTerm** by mutual recursion in terms of types **Tp** as follows, where Λ is the metalinguistic empty string:

(121)
$$\begin{array}{rcl}\mathbf{Config} & ::= & \Lambda \mid \mathbf{TreeTerm}, \mathbf{Config} \\ \mathbf{TreeTerm} & ::= & \mathbf{Tp} \mid [\mathbf{Config}]\end{array}$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta(C) \Rightarrow D}{\Delta(\Gamma, A\backslash C) \Rightarrow D}\ \backslash L \qquad \frac{\Gamma \Rightarrow B \qquad \Delta(C) \Rightarrow D}{\Delta(C/B, \Gamma) \Rightarrow D}\ \backslash L$$

$$\frac{A, \Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C}\ \backslash R \qquad \frac{\Gamma, B \Rightarrow C}{\Gamma \Rightarrow C/B}\ /R$$

$$\frac{\Delta(A, B) \Rightarrow C}{\Delta(A\bullet B) \Rightarrow C}\ \bullet L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A\bullet B}\ \bullet R$$

$$\frac{\Delta(A) \Rightarrow D}{\Delta(A\&B) \Rightarrow D}\ \&L_1 \qquad \frac{\Delta(B) \Rightarrow D}{\Delta(A\&B) \Rightarrow D}\ \&L_2 \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A\&B}\ \&R$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A\oplus B}\ \oplus R_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A\oplus B}\ \oplus R_2 \qquad \frac{\Delta(A) \Rightarrow D \qquad \Delta(B) \Rightarrow D}{\Delta(A\oplus B) \Rightarrow D}\ \oplus L$$

$$\frac{\Gamma(A) \Rightarrow B}{\Gamma([[\,]^{-1}A]) \Rightarrow B}\ [\,]^{-1}L \qquad \frac{[\Gamma] \Rightarrow A}{\Gamma \Rightarrow [\,]^{-1}A}\ [\,]^{-1}R$$

$$\frac{\Gamma([A]) \Rightarrow B}{\Gamma(\langle\rangle A) \Rightarrow B}\ \langle\rangle L \qquad \frac{\Gamma \Rightarrow A}{[\Gamma] \Rightarrow \langle\rangle A}\ \langle\rangle R$$

$$\frac{\Delta(A) \Rightarrow D}{\Delta(!A) \Rightarrow D}\ !L \qquad \frac{A_1, \ldots, A_n \Rightarrow A}{A_1, \ldots, A_n \Rightarrow !A}\ !R$$

$$\frac{\Delta(\Gamma, !A) \Rightarrow D}{\Delta(!A, \Gamma) \Rightarrow D}\ !P_1 \qquad \frac{\Delta(!A, \Gamma) \Rightarrow D}{\Delta(\Gamma, !A) \Rightarrow D}\ !P_2 \qquad \frac{\Delta(!A_0, \ldots, !A_n, [!A_0, \ldots, !A_n, \Gamma]) \Rightarrow D}{\Delta(!A_0, \ldots, !A_n, \Gamma) \Rightarrow D}\ !C$$

$$\frac{\Delta(A) \Rightarrow D \qquad \Delta(A, A) \Rightarrow D \qquad \ldots}{\Delta(?A) \Rightarrow D}\ ?L \qquad \frac{\Gamma_1 \Rightarrow C \qquad \Gamma_2 \Rightarrow ?C}{\Gamma_1, \Gamma_2 \Rightarrow ?C}\ ?M$$

Figure 10.1: Rules for the categorial logic fragment

The rules for the fragment of categorial logic are as shown in Figure 10.1. For tree terms and configurations, counts are as follows:

$$
\begin{array}{rcl}
\#^{!}_{m,Q}(\zeta_1, \zeta_2) & = & \#^{!}_{m,Q}(\zeta_1) + \#^{!}_{m,Q}(\zeta_2) \\
\#^{!}_{min,Q}(A) & = & X^{-}(\overset{\bullet}{\#}_{min,Q}(A)) \\
\#^{!}_{max,P}(A) & = & X^{+}(\overset{\bullet}{\#}_{max,P}(A)) \\
\#^{!}_{max,[]}(A) & = & \top \\
\#^{!}_{m,Q}(0) & = & 0 \\
\overset{\bullet}{\#}_{m,Q}(\Gamma, \Delta) & = & \overset{\bullet}{\#}_{m,Q}(\Gamma) + \overset{\bullet}{\#}_{m,Q}(\Delta) \\
\overset{\bullet}{\#}_{m,[]}([\Gamma]) & = & \overset{\bullet}{\#}_{m,[]}(\Gamma)' \\
\#^{\bullet}{}_{m,P}([\Gamma]) & = & \#^{\bullet}{}_{m,P}(\Gamma) \\
\overset{\bullet}{\#}_{m,Q}(A) & = & \overset{\bullet}{\#}_{m,Q}(A) \text{ for } A \in \mathbf{Tp} \\
\overset{\bullet}{\#}_{m,Q}(\Lambda) & = & 0
\end{array}
$$

10.3 Theorem and proof

Definition

We say that $t \leq 0$ (and that $0 \geq t$) if and only if $t = *$ or $t = \bot$ or $t \leq 0$, and we say that $0 \leq t$ (and that $t \geq 0$) if and only if $t = *$ or $t = \top$ or $0 \leq t$.

Theorem

$$\vdash \Sigma \Longrightarrow \forall Q \in \mathcal{P} \cup \{[]\}, \#_{min,Q}(\Sigma) \leq 0 \leq \#_{max,Q}(\Sigma)$$

Proof By induction on the length of derivations.

10.3.1 Multiplicatives

- $\dfrac{\Gamma \Rightarrow A \qquad \Delta(C) \Rightarrow D}{\Delta(\Gamma, A\backslash C) \Rightarrow D}\backslash L$

  For every atom or bracket,

  $$
  \begin{array}{l}
  \#_m(\Delta(\Gamma, A\backslash C) \Rightarrow D) = \\
  \overset{\circ}{\#}_m(D) - \overset{\bullet}{\#}_{\overline{m}}(\Delta) - \overset{\bullet}{\#}_{\overline{m}}(\Gamma) - \overset{\bullet}{\#}_{\overline{m}}(A\backslash C) = \\
  \overset{\circ}{\#}_m(D) - \overset{\bullet}{\#}_{\overline{m}}(\Delta) - \overset{\bullet}{\#}_{\overline{m}}(\Gamma) - \overset{\bullet}{\#}_{\overline{m}}(C) + \overset{\circ}{\#}_m(A) = \\
  \overset{\circ}{\#}_m(A) - \overset{\bullet}{\#}_{\overline{m}}(\Gamma) + \overset{\circ}{\#}_m(D) - \overset{\bullet}{\#}_{\overline{m}}(\Delta) - \overset{\bullet}{\#}_{\overline{m}}(C) = \\
  \#_m(\Gamma \Rightarrow A) + \#_m(\Delta(C) \Rightarrow D)
  \end{array}
  $$

  By induction hypothesis (i.h.), $\#_{min}(\Gamma \Rightarrow A) \leq 0$ and $\#_{min}(\Delta(C) \Rightarrow D) \leq 0$. Therefore $\#_{min}(\Delta(\Gamma, A\backslash C) \Rightarrow D) = \#_{min}(\Gamma \Rightarrow A) + \#_{min}(\Delta(C) \Rightarrow D) \leq 0$. Similarly, $0 \leq \#_{max}(\Delta(\Gamma, A\backslash C) \Rightarrow D) = \#_{max}(\Gamma \Rightarrow A) + \#_{max}(\Delta(C) \Rightarrow D)$.

Therefore:

$\#_{min}(\Delta(\Gamma, A\backslash C) \Rightarrow D) \leq 0 \leq \#_{max}(\Delta(\Gamma, A\backslash C) \Rightarrow D)$

- $\dfrac{\Gamma \Rightarrow B \qquad \Delta(C) \Rightarrow D}{\Delta(C/B, \Gamma) \Rightarrow D}\ \backslash L$

  Like $\backslash L$.

- $\dfrac{A, \Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C}\ \backslash R$

  For every atom or bracket,

  $\#_m(\Gamma \Rightarrow A\backslash C) =$
  $\overset{\circ}{\#}_m(A\backslash C) - \overset{\bullet}{\#}_{\overline{m}}(\Gamma) =$
  $\overset{\circ}{\#}_m(C) - \overset{\bullet}{\#}_{\overline{m}}(A) - \overset{\bullet}{\#}_{\overline{m}}(\Gamma) =$
  $\#_m(A, \Gamma \Rightarrow C)$

  Therefore by i.h.,

  $\#_{min}(\Gamma \Rightarrow A\backslash C) \leq 0 \leq \#_{max}(\Gamma \Rightarrow A\backslash C)$

- $\dfrac{\Gamma, B \Rightarrow C}{\Gamma \Rightarrow C/B}\ /R$

  Like $\backslash R$.

- $\dfrac{\Delta(A, B) \Rightarrow C}{\Delta(A\bullet B) \Rightarrow C}\ \bullet L$

  For every atom or bracket,

  $\#_m(\Delta(A\bullet B) \Rightarrow C) =$
  $\overset{\circ}{\#}_m(C) - \overset{\bullet}{\#}_{\overline{m}}(\Delta) - \overset{\bullet}{\#}_{\overline{m}}(A\bullet B) =$
  $\overset{\circ}{\#}_m(C) - \overset{\bullet}{\#}_{\overline{m}}(\Delta) - \overset{\bullet}{\#}_{\overline{m}}(A) - \overset{\bullet}{\#}_{\overline{m}}(B) =$
  $\#_m(\Delta(A, B) \Rightarrow C)$

  Therefore by i.h.,

  $\#_{min}(\Delta(A\bullet B) \Rightarrow C) \leq 0 \leq \#_{max}(\Delta(A\bullet B) \Rightarrow C)$

- $\dfrac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A\bullet B}\ \bullet R$

  For every atom or bracket,

  $\#_m(\Gamma_1, \Gamma_2 \Rightarrow A\bullet B) =$
  $\overset{\circ}{\#}_m(A\bullet B) - \overset{\bullet}{\#}_{\overline{m}}(\Gamma_1, \Gamma_2) =$
  $\overset{\circ}{\#}_m(A) - \overset{\bullet}{\#}_{\overline{m}}(\Gamma_1) + \overset{\circ}{\#}_m(B) - \overset{\bullet}{\#}_{\overline{m}}(\Gamma_2) =$
  $\#_m(\Gamma_1 \Rightarrow A) + \#_m(\Gamma_2 \Rightarrow B)$

Therefore by i.h.,

$$\#_{min}(\Gamma_1, \Gamma_2 \Rightarrow A \bullet B) \leq 0 \leq \#_{max}(\Gamma_1, \Gamma_2 \Rightarrow A \bullet B)$$

10.3.2 Additives

- $$\frac{\Delta(A) \Rightarrow D}{\Delta(A \& B) \Rightarrow D} \& L_1$$

  For every atom or bracket,

$$\begin{array}{l}
\#_{min}(\Delta(A\&B) \Rightarrow D) = \\
\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta) - \overset{\bullet}{\#}_{max}(A\&B) = \\
\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta) - max(\overset{\bullet}{\#}_{max}(A), \overset{\bullet}{\#}_{max}(B)) \leq \\
\#_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta) - \overset{\bullet}{\#}_{max}(A) = \\
\#_{min}(\Delta(A) \Rightarrow D) \leq 0 \; i.h.
\end{array}$$

  And

$$\begin{array}{l}
\#_{max}(\Delta(A\&B) \Rightarrow D) = \\
\overset{\circ}{\#}_{max}(D) - \overset{\bullet}{\#}_{min}(\Delta) - \overset{\bullet}{\#}_{min}(A\&B) = \\
\overset{\circ}{\#}_{max}(D) - \overset{\bullet}{\#}_{min}(\Delta) - min(\overset{\bullet}{\#}_{min}(A), \overset{\bullet}{\#}_{min}(B)) \geq \\
\overset{\circ}{\#}_{max}(D) - \overset{\bullet}{\#}_{min}(\Delta) - \overset{\bullet}{\#}_{min}(A) = \\
\#_{max}(\Delta(A) \Rightarrow D) \geq 0 \; i.h.
\end{array}$$

  Therefore:

$$\#_{min}(\Delta(A\&B) \Rightarrow D) \leq 0 \leq \#_{max}(\Delta(A\&B) \Rightarrow D)$$

- $$\frac{\Delta(B) \Rightarrow D}{\Delta(A \& B) \Rightarrow D} \& L_2$$

  Like $\& L_1$.

- $$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \& B} \& R$$

$$\begin{array}{l}
\#_{min}(\Gamma \Rightarrow A\&B) = \\
\overset{\circ}{\#}_{min}(A\&B) - \overset{\bullet}{\#}_{max}(\Gamma) = \\
max(\overset{\circ}{\#}_{min}(A), \overset{\circ}{\#}_{min}(B)) - \overset{\bullet}{\#}_{max}(\Gamma) = \\
max(\overset{\circ}{\#}_{min}(A) - \overset{\bullet}{\#}_{max}(\Gamma), \overset{\circ}{\#}_{min}(B) - \overset{\bullet}{\#}_{max}(\Gamma)) = \\
\underbrace{max(\underbrace{\#_{min}(\Gamma \Rightarrow A)}_{\leq 0 \; i.h.}, \underbrace{\#_{min}(\Gamma \Rightarrow B)}_{\leq 0 \; i.h.})}_{\leq 0}
\end{array}$$

  And

$$
\begin{aligned}
&\#_{max}(\Gamma \Rightarrow A\&B) = \\
&\overset{\circ}{\#}_{max}(A\&B) - \overset{\bullet}{\#}_{min}(\Gamma) = \\
&min(\overset{\circ}{\#}_{max}(A), \overset{\circ}{\#}_{max}(B)) - \overset{\bullet}{\#}_{min}(\Gamma) = \\
&min(\overset{\circ}{\#}_{max}(A) - \overset{\bullet}{\#}_{min}(\Gamma), \overset{\circ}{\#}_{max}(B) - \overset{\bullet}{\#}_{min}(\Gamma)) = \\
&\underbrace{min(\underbrace{\#_{max}(\Gamma \Rightarrow A)}_{0 \leq \ i.h.}, \underbrace{\#_{max}(\Gamma \Rightarrow B)}_{0 \leq \ i.h.})}_{0 \leq}
\end{aligned}
$$

Therefore:

$\#_{min}(\Gamma \Rightarrow A\&B) \leq 0 \leq \#_{max}(\Gamma \Rightarrow A\&B)$.

- $\dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \oplus B} \oplus R_1$

$$
\begin{aligned}
&\#_{min}(\Gamma \Rightarrow A \oplus B) = \\
&\overset{\circ}{\#}_{min}(A \oplus B) - \overset{\bullet}{\#}_{max}(\Gamma) = \\
&min(\overset{\circ}{\#}_{min}(A), \overset{\bullet}{\#}_{min}(B)) - \overset{\bullet}{\#}_{max}(\Gamma) \leq \\
&\overset{\circ}{\#}_{min}(A) - \overset{\bullet}{\#}_{max}(\Gamma) = \\
&\#_{min}(\Gamma \Rightarrow A) \leq 0 \text{ i.h.}
\end{aligned}
$$

And

$$
\begin{aligned}
&\#_{max}(\Gamma \Rightarrow A \oplus B) = \\
&\overset{\circ}{\#}_{max}(A \oplus B) - \overset{\bullet}{\#}_{min}(\Gamma) = \\
&max(\overset{\bullet}{\#}_{max}(A), \overset{\bullet}{\#}_{max}(B)) - \overset{\bullet}{\#}_{min}(\Gamma) \geq \\
&\overset{\bullet}{\#}_{max}(A) - \overset{\bullet}{\#}_{min}(\Gamma) = \\
&\#_{max}(\Gamma \Rightarrow A) \geq 0 \text{ i.h.}
\end{aligned}
$$

- $\dfrac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \oplus B} \oplus R_2$

Like $\oplus R_1$.

- $\dfrac{\Delta(A) \Rightarrow D \qquad \Delta(B) \Rightarrow D}{\Delta(A \oplus B) \Rightarrow D} \oplus L$

For every atom or bracket,

$$
\begin{aligned}
&\#_{min}(\Delta(A \oplus B) \Rightarrow D) = \\
&\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta) - \overset{\bullet}{\#}_{max}(A \oplus B) = \\
&\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta) - min(\overset{\bullet}{\#}_{max}(A), \overset{\bullet}{\#}_{max}(B)) = \\
&max(\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta) - \overset{\bullet}{\#}_{max}(A), \overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta) - \overset{\bullet}{\#}_{max}(B)) = \\
&\underbrace{max(\underbrace{\#_{min}(\Delta(A) \Rightarrow D)}_{\leq 0 \text{ i.h.}}, \underbrace{\#_{min}(\Delta(B) \Rightarrow D)}_{\leq 0 \text{ i.h.}})}_{\leq 0}
\end{aligned}
$$

$0 \leq \#_{max}(\Xi(A \oplus B) \Rightarrow D)$ similarly

10.3.3 Bracket modalities

- $$\frac{\Gamma(A) \Rightarrow B}{\Gamma([[\,]^{-1}A]) \Rightarrow B}\,[\,]^{-1}L$$

  For atoms:

  $\#_{m,P}(\Gamma([[\,]^{-1}A]) \Rightarrow B) =$
  $\overset{\circ}{\#}_{m,P}(B) - \overset{\bullet}{\#}_{\overline{m},P}(\Gamma([[\,]^{-1}A])) =$
  $\overset{\circ}{\#}_{m,P}(B) - \overset{\bullet}{\#}_{\overline{m},P}(\Gamma) - \overset{\bullet}{\#}_{\overline{m},P}([[\,]^{-1}A])) =$
  $\overset{\circ}{\#}_{m,P}(B) - \overset{\bullet}{\#}_{\overline{m},P}(\Gamma) - \overset{\bullet}{\#}_{\overline{m},P}([\,]^{-1}A)) =$
  $\overset{\circ}{\#}_{m,P}(B) - \overset{\bullet}{\#}_{\overline{m},P}(\Gamma) - \overset{\bullet}{\#}_{\overline{m},P}(A)) =$
  $\overset{\circ}{\#}_{m,P}(B) - \overset{\bullet}{\#}_{\overline{m},P}(\Gamma(A)) =$
  $\#_{m,P}(\Gamma(A) \Rightarrow B)$

  I.e. the property for the conclusion follows from the induccion hypothesis for the premise since brackets and bracket modalities are transparent to atom count.

  For brackets:

  $\#_{m,[]}(\Gamma([[\,]^{-1}A]) \Rightarrow B) =$
  $\overset{\circ}{\#}_{m,[]}(B) - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma([[\,]^{-1}A])) =$
  $\overset{\circ}{\#}_{m,[]}(B) - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) - \overset{\bullet}{\#}_{\overline{m},[]}([[\,]^{-1}A])) =$
  $\overset{\circ}{\#}_{m,[]}(B) - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) - \overset{\bullet}{\#}_{\overline{m},[]}([\,]^{-1}A) - 1 =$
  $\overset{\circ}{\#}_{m,[]}(B) - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) - \overset{\bullet}{\#}_{\overline{m},[]}(A) + 1 - 1 =$
  $\overset{\circ}{\#}_{m,[]}(B) - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) - \overset{\bullet}{\#}_{\overline{m},[]}(A)) =$
  $\overset{\circ}{\#}_{m,[]}(B) - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma(A)) =$
  $\#_{m,[]}(\Gamma(A) \Rightarrow B) =$

  Therefore by i.h.,

  $\#_{min}(\Gamma([[\,]^{-1}A]) \Rightarrow B) \leq 0 \leq \#_{max}(\Gamma([[\,]^{-1}A]) \Rightarrow B)$

- $$\frac{[\Gamma] \Rightarrow A}{\Gamma \Rightarrow [\,]^{-1}A}\,[\,]^{-1}R$$

  For atoms:

  $\#_{m,P}(\Gamma \Rightarrow [\,]^{-1}A) = \#_{m,P}(\Gamma \Rightarrow A) = \#_{m,P}([\Gamma] \Rightarrow A)$

  Since brackets and bracket modalities are transparant to atom count.

  For brackets:

$$
\begin{array}{l}
\#_{m,[]}(\Gamma \Rightarrow [\,]^{-1}A) = \\
\overset{\circ}{\#}_{m,[]}([\,]^{-1}A) - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) = \\
\overset{\circ}{\#}_{m,[]}(A) - 1 - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) = \\
\overset{\circ}{\#}_{m,[]}(A) - (\overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) + 1) = \\
\overset{\circ}{\#}_{m,[]}(A) - \overset{\bullet}{\#}_{\overline{m},[]}([\Gamma]) = \\
\#_{m,[]}([\Gamma] \Rightarrow A)
\end{array}
$$

Therefore by i.h.

$$\#_{min}(\Gamma \Rightarrow [\,]^{-1}A) \leq 0 \leq \#_{max}(\Gamma \Rightarrow [\,]^{-1}A)$$

- $\dfrac{\Gamma([A]) \Rightarrow B}{\Gamma(\langle\rangle A) \Rightarrow B}\,\langle\rangle L$

  For atoms,

  $$\#_{m,P}(\Gamma(\langle\rangle A) \Rightarrow B) = \#_{m,P}(\Gamma([A]) \Rightarrow B)$$

  since brackets and bracket modalities are transparent to atom count.

  For brackets,

$$
\begin{array}{l}
\#_{m,[]}(\Gamma(\langle\rangle A) \Rightarrow B) = \\
\overset{\circ}{\#}_{m,[]}(B) - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) - \overset{\bullet}{\#}_{\overline{m},[]}(\langle\rangle A) = \\
\overset{\circ}{\#}_{m,[]}(B) - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) - (\overset{\bullet}{\#}_{\overline{m},[]}(A) + 1) = \\
\overset{\circ}{\#}_{m,[]}(B) - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) - \overset{\bullet}{\#}_{\overline{m},[]}([A]) = \\
\#_{m,[]}(\Gamma([A]) \Rightarrow B)
\end{array}
$$

  Therefore by i.h.

  $$\#_{min}(\Gamma(\langle\rangle A) \Rightarrow B) \leq 0 \leq \#_{max}(\Gamma(\langle\rangle A) \Rightarrow B)$$

- $\dfrac{\Gamma \Rightarrow A}{[\Gamma] \Rightarrow \langle\rangle A}\,\langle\rangle R$

  For atoms,

  $$\#_{m,P}([\Gamma] \Rightarrow \langle\rangle A) = \#_{m,P}(\Gamma \Rightarrow A)$$

  since brackets and bracket modalities are transparent to atom count.

  For brackets,

$$\begin{array}{l}
\#_{m,[]}([\Gamma] \Rightarrow \langle\rangle A) = \\
\overset{\circ}{\#}_{m,[]}(\langle\rangle A) - \overset{\bullet}{\#}_{\overline{m},[]}([\Gamma]) = \\
\overset{\circ}{\#}_{m,[]}(A) + 1 - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) - 1 = \\
\overset{\circ}{\#}_{m,[]}(A) - \overset{\bullet}{\#}_{\overline{m},[]}(\Gamma) = \\
\#_{m,[]}(\Gamma \Rightarrow A)
\end{array}$$

Therefore by i.h.:

$$\#_{min}([\Gamma] \Rightarrow \langle\rangle A) \leq 0 \leq \#_{max}([\Gamma] \Rightarrow \langle\rangle A)$$

10.3.4 Subexponentials

- $\dfrac{\Delta(A) \Rightarrow D}{\Delta(!A) \Rightarrow D}\ !L$

  For atoms,

$$\begin{array}{l}
\#_{min,P}(\Delta(!A) \Rightarrow D) = \\
\overset{\circ}{\#}_{min,P}(D) - \overset{\bullet}{\#}_{max,P}(\Delta) - \overset{\bullet}{\#}_{max,P}(!A) = \\
\overset{\circ}{\#}_{min,P}(D) - \overset{\bullet}{\#}_{max,P}(\Delta) - X^{+}(\overset{\bullet}{\#}_{max,P}(A)) \leq \\
\overset{\circ}{\#}_{min,P}(D) - \overset{\bullet}{\#}_{max,P}(\Delta) - \overset{\bullet}{\#}_{max,P}(A) = \\
\#_{min,P}(\Delta(A) \Rightarrow D) \leq 0 \text{ i.h.}
\end{array}$$

  And

$$\begin{array}{l}
\#_{max,P}(\Delta(!A) \Rightarrow D) = \\
\overset{\circ}{\#}_{max,P}(D) - \overset{\bullet}{\#}_{min,P}(\Delta) - \overset{\bullet}{\#}_{min,P}(!A) = \\
\overset{\circ}{\#}_{max,P}(D) - \overset{\bullet}{\#}_{min,P}(\Delta) - X^{-}(\overset{\bullet}{\#}_{min,P}(A)) \geq \\
\overset{\circ}{\#}_{max,P}(D) - \overset{\bullet}{\#}_{min,P}(\Delta) - \overset{\bullet}{\#}_{min,P}(A) = \\
\#_{max,P}(\Delta(A) \Rightarrow D) \geq 0\ i.h.
\end{array}$$

  For brackets,

$$\begin{array}{l}
\#_{min,[]}(\Delta(!A) \Rightarrow D) = \\
\overset{\circ}{\#}_{min,[]}(D) - \overset{\bullet}{\#}_{max,[]}(\Delta) - \overset{\bullet}{\#}_{max,[]}(!A) = \\
\overset{\circ}{\#}_{min,[]}(D) - \overset{\bullet}{\#}_{max,[]}(\Delta) - \top \leq \\
\overset{\circ}{\#}_{min,[]}(D) - \overset{\bullet}{\#}_{max,[]}(\Delta) - \overset{\bullet}{\#}_{max,[]}(A) = \\
\#_{min,[]}(\Delta(A) \Rightarrow D) \leq 0\ i.h.
\end{array}$$

  And

$$\begin{array}{l}
\#_{max,[]}(\Delta(!A) \Rightarrow D) = \\
\overset{\circ}{\#}_{max,[]}(D) - \overset{\bullet}{\#}_{min,[]}(\Delta) - \overset{\bullet}{\#}_{min,[]}(!A) = \\
\overset{\circ}{\#}_{max,[]}(D) - \overset{\bullet}{\#}_{min,[]}(\Delta) - X^{-}(\overset{\bullet}{\#}_{min,[]}(A)) \geq \\
\overset{\circ}{\#}_{max,[]}(D) - \overset{\bullet}{\#}_{min,[]}(\Delta) - \overset{\bullet}{\#}_{min,[]}(A) = \\
\#_{max,[]}(\Delta(A) \Rightarrow D) \geq 0 \text{ i.h.}
\end{array}$$

- $\dfrac{!A_1, \ldots, !A_n \Rightarrow A}{!A_1, \ldots, !A_n \Rightarrow !A}\ !R$

For atoms and brackets,

$$\begin{array}{l}
\#_{min,Q}(!A_1, \dots, !A_n \Rightarrow !A) = \\
\overset{\circ}{\#}_{min,Q}(!A) - \overset{\bullet}{\#}_{max,Q}(!A_1, \dots, !A_n) = \\
X^-(\overset{\circ}{\#}_{min,Q}(A)) - \overset{\bullet}{\#}_{max,Q}(!A_1, \dots, !A_n) \leq \\
\overset{\circ}{\#}_{min,Q}(A) - \overset{\bullet}{\#}_{max,Q}(!A_1, \dots, !A_n) = \\
\#_{min,Q}(!A_1, \dots, !A_n \Rightarrow A) \leq 0 \text{ i.h.}
\end{array}$$

And,

$$\begin{array}{l}
\#_{max,Q}(!A_1, \dots, !A_n \Rightarrow !A) = \\
\overset{\circ}{\#}_{max,Q}(!A) - \overset{\bullet}{\#}_{min,Q}(!A_1, \dots, !A_n) = \\
X^+(\overset{\circ}{\#}_{max,Q}(A)) - \overset{\bullet}{\#}_{min,Q}(!A_1, \dots, !A_n) \geq \\
\overset{\circ}{\#}_{max,Q}(A) - \overset{\bullet}{\#}_{min,Q}(!A_1, \dots, !A_n) = \\
\#_{max,Q}(!A_1, \dots, !A_n \Rightarrow A) \geq 0 \text{ i.h.}
\end{array}$$

- $$\frac{\Delta(!A_0, \dots, !A_n, [!A_0, \dots, !A_n, \Gamma]) \Rightarrow D}{\Delta(!A_0, \dots, !A_n, \Gamma) \Rightarrow D}\ !C$$

For atoms,

$$\begin{array}{l}
\#_{min}(\Delta(!A_0, \dots, !A_n, \Gamma) \Rightarrow D) = \\
\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta, \Gamma) - \overset{\bullet}{\#}_{max}(!A_0) - \cdots - \overset{\bullet}{\#}_{max}(!A_n) = \\
\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta, \Gamma) - X^+(\overset{\bullet}{\#}_{max}(A_0)) - \cdots - X^+(\overset{\bullet}{\#}_{max}(A_n)) \leq \\
\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta, [\Gamma]) - \\
X^+(\overset{\bullet}{\#}_{max}(A_0)) - \cdots - X^+(\overset{\bullet}{\#}_{max}(A_n)) - X^+(\overset{\bullet}{\#}_{max}(A_0)) - \cdots - X^+(\overset{\bullet}{\#}_{max}(A_n)) = \\
\#_{min}(\Delta(!A_0, \dots, !A_n, [!A_0, \dots, !A_n, \Gamma]) \Rightarrow D)
\end{array}$$

For brackets,

$$\begin{array}{l}
\#_{min}(\Delta(!A_0, \dots, !A_n, \Gamma) \Rightarrow D) = \\
\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta, \Gamma) - \overset{\bullet}{\#}_{max}(!A_0) - \cdots - \overset{\bullet}{\#}_{max}(!A_n) = \\
\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta, \Gamma) - \top - \cdots - \top \leq \\
\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta, [\Gamma]) - \top - \cdots - \top - \top - \cdots - \top = \\
\#_{min}(\Delta(!A_0, \dots, !A_n, [!A_0, \dots, !A_n, \Gamma]) \Rightarrow D)
\end{array}$$

And for atoms and brackets,

$$\begin{array}{l}
\#_{max}(\Delta(!A_0, \dots, !A_n, \Gamma) \Rightarrow D) = \\
\overset{\circ}{\#}_{max}(D) - \overset{\bullet}{\#}_{min}(\Delta, \Gamma) - \overset{\bullet}{\#}_{min}(!A_0) - \cdots - \overset{\bullet}{\#}_{min}(!A_n) = \\
\overset{\circ}{\#}_{max}(D) - \overset{\bullet}{\#}_{min}(\Delta, \Gamma) - X^-(\overset{\bullet}{\#}_{min}(A_0)) - \cdots - X^-(\overset{\bullet}{\#}_{min}(A_n)) \leq \\
\overset{\circ}{\#}_{max}(D) - \overset{\bullet}{\#}_{min}(\Delta, [\Gamma]) - \\
X^-(\overset{\bullet}{\#}_{min}(A_0)) - \cdots - X^-(\overset{\bullet}{\#}_{min}(A_n)) - X^+(\overset{\bullet}{\#}_{min}(A_0)) - \cdots - X^-(\overset{\bullet}{\#}_{min}(A_n)) = \\
\#_{max}(\Delta(!A_0, \dots, !A_n, [!A_0, \dots, !A_n, \Gamma]) \Rightarrow D)
\end{array}$$

- $$\frac{\Delta(A) \Rightarrow D \qquad \Delta(A, A) \Rightarrow D \dots}{\Delta(?A) \Rightarrow D}\ ?L$$

For atoms and brackets,

$$
\begin{aligned}
&\#_{min}(\Delta(?A) \Rightarrow D) = \\
&\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta) - \overset{\bullet}{\#}_{max}(?A) = \\
&\overset{\circ}{\#}_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta) - X^{+}(\overset{\bullet}{\#}_{max}(A)) \leq \\
&\#_{min}(D) - \overset{\bullet}{\#}_{max}(\Delta) - \overset{\bullet}{\#}_{max}(A) = \\
&\#_{min}(\Delta(A) \Rightarrow D) \leq 0 \text{ i.h.}
\end{aligned}
$$

And

$$
\begin{aligned}
&\#_{max}(\Delta(?A) \Rightarrow D) = \\
&\overset{\circ}{\#}_{max}(D) - \overset{\bullet}{\#}_{min}(\Delta) - \overset{\bullet}{\#}_{min}(?A) = \\
&\overset{\circ}{\#}_{max}(D) - \overset{\bullet}{\#}_{min}(\Delta) - X^{-}(\overset{\bullet}{\#}_{min}(A)) \geq \\
&\overset{\circ}{\#}_{max}(D) - \overset{\bullet}{\#}_{min}(\Delta) - \overset{\bullet}{\#}_{min}(A) = \\
&\#_{max}(\Delta(A) \Rightarrow D) \geq 0 \text{ i.h.}
\end{aligned}
$$

- $$\frac{\Gamma_0 \Rightarrow C \qquad \cdots \qquad \Gamma_n \Rightarrow C}{\Gamma_1, \ldots, \Gamma_n \Rightarrow ?C}\ ?R$$

For atoms and brackets,

$$
\begin{aligned}
&\#_{min}(\Gamma_0, \ldots, \Gamma_n \Rightarrow ?A) = \\
&\overset{\circ}{\#}_{min}(?A) - \overset{\bullet}{\#}_{max}(\Gamma_0) - \cdots - \overset{\bullet}{\#}_{max}(\Gamma_n) = \\
&X^{-}(\overset{\circ}{\#}_{min}(A)) - \overset{\bullet}{\#}_{max}(\Gamma_0) - \cdots - \overset{\bullet}{\#}_{max}(\Gamma_n) \leq \\
&n \cdot (\overset{\circ}{\#}_{min}(A)) - \overset{\bullet}{\#}_{max}(\Gamma_0) - \cdots - \overset{\bullet}{\#}_{max}(\Gamma_n) \leq \\
&\underbrace{\underbrace{\overset{\circ}{\#}_{min}(A) - \overset{\bullet}{\#}_{max}(\Gamma_0)}_{\leq 0 \text{ i.h.}} + \cdots + \underbrace{\overset{\circ}{\#}_{min}(A) - \overset{\bullet}{\#}_{max}(\Gamma_n)}_{\leq 0 \text{ i.h.}}}_{\leq 0}
\end{aligned}
$$

And,

$$
\begin{aligned}
&\#_{max}(\Gamma_0, \ldots, \Gamma_n \Rightarrow ?A) = \\
&\overset{\circ}{\#}_{max}(?A) - \overset{\bullet}{\#}_{min}(\Gamma_0) - \cdots - \overset{\bullet}{\#}_{min}(\Gamma_n) = \\
&X^{-}(\overset{\circ}{\#}_{max}(A)) - \overset{\bullet}{\#}_{min}(\Gamma_0) - \cdots - \overset{\bullet}{\#}_{min}(\Gamma_n) \geq \\
&n \cdot (\overset{\circ}{\#}_{max}(A)) - \overset{\bullet}{\#}_{min}(\Gamma_0) - \cdots - \overset{\bullet}{\#}_{min}(\Gamma_n) = \\
&\underbrace{\underbrace{\overset{\circ}{\#}_{max}(A) - \overset{\bullet}{\#}_{min}(\Gamma_0)}_{\geq 0 \text{ i.h.}} + \cdots + \underbrace{\overset{\circ}{\#}_{max}(A) - \overset{\bullet}{\#}_{min}(\Gamma_n)}_{\geq 0 \text{ i.h.}}}_{\geq 0}
\end{aligned}
$$

Part IV

GRAMMAR

In this part we exemplify the applications to grammar of categorial logic. Chapter 11 contains the lexicon. The subsequent chapters cover linguistic applications including Montague grammar, coordination, discontinuity and relativisation.

Chapter 11

Lexicon

007 : $\blacksquare\forall g Nt(s(g)) : 007$
a : $\blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda A\lambda B\exists C[(A\ C) \wedge (B\ C)]$
admire : $\Box((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa) : \text{ˆ}\lambda A\lambda B(Pres\ ((\text{ˇ}admire\ A)\ B))$
And : $\blacksquare\forall f(Sf/Sf) : \lambda AA$
and : $\blacksquare\forall f((?\blacksquare Sf\backslash[]^{-1}[]^{-1}Sf)/\blacksquare Sf) : (\mathbf{\Phi^{n+}}\ 0\ and)$
and : $\blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)) : (\mathbf{\Phi^{n+}}\ (s\ 0)\ and)$
and : $\blacksquare\forall a\forall f((?\blacksquare(Sf/!Na)\backslash[]^{-1}[]^{-1}(Sf/!Na))/\blacksquare(Sf/!Na)) : (\mathbf{\Phi^{n+}}\ (s\ 0)\ and)$
and : $\blacksquare\forall f((?\blacksquare(Sf/\exists aNa)\backslash[]^{-1}[]^{-1}(Sf/\exists aNa))/\blacksquare(Sf/\exists aNa)) : (\mathbf{\Phi^{n+}}\ (s\ 0)\ and)$
and : $\blacksquare\forall w\forall a\forall b\forall f((\blacksquare((Sf\uparrow(((\langle\rangle Na\backslash Sf)\circ\!-Ww)/Nb))\text{⫯}_2 Ww)\backslash[]^{-1}[]^{-1}((Sf\uparrow(((\langle\rangle Na\backslash Sf)\circ\!-Ww)/Nb))\text{⫯}_2 Ww))/$
$\text{ˆˆ}\blacksquare((Sf\uparrow(((\langle\rangle Na\backslash Sf)\circ\!-Ww)/Nb))\text{⫯}_2 Ww)) : \lambda A\lambda B\lambda C[(B\ C) \wedge (A\ C)]$
and : $\blacksquare\forall f\forall a((?\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash[]^{-1}[]^{-1}((\langle\rangle Na\backslash Sf)/\exists bNb))/\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb)) : (\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and)$
and : $\blacksquare\forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/$
$\exists bNb)\backslash(\langle\rangle Na\backslash Sf))) : (\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and)$
and : $\blacksquare\forall f\forall a((\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))))\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/$
$(\exists bNb\oplus\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))))\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g((CNg/CNg)\sqcup$
$(CNg\backslash CNg))))\backslash(\langle\rangle Na\backslash Sf))) : \lambda A\lambda B\lambda C\lambda D[((B\ C)\ D) \wedge ((A\ C)\ D)]$
and : $\blacksquare\forall a\forall b\forall f((?\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists cNc\oplus CPb))\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/(\exists cNc\oplus CPb))\backslash(\langle\rangle Na\backslash Sf)))/$
$\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists cNc\oplus CPb))\backslash(\langle\rangle Na\backslash Sf))) : (\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and)$
and : $\blacksquare\forall a\forall b\forall f((?\blacksquare((\langle\rangle Na\backslash Sf)/PPb)\backslash[]^{-1}[]^{-1}((\langle\rangle Na\backslash Sf)/PPb))/\blacksquare((\langle\rangle Na\backslash Sf)/PPb)) : (\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and)$
and : $\blacksquare\forall a\forall b\forall f((?\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf)))/$
$\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf))) : (\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and)$
and : $\blacksquare\forall a\forall b\forall f((?\blacksquare((\langle\rangle Na\backslash Sf)/!Nb)\backslash[]^{-1}[]^{-1}((\langle\rangle Na\backslash Sf)/!Nb))/\blacksquare((\langle\rangle Na\backslash Sf)/!Nb)) : (\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and)$
ate : $\Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa) : \text{ˆ}\lambda A\lambda B(Past\ ((\text{ˇ}eat\ A)\ B))$
bagels : $\Box(Nt(p(n))\&CNp(n)) : \text{ˆ}((gen\ \text{ˇ}bagels), \text{ˇ}bagels)$
barn : $\Box CNs(n) : barn$
be : $\Box((\langle\rangle W[there]\multimap Sb)/\exists aNa) : \text{ˆ}\lambda A(\text{ˇ}be\ A)$
before : $\blacksquare(\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))/Sf) : \lambda A\lambda B\lambda C((before\ A)\ (B\ C))$
beginning : $\Box CNs(n) : beginning$
believes : $\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/(CPthat\sqcup\Box Sf)) : \text{ˆ}\lambda A\lambda B(Pres\ ((\text{ˇ}believe\ A)\ B))$
bill : $\blacksquare Nt(s(m)) : b$
bond : $\blacksquare Nt(s(m)) : b$
book : $\Box CNs(n) : book$
bought : $\Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\bullet\exists aNa)) : \text{ˆ}\lambda A\lambda B(Past\ (((\text{ˇ}buy\ \pi_1 A)\ \pi_2 A)\ B))$
bought : $\Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa) : \text{ˆ}\lambda A\lambda B(Past\ ((\text{ˇ}buy\ A)\ B))$
by : $\blacksquare\forall a(((\langle\rangle Na\backslash S-)\backslash(\langle\rangle Na\backslash S-))/Na) : \lambda A\lambda B\lambda C[[C = A] \wedge (B\ C)]$
by : $\Box(\forall n(CNn\backslash CNn)/\exists aNa) : \text{ˆ}\lambda A\lambda B((\text{ˇ}by\ A)\ B)$
buys : $\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\bullet\exists aNa)) : \text{ˆ}\lambda A\lambda B(Pres\ (((\text{ˇ}buy\ \pi_1 A)\ \pi_2 A)\ B))$
calls : $\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists a((W[up]\text{◐}Na)\sqcup(Na\text{◑}W[up]))) : \text{ˆ}\lambda A\lambda B((\text{ˇ}phone\ A)\ B)$
catch : $\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa) : \text{ˆ}\lambda A\lambda B((\text{ˇ}catch\ A)\ B)$

cezanne : $\blacksquare Nt(s(m))$: c

cd : $\square CNs(n)$: cd

charles : $\blacksquare Nt(s(m))$: c

clark : $\blacksquare\forall gNt(s(g))$: c

coffee : $\square(Nt(s(n))\&CNs(n))$: $^\wedge((gen\ ^\vee coffee), ^\vee coffee)$

created : $\square((\langle\rangle\exists aNa\backslash Sf)/\exists aNa)$: $^\wedge\lambda A\lambda B(Past\ ((^\vee create\ A)\ B))$

darkness : $\square(CNs(n)\&Nt(s(n)))$: $^\wedge(^\vee darkness, (gen\ ^\vee darkness))$

deep : $\square CNs(n)$: $deep$

did : $\blacksquare\forall a\forall g\forall b\forall h((((\langle\rangle Na\backslash Sg)\uparrow(\langle\rangle Nb\backslash Sh))/(\exists c\langle\rangle Nc\backslash Sf))\backslash((\langle\rangle Na\backslash Sg)\uparrow(\langle\rangle Nb\backslash Sh)))$: $\lambda A\lambda B((A\ B)\ B)$

did+too : $(((\langle\rangle NA\backslash SB)\uparrow(\langle\rangle NC\backslash SD))/(\langle\rangle NE\backslash SF))\backslash((\langle\rangle NG\backslash SH)\uparrow(NI\backslash SJ))$: $\lambda K\lambda L((K\ L)\ L)$

doesnt : $\blacksquare\forall g\forall a((Sg\uparrow((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)))\downarrow Sg)$: $\lambda A\neg(A\ \lambda B\lambda C(B\ C))$

dog : $\square CNs(n)$: dog

donuts : $\square(Nt(p(n))\&CNp(n))$: $^\wedge((gen\ ^\vee donuts), ^\vee donuts)$

earth : $\square CNs(n)$: $earth$

eat : $\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa)$: $^\wedge\lambda A\lambda B((^\vee eat\ A)\ B)$

edinburgh : $\blacksquare Nt(s(n))$: e

editor : $\square(\forall gCNs(g)/PPof)$: $editor$

every : $\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g))$: $\lambda A\lambda B\forall C[(A\ C)\rightarrow(B\ C)]$

everyone : $\square\forall f((Sf\uparrow\forall gNt(g))\downarrow Sf)$: $^\wedge\lambda A\forall B[(^\vee person\ B)\rightarrow(A\ B)]$

face : $\square CNs(n)$: $face$

fell : $\square(\exists a\langle\rangle Na\backslash Sf)$: $^\wedge\lambda A(Past\ (^\vee fall\ A))$

filed : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)$: $^\wedge\lambda A\lambda B(Past\ ((^\vee file\ A)\ B))$

finds : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)$: $^\wedge\lambda A\lambda B(Pres\ ((^\vee find\ A)\ B))$

fish : $\square CNs(n)$: $fish$

for : $\blacksquare(PPfor/\exists aNa)$: λAA

form : $\square(CNs(n)\&Nt(s(n)))$: $^\wedge(^\vee form, (gen\ ^\vee form))$

fortunately : $\square\forall f(^\vee Sf\downarrow Sf)$: $fortunately$

friends : $\square(CNp/PPof)$: $friends$

from : $\square((\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))\&\forall n(CNn\backslash CNn))/\exists bNb)$: $^\wedge\lambda A((^\vee fromadv\ A), (^\vee fromadn\ A))$

gave : $\square((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto))$: $^\wedge\lambda A\lambda B(Past\ (((^\vee give\ \pi_2 A)\ \pi_1 A)\ B))$

gave : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa◑W[the, cold, shoulder]))$: $^\wedge\lambda A\lambda B(Past\ ((^\vee shun\ A)\ B))$

gave : $\square(((\langle\rangle\exists aNa\backslash Sf)/\exists aNa)/\exists aNa)$: $^\wedge\lambda A\lambda B\lambda C(Past\ (((^\vee give\ A)\ B)\ C))$

girl : $\square CNs(f)$: $girl$

gives : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa◑W[the, cold, shoulder]))$: $^\wedge\lambda A\lambda B(Pres\ ((^\vee shun\ A)\ B))$

God : $\blacksquare Nt(s(m))$: God

good : $\square\forall n(CNn/CNn)$: $good$

has : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)$: $^\wedge\lambda A\lambda B(Pres\ ((^\vee have\ A)\ B))$

he : $\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))$: λAA

heaven : $\square CNs(n)$: $heaven$

her : $\blacksquare\forall g\forall a(((\langle\rangle Na\backslash Sg)\uparrow\blacksquare Nt(s(f)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f))))$: λAA

himself : $\blacksquare\forall f(((\langle\rangle Nt(s(m))\backslash Sf)\uparrow Nt(s(m)))\downarrow(\langle\rangle Nt(s(m))\backslash Sf))$: $\lambda A\lambda B((A\ B)\ B)$

horse : $\square CNs(n)$: $horse$

in : $\square(\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))/\exists aNa)$: $^\wedge\lambda A\lambda B\lambda C((^\vee in\ A)\ (B\ C))$

in : $\square(\forall f(Sf\div Sf)/\exists aNa)$: in

is : $\blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I)))$: $\lambda A\lambda B(Pres\ (A{-}{>}C.[B = C]; D.((D\ \lambda E[E = B])\ B)))$

it : $\blacksquare W[it]$: 0

it : $\blacksquare\forall f\forall a(((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n))))$: λAA

it : $\blacksquare[]^{-1}\forall f((\blacksquare Sf|\blacksquare Nt(s(n)))/(\langle\rangle Nt(s(n))\backslash Sf))$: λAA

jogs : $\square(\langle\rangle\exists gNt(s(g))\backslash Sf)$: $^\wedge\lambda A(Pres\ (^\vee jog\ A))$

john : $\blacksquare Nt(s(m))$: j

laughs : $\square(\langle\rangle\exists gNt(s(g))\backslash Sf)$: $^\wedge\lambda A(Pres\ (^\vee laugh\ A))$

left : $\square(\langle\rangle\exists gNt(s(g))\backslash Sf)$: $^\wedge\lambda A(Pres\ (^\vee leave\ A))$

let : $\square(Sim/Sb)$: let

light : $\square(CNs(n)\&Nt(s(n)))$: ${}^{\wedge}({}^{\vee}light, (gen\ {}^{\vee}light))$
likes : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)$: ${}^{\wedge}\lambda A\lambda B(Pres\ (({}^{\vee}like\ A)\ B))$
logic : $\square(Nt(s(n))\&CNs(n))$: ${}^{\wedge}((gen\ {}^{\vee}logic), {}^{\vee}logic)$
london : $\blacksquare Nt(s(n))$: l
loses : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)$: ${}^{\wedge}\lambda A\lambda B(Pres\ (({}^{\vee}lose\ A)\ B))$
love : $\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa)$: ${}^{\wedge}\lambda A\lambda B(({}^{\vee}love\ A)\ B)$
loved : $\square\forall a\forall b(((\langle\rangle Na\backslash S-)\uparrow Nb)\odot(((\langle\rangle Na\backslash S-)\uparrow Nb)\downarrow\forall g(CNg\backslash CNg)))$: ${}^{\wedge}({}^{\vee}love, \lambda A\lambda B\lambda C[(B\ C)\wedge\exists D((A\ C)\ D)])$
loves : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)$: ${}^{\wedge}\lambda A\lambda B(Pres\ (({}^{\vee}love\ A)\ B))$
man : $\square CNs(m)$: man
mary : $\blacksquare Nt(s(f))$: m
met : $\square((\langle\rangle\exists aNa\backslash Sf)/\exists aNa)$: ${}^{\wedge}\lambda A\lambda B(Past\ (({}^{\vee}meet\ A)\ B))$
more : $\blacksquare\forall h\forall g\forall f((Sf\uparrow(((Sh\uparrow Nt(p(g)))\downarrow Sh)/CNp(g)))\downarrow(Sf/{}^{\wedge}(CPthan\uparrow\blacksquare(((Sh\uparrow Nt(p(g)))\downarrow Sh)/CNp(g)))))$:
$\lambda A\lambda B[|\lambda C(A\ \lambda D\lambda E[(D\ C)\wedge(E\ C)])| > |\lambda F{}^{\vee}(B\ \lambda G\lambda H[(G\ F)\wedge(H\ F)])|]$
mountain : $\square CNs(n)$: $mountain$
moved : $\square(\langle\rangle\exists aNa\backslash Sf)$: ${}^{\wedge}\lambda A(Past\ ({}^{\vee}move\ A))$
necessarily : $\blacksquare(SA/\square SA)$: Nec
of : $\square((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa))$: ${}^{\wedge}({}^{\vee}of, \lambda AA)$
or : $\blacksquare\forall f((?\blacksquare Sf\backslash[]^{-1}[]^{-1}Sf)/\blacksquare Sf)$: $(\mathbf{\Phi^{n+}}\ 0\ or)$
or : $\blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf))$: $(\mathbf{\Phi^{n+}}\ (s\ 0)\ or)$
or : $\blacksquare\forall f((?\blacksquare(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf))\backslash[]^{-1}[]^{-1}(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf)))/\blacksquare(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf)))$: $(\mathbf{\Phi^{n+}}\ (s\ 0)\ or)$
or : $\blacksquare\forall a\forall f((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)/\exists bNb)\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)/\exists bNb))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)/\exists bNb))$:
$(\mathbf{\Phi^{n+}}\ (s\ (s\ (s\ 0)))\ or)$
painting : $\square(CNs(n)/PPof)$: ${}^{\wedge}\lambda A(({}^{\vee}of\ A)\ {}^{\vee}painting)$
paper : $\square CNs(n)$: $paper$
park : $\square CNs(n)$: $park$
past : $\square\forall a\forall f(((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))/\exists bNb)$: ${}^{\wedge}\lambda A\lambda B\lambda C(({}^{\vee}past\ A)\ (B\ C))$
perseverance : $\square(Nt(s(n))\&CNs(n))$: ${}^{\wedge}((gen\ {}^{\vee}perseverance), {}^{\vee}perseverance)$
peter : $\blacksquare Nt(s(m))$: p
phonetics : $\square(Nt(s(n))\&CNs(n))$: ${}^{\wedge}((gen\ {}^{\vee}phonetics), {}^{\vee}phonetics)$
praises : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)$: ${}^{\wedge}\lambda A\lambda B(Pres\ (({}^{\vee}praise\ A)\ B))$
raced : $\square(\langle\rangle\exists aNa\backslash Sf)$: ${}^{\wedge}\lambda A(Past\ ({}^{\vee}race\ A))$
raced : $\square\forall a\forall b(((\langle\rangle Na\backslash S-)\uparrow Nb)\odot(((\langle\rangle Na\backslash S-)\uparrow Nb)\downarrow\forall g(CNg\backslash CNg)))$: ${}^{\wedge}({}^{\vee}race2, \lambda A\lambda B\lambda C[(B\ C)\wedge\exists D((A\ C)\ D)])$
rains : $\square(\langle\rangle W[it]\multimap Sf)$: ${}^{\wedge}(Pres\ {}^{\vee}itrains)$
reading : $\square((\langle\rangle\exists aNa\backslash Spsp)/\exists aNa)$: ${}^{\wedge}\lambda A\lambda B(({}^{\vee}read\ A)\ B)$
robin : $\blacksquare\forall gNt(s(g))$: r
said : $\square((\langle\rangle\exists aNa\backslash Sf)/Sim)$: ${}^{\wedge}\lambda A\lambda B(Past\ (({}^{\vee}say\ A)\ B))$
saw : $\square((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat))$: ${}^{\wedge}\lambda A\lambda B(Past\ ((A{-}{>}C.({}^{\vee}seee\ C); D.({}^{\vee}seet\ D))\ B))$
seeks : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\square\forall a\forall f(((Na\backslash Sf)/\exists bNb)\backslash(Na\backslash Sf)))$: ${}^{\wedge}\lambda A\lambda B(({}^{\vee}tries\ {}^{\wedge}(({}^{\vee}A\ {}^{\vee}find)\ B))\ B)$
sees : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)$: ${}^{\wedge}\lambda A\lambda B(Pres\ (({}^{\vee}see\ A)\ B))$
sent : $\square((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto))$: ${}^{\wedge}\lambda A\lambda B(Past\ ((({}^{\vee}sent\ \pi_2 A)\ \pi_1 A)\ B))$
sent : $\square(((\langle\rangle\exists aNa\backslash Sf)/\exists aNa)/\exists aNa)$: ${}^{\wedge}\lambda A\lambda B\lambda C(Past\ ((({}^{\vee}send\ A)\ B)\ C))$
she : $\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(f)))/(\langle\rangle Nt(s(f))\backslash Sg))$: λAA
sings : $\square(\langle\rangle\exists gNt(s(g))\backslash Sf)$: ${}^{\wedge}\lambda A(Pres\ ({}^{\vee}sing\ A))$
slept : $\square(\langle\rangle\exists gNt(s(g))\backslash Sf)$: ${}^{\wedge}\lambda A(Past\ ({}^{\vee}sleep\ A))$
slowly : $\square\forall a\forall f(\square(\langle\rangle Na\backslash Sf)\backslash(\langle\rangle\square Na\backslash Sf))$: ${}^{\wedge}\lambda A\lambda B({}^{\vee}slowly\ {}^{\wedge}({}^{\vee}A\ {}^{\vee}B))$
sneezed : $\square(\langle\rangle\exists gNt(s(g))\backslash Sf)$: ${}^{\wedge}\lambda A(Past\ ({}^{\vee}sneeze\ A))$
sold : $\square((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPfor))$: ${}^{\wedge}\lambda A\lambda B(Past\ ((({}^{\vee}sell\ \pi_2 A)\ \pi_1 A)\ B))$
someone : $\square\forall f((Sf\uparrow\blacksquare\forall gNt(g))\downarrow Sf)$: ${}^{\wedge}\lambda A\exists B[({}^{\vee}person\ B)\wedge(A\ B)]$
Spirit : $\square CNs(m)$: $Spirit$
studies : $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)$: ${}^{\wedge}\lambda A\lambda B(Pres\ (({}^{\vee}study\ A)\ B))$
such+that : $\blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n)))$: $\lambda A\lambda B\lambda C[(B\ C)\wedge(A\ C)]$
suzy : $\blacksquare Nt(s(f))$: s
talks : $\square(\langle\rangle\exists gNt(s(g))\backslash Sf)$: ${}^{\wedge}\lambda A(Pres\ ({}^{\vee}talk\ A))$
tall : $\square\forall g(CNg/CNg)$: $tall$

teetotal : $\Box\forall n(CNn/CNn)$: $\hat{}\lambda A\lambda B[(A\ B) \wedge (\check{}teetotal\ B)]$
tenmilliondollars : $\Box Nt(s(n))$: *tenmilliondollars*
than : $\blacksquare(CPthan/\Box Sf)$: λAA
that : $\blacksquare(CPthat/\Box Sf)$: λAA
that : $\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf))$: $\lambda A\lambda B\lambda C[(B\ C) \wedge (A\ C)]$
the : $\blacksquare\forall n(Nt(n)/CNn)$: ι
the+cold+shoulder : $\blacksquare W[the, cold, shoulder]$: *0*
there : $\blacksquare W[there]$: *0*
thinks : $\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/(CPthat\sqcup\Box Sf))$: $\hat{}\lambda A\lambda B(Pres\ ((\check{}think\ A)\ B))$
to : $\blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb)))$: λAA
today : $\Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))$: $\hat{}\lambda A\lambda B(\check{}today\ (A\ B))$
tries : $\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\Box(\langle\rangle\exists gNt(s(g))\backslash Si))$: $\hat{}\lambda A\lambda B((\check{}tries\ \hat{}(\check{}A\ B))\ B)$
unicorn : $\Box CNs(n)$: *unicorn*
up : $\blacksquare W[up]$: *0*
upon : $\Box((\forall b\forall f((\langle\rangle Nb\backslash Sf)\backslash(\langle\rangle Nb\backslash Sf))\&\forall g(CNg\backslash CNg))/\exists aNa)$: $\hat{}\lambda A((\check{}uponadv\ A), (\check{}uponadn\ A))$
void : $\Box\forall g(CNg/CNg)$: *void*
walk : $\Box(\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)$: $\hat{}\lambda A(Pres\ (\check{}walk\ A))$
walk : $\Box(\langle\rangle\exists aNa\backslash Sb)$: $\hat{}\lambda A(\check{}walk\ A)$
walks : $\Box(\langle\rangle\exists gNt(s(g))\backslash Sf)$: $\hat{}\lambda A(Pres\ (\check{}walk\ A))$
was : $\blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg)) - I)))$: $\lambda A\lambda B(Past\ (A -\!> C.[B = C]; D.((D\ \lambda E[E = B])\ B)))$
was : $\Box((\langle\rangle W[there] \multimap Sf)/\exists aNa)$: $\hat{}\lambda A(Past\ (\check{}be\ A))$
waters : $\Box CNp(n)$: *waters*
which : $\blacksquare\forall n\forall m((Nt(n)\uparrow Nt(m))\downarrow([]^{-1}[]^{-1}(CNm\backslash CNm)/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf)))$: $\lambda A\lambda B\lambda C\lambda D[(C\ D) \wedge (B\ (A\ D))]$
who : $\blacksquare\forall h\forall n([]^{-1}[]^{-1}(Nt(n)\backslash((Sh\uparrow Nt(n))\downarrow Sh))/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf))$: $\lambda A\lambda B\lambda C[(A\ B) \wedge (C\ B)]$
will : $\blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb))$: $\lambda A\lambda B(Fut\ (A\ B))$
without : $\Box(\forall g(CNg\backslash CNg)/\exists aNa)$: $\hat{}\lambda A\lambda B\lambda C[(B\ C) \wedge \neg((\check{}with\ A)\ C)]$
without : $\blacksquare\forall a\forall f([]^{-1}((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))/(\langle\rangle Na\backslash Spsp))$: $\lambda A\lambda B\lambda C[(B\ C) \wedge \neg(A\ C)]$
woman : $\Box CNs(f)$: *woman*
yesterday : $\Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))$: $\hat{}\lambda A\lambda B(\check{}yesterday\ (A\ B))$

Chapter 12

Initial Examples

The derivations we give have been computer-generated from the lexicon given in Chapter 11 and a parser for the categorial logic. The implementation is a categorial parser/theorem-prover CatLog2 comprising 6000 lines of Prolog using backward chaining proof-search in the Gentzen sequent calculus (Morrill 2011[63]), and the focusing of Andreoli (1992[5]), see Chapter 9; in addition to focusing, the implementation exploits count-invariance (van Benthem 1991[91]; Valentín, Serret and Morrill (2013[90]), see Chapter 10. In focusing, proofs are built in alternating phases of don't care non-deterministic invertible/asynchronous rule application and focused noninvertible/synchronous rule application. The boxes in our derivations mark the focused types, which are the active types of synchronous rule application. All the reader needs to have in mind is that a boxed type in the conclusion of an inference step is always the active type of that inference step. The first example is as follows:[1]

(122) **[john]+walks** : Sf

Note that in our syntactical form the subject is a bracketed domain, and this will always be the case — implementing that subjects are weak islands. Lookup in our lexicon yields the following semantically labelled sequent:

(123) $[\blacksquare Nt(s(m)) : j], \Box(\langle\rangle\exists g Nt(s(g))\backslash Sf) : {}^\wedge\lambda A(Pres\ ({}^\vee walk\ A)) \ \Rightarrow\ Sf$

The lexical types are semantically modalised outermost, and this will always be the case — implementing that word meanings are intensions/senses; the modality of the proper name subject is semantically inactive (we take proper names to be rigid designators), while the modality of the tensed verb is semantically active (the interpretation of tensed verbs depends on the temporal reference points). The verb projects a finite sentence (feature f) when it combines with a third person singular (bracketed) subject of any gender (the existential quantification); the actual subject is masculine (feature m).

The derivation is as follows:

(124)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\boxed{Nt(s(m))} \ \Rightarrow\ Nt(s(m))}}{\boxed{\blacksquare Nt(s(m))} \ \Rightarrow\ Nt(s(m))}\blacksquare L}{\blacksquare Nt(s(m)) \ \Rightarrow\ \boxed{\exists g Nt(s(g))}}\exists R}{[\blacksquare Nt(s(m))] \ \Rightarrow\ \boxed{\langle\rangle\exists g Nt(s(g))}}\langle\rangle R \qquad \overline{\boxed{Sf} \ \Rightarrow\ Sf}}{[\blacksquare Nt(s(m))], \boxed{\langle\rangle\exists g Nt(s(g))\backslash Sf} \ \Rightarrow\ Sf}\backslash L}{[\blacksquare Nt(s(m))], \boxed{\Box(\langle\rangle\exists g Nt(s(g))\backslash Sf)} \ \Rightarrow\ Sf}\Box L$$

The flow of information in the semantic reading of derivations can be illustrated for the case in hand as follows; note that in practice the steps of this information flow are implemented by unification stepwise with derivation. First, variables for the antecedent semantics are added in the endsequent:

[1] Note how in the input to CatLog brackets mark islands: single brackets for weak islands such as subjects and double brackets for strong islands such as relative clauses and coordinate structures, Morrill (2011[81], Chapter 5). Morrill, Kuznetsov, Kanovich and Scedrov (2018[68]) considers induction of such brackets.

(125) $[\blacksquare Nt(s(m)) : x], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : y \ \Rightarrow\ Sf$

Reading bottom-up, at the lowest inference step (□L) the verb semantics is replaced by the extension z and the subject semantics x is carried over:

(126)
$$\frac{[\blacksquare Nt(s(m)) : x], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} : z \ \Rightarrow\ Sf}{[\blacksquare Nt(s(m)) : x], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : y \ \Rightarrow\ Sf}\,\Box L$$

At the second inference we propagate the subject semantics on the argument branch:

(127)
$$\frac{\dfrac{[\blacksquare Nt(s(m)) : x] \ \Rightarrow\ \boxed{\langle\rangle\exists gNt(s(g))} \qquad \boxed{Sf} \ \Rightarrow\ Sf}{[\blacksquare Nt(s(m)) : x], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} : z \ \Rightarrow\ Sf}\,\backslash L}{[\blacksquare Nt(s(m)) : x], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : y \ \Rightarrow\ Sf}\,\Box L$$

The next three inferences involve semantically transparent copying of the antecedent semantics:

(128)
$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(m))} : \mathrm{x} \ \Rightarrow\ Nt(s(m))}}{\boxed{\blacksquare Nt(s(m)) : x} \ \Rightarrow\ Nt(s(m))}\,\blacksquare L}{\blacksquare Nt(s(m)) : x \ \Rightarrow\ \boxed{\exists gNt(s(g))}}\,\exists R}{[\blacksquare Nt(s(m)) : x] \ \Rightarrow\ \boxed{\langle\rangle\exists gNt(s(g))}}\,\langle\rangle R \qquad \dfrac{}{\boxed{Sf} \ \Rightarrow\ Sf}}{\dfrac{[\blacksquare Nt(s(m)) : x], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} : z \ \Rightarrow\ Sf}{[\blacksquare Nt(s(m)) : x], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : y \ \Rightarrow\ Sf}\,\Box L}\,\backslash L$$

At the identity axiom the antecedent semantics is copied to the succedent:

(129)
$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(m)) : x} \ \Rightarrow\ Nt(s(m)) : x}}{\boxed{\blacksquare Nt(s(m)) : x} \ \Rightarrow\ Nt(s(m))}\,\blacksquare L}{\blacksquare Nt(s(m)) : x \ \Rightarrow\ \boxed{\exists gNt(s(g))}}\,\exists R}{[\blacksquare Nt(s(m)) : x] \ \Rightarrow\ \boxed{\langle\rangle\exists gNt(s(g))}}\,\langle\rangle R \qquad \dfrac{}{\boxed{Sf} \ \Rightarrow\ Sf}}{\dfrac{[\blacksquare Nt(s(m)) : x], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} : z \ \Rightarrow\ Sf}{[\blacksquare Nt(s(m)) : x], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : y \ \Rightarrow\ Sf}\,\Box L}\,\backslash L$$

In a following phase the succedent semantics is copied from premises to conclusions as far as the root of the argument branch:

(130)
$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(m)) : x} \ \Rightarrow\ Nt(s(m)) : x}}{\boxed{\blacksquare Nt(s(m)) : x} \ \Rightarrow\ Nt(s(m)) : x}\,\blacksquare L}{\blacksquare Nt(s(m)) : x \ \Rightarrow\ \boxed{\exists gNt(s(g))} : x}\,\exists R}{[\blacksquare Nt(s(m)) : x] \ \Rightarrow\ \boxed{\langle\rangle\exists gNt(s(g)) : x}}\,\langle\rangle R \qquad \dfrac{}{\boxed{Sf} \ \Rightarrow\ Sf}}{\dfrac{[\blacksquare Nt(s(m)) : x], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} : z \ \Rightarrow\ Sf}{[\blacksquare Nt(s(m)) : x], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : y \ \Rightarrow\ Sf}\,\Box L}\,\backslash L$$

Now the functor value semantics in the antecedent of the value branch is labelled with a new variable w:

(131)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(m)):x} \Rightarrow Nt(s(m)):x}}{\boxed{\blacksquare Nt(s(m)):x} \Rightarrow Nt(s(m)):x}\blacksquare L}{\blacksquare Nt(s(m)):x \Rightarrow \boxed{\exists gNt(s(g))}:x}\exists R}{[\blacksquare Nt(s(m)):x] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g)):x}}\langle\rangle R \quad \dfrac{}{\boxed{Sf:w} \Rightarrow Sf}}{\dfrac{[\blacksquare Nt(s(m)):x], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf}:z \Rightarrow Sf}{[\blacksquare Nt(s(m)):x], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf):y \Rightarrow Sf}\Box L}\backslash L$$

At the *id* axiom this semantics is copied from antecedent to succedent:

(132)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(m)):x} \Rightarrow Nt(s(m)):x}}{\boxed{\blacksquare Nt(s(m)):j} \Rightarrow Nt(s(m)):j}\blacksquare L}{\blacksquare Nt(s(m)):x \Rightarrow \boxed{\exists gNt(s(g))}:x}\exists R}{[\blacksquare Nt(s(m)):x] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g)):x}}\langle\rangle R \quad \dfrac{}{\boxed{Sf:w} \Rightarrow Sf:w}}{\dfrac{[\blacksquare Nt(s(m)):x], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf}:z \Rightarrow Sf}{[\blacksquare Nt(s(m)):x], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf):y \Rightarrow Sf}\Box L}\backslash L$$

In the $\backslash L$ conclusion succedent the semantics of the major premise is subject to the substitution of w by the functional application of the functor z to the argument x:

(133)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(m)):x} \Rightarrow Nt(s(m)):x}}{\boxed{\blacksquare Nt(s(m)):x} \Rightarrow Nt(s(m)):x}\blacksquare L}{\blacksquare Nt(s(m)):x \Rightarrow \boxed{\exists gNt(s(g))}:x}\exists R}{[\blacksquare Nt(s(m)):x] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g)):x}}\langle\rangle R \quad \dfrac{}{\boxed{Sf:w} \Rightarrow Sf:w}}{[\blacksquare Nt(s(m)):x], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf}:z \Rightarrow Sf:w\{(z\ x)/w\} = (z\ x)}\backslash L}{[\blacksquare Nt(s(m)):x], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf):y \Rightarrow Sf}\Box L$$

And thence to the conclusion of the endsequent:

(134)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(m)):x} \Rightarrow Nt(s(m)):x}}{\boxed{\blacksquare Nt(s(m)):x} \Rightarrow Nt(s(m)):x}\blacksquare L}{\blacksquare Nt(s(m)):x \Rightarrow \boxed{\exists gNt(s(g))}:x}\exists R}{[\blacksquare Nt(s(m)):x] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g)):x}}\langle\rangle R \quad \dfrac{}{\boxed{Sf:w} \Rightarrow Sf:w}}{[\blacksquare Nt(s(m)):x], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf}:z \Rightarrow Sf:(z\ x)}\backslash L}{[\blacksquare Nt(s(m)):x], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf):y \Rightarrow Sf:(z\ x)\{{}^{\vee}y/z\} = ({}^{\vee}y\ x)}\Box L$$

Now we can substitute in the lexical semantics j for **John** (x) and ${}^{\wedge}\lambda A(Pres\ ({}^{\vee}walk\ A))$ for **walks** (y) and evaluate:[2]

(135) $({}^{\vee\wedge}\lambda A(Pres\ ({}^{\vee}walk\ A))\ j) =$
$(\lambda A(Pres\ ({}^{\vee}walk\ A))\ j) =$
$(Pres\ ({}^{\vee}walk\ j))$

[2]Montague's Intensional Logic assigned nonlogical constants of type τ a denotation in the intension of τ and then interpreted a constant with respect to a world as its extension in that world. By contrast our semantic representation language assigns constants denotations in their own type, so our semantic representations have explicit extensionalisations of intensional constants.

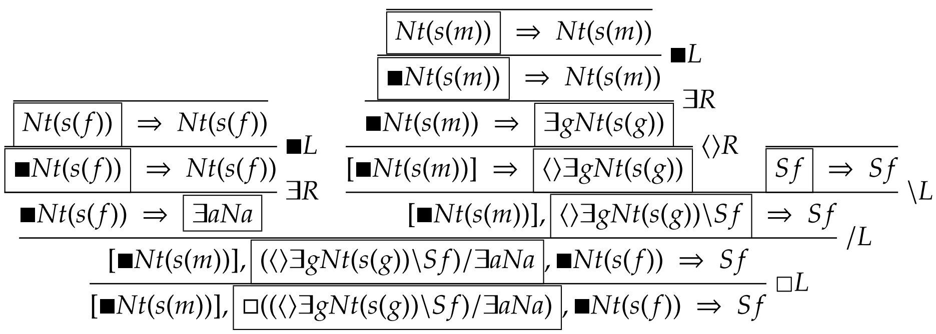


Figure 12.1: Derivation for **John loves Mary**

(As we have said, this elucidation is not exactly how CatLog2 extracts semantics; CatLog2 uses unification and instantiation of metavariables to deliver in a single pass the unevaluated semantics of the upwards and downward phases, and then normalises.)

By way of a second example, the following is a simple transitive sentence:

(136) **[john]+loves+mary** : Sf

Lexical lookup yields:

(137) $[\blacksquare Nt(s(m)) : j], \Box((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na) : \hat{}\lambda A\lambda B(Pres\ ((\check{}love\ A)\ B)), \blacksquare Nt(s(f)) : m \Rightarrow Sf$

There is the derivation given in Figure 12.1. Reading upwards from the endsequent, the first inference removes the intensionality modality from the transitive verb, and then over left selects the object to analyse as the argument of the transitive verb; this is done by existential right instantiating the agreement feature to third person singular feminine, followed by (semantically inactive) intensionality modality left. The right hand branch is the same as for example (122) after the first inference. All this delivers semantics:

(138) $(Pres\ ((\check{}love\ m)\ j))$

The next example has a subordinate clause:

(139) **[john]+thinks+[mary]+walks** : Sf

Lexical lookup yields the following; note that the propositional attitude verb is polymorphic with respect to a complementised or uncomplementised sentential argument, expressed with a semantically inactive additive disjunction:

(140) $[\blacksquare Nt(s(m)) : j], \Box((\langle\rangle\exists g Nt(s(g))\backslash Sf)/(CPthat\sqcup\Box Sf)) : \hat{}\lambda A\lambda B(Pres\ ((\check{}think\ A)\ B)),$
$[\blacksquare Nt(s(f)) : m], \Box(\langle\rangle\exists g Nt(s(g))\backslash Sf) : \hat{}\lambda C(Pres\ (\check{}walk\ C)) \Rightarrow Sf$

This has the derivation given in Figure 12.2. Reading bottom-up, following elimination of the intensionality modality on the propositional attitude verb, over left partitions in such a way as to supply the subordinate clause as the propositional argument. Again, the righthand subtree is the same as for example (12.1) after the first inference. In the lefthand subtree semantically inactive additive conjunction right selects the modalised uncomplementized sentence type. The succedent modality is removed, this being licensed by the fact that all the antecedent types are modalised, and the remaining derivation is also like that for example (12.1). The derivation delivers semantics:

(141) $(Pres\ ((\check{}think\ \hat{}(Pres\ (\check{}walk\ m)))\ j))$

The following example involves a ditransitive verb:

(142) **[mary]+buys+john+coffee** : Sf

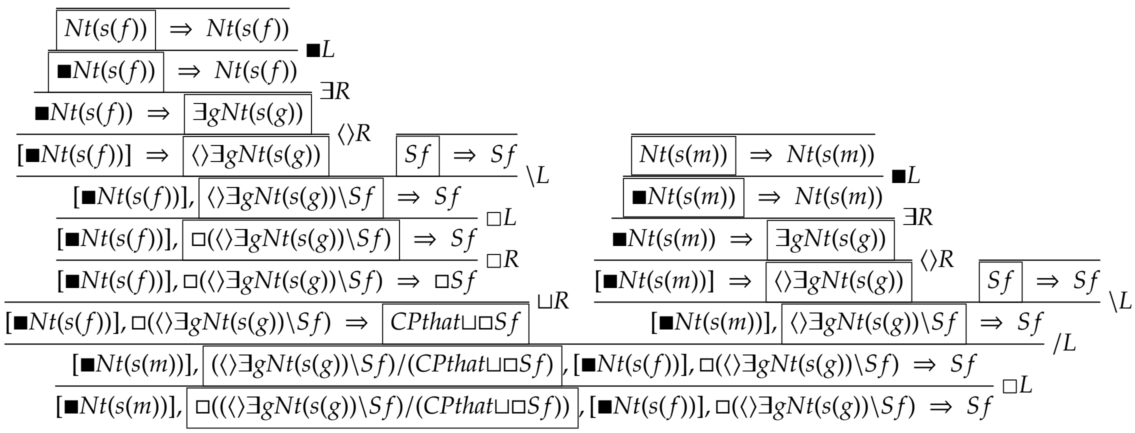


Figure 12.2: Derivation for **John thinks Mary walks**

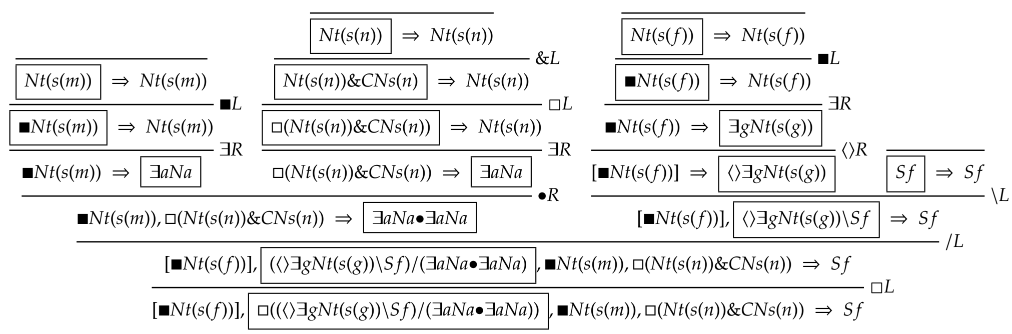


Figure 12.3: Derivation for **Mary buys John coffee**

Lexical lookup is as follows; note the use of product (multiplicative conjunction) for the ditransitive verb, and the use of additive conjunction for the polymorphism of the mass noun *coffee* which can appear either as a bare nominal or with an article:

(143) $[\blacksquare Nt(s(f)) : m], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa{\bullet}\exists aNa)) : \hat{}\lambda A\lambda B(Pres\ (((\check{}buy\ \pi_1 A)\ \pi_2 A)\ B)),$
$\blacksquare Nt(s(m)) : j, \Box(Nt(s(n))\&CNs(n)) : \hat{}((gen\ \check{}coffee), \check{}coffee) \ \Rightarrow\ Sf$

There is the derivation given in Figure 12.3. After removal of the outer modality of the ditransitive verb, the partitioning of over left selects the two objects as the verb's product argument, partitioned in turn by continuous product right. The indirect object **John** is analysed by existential right and inactive modality left inferences; the direct object **coffee** is analysed by existential right and (active) modality left inferences followed by selection of the bare noun type by additive conjunction left. The rightmost subtree is as usual for an intransitive sentence. This delivers semantics as follows in which a 'generic' operator applies to *coffee*:

(144) $(Pres\ (((\check{}buy\ j)\ (gen\ \check{}coffee))\ m))$

The next example includes a definite article:

(145) **[the+man]+walks** : Sf

We treat the definite article simply as an iota operator which returns the unique individual in the context of discourse satisfying its common noun argument (Carpenter 1997[11]); this unicity is presupposed by the use of the definite. Lexical lookup yields the semantically labelled sequent:

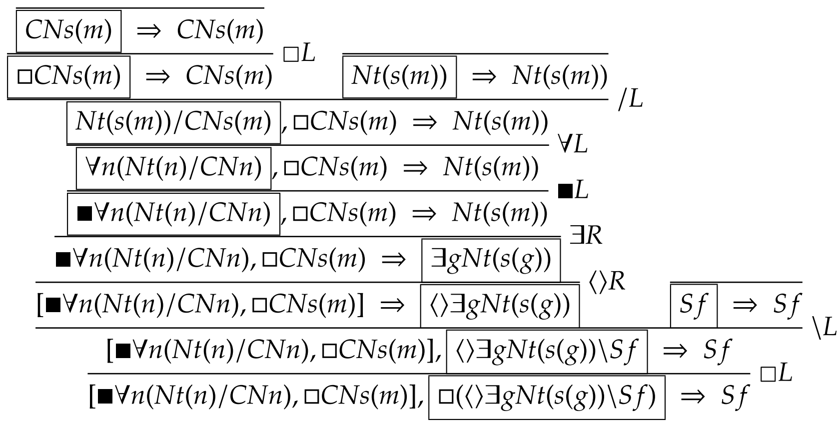


Figure 12.4: Derivation for **The man walks**

(146) $[\blacksquare\forall n(Nt(n)/CNn) : \iota, \Box CNs(m) : man], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda A(Pres\ (\check{}walk\ A)) \Rightarrow Sf$

There is the derivation given in Figure 12.4. This is like the derivation of an intransitive sentence before, but with the analysis of the definite noun phrase subject at the top left. The derivation delivers semantics:

(147) $(Pres\ (\check{}walk\ (\iota\ \check{}man)))$

The next two examples have adverbial and adnominal prepositional modification respectively. We consider the adverbial case first:

(148) **[john]+walks+from+edinburgh** : Sf

Lexical lookup inserts a single value-polymorphic prepositional type, which uses semantically active additive conjunction:

(149) $[\blacksquare Nt(s(m)) : j], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda A(Pres\ (\check{}walk\ A)), \Box((\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash$
$(\langle\rangle Na\backslash Sf))\&\forall n(CNn\backslash CNn))/\exists bNb) : \hat{}\lambda B((\check{}fromadv\ B), (\check{}fromadn\ B)), \blacksquare Nt(s(n)) : e \Rightarrow Sf$

There is the derivation given in Figure 12.5. After elimination of the outer modality of the preposition, over left selects as the prepositional argument the prepositional object, which is analysed in the leftmost subtree. In the sister subtree additive conjunction left selects the adverbial type for the prepositional phrase and for all left instantiates the subject agreement and verb form features to third person singular masculine, and finite. Following under left, in the middle subtree **walks** is analysed as the intransitive verb second argument of the adverbial preposition; note the analysis of the higher-order type by the under right rule, which lowers the conclusion succedent hypothetical subtype into the premise antecedent. The rightmost subtree is an intransitive sentence case again. All this delivers the semantics:

(150) $(((\check{}fromadv\ e)\ \lambda B(Pres\ (\check{}walk\ B)))\ j)$

The adnominal case is:

(151) **[the+man+from+edinburgh]+walks** : Sf

Lexical lookup yields:

(152) $[\blacksquare\forall n(Nt(n)/CNn) : \iota, \Box CNs(m) : man, \Box((\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))\&$
$\forall n(CNn\backslash CNn))/\exists bNb) : \hat{}\lambda A((\check{}fromadv\ A), (\check{}fromadn\ A)), \blacksquare Nt(s(n)) : e],$
$\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda B(Pres\ (\check{}walk\ B)) \Rightarrow Sf$

There is the derivation given in Figure 12.6. In the first two steps the intransitive verb **walks** is prepared to apply to the complex subject. Bracket right and exists right follow, then (inactive) modality left and for all left on the determiner, which then applies to the complex common noun. The result of modality left on the preposition applies to the prepositional object and in the major premise additive conjunction left selects the adnominal prepositional type. The semantics delivered is:

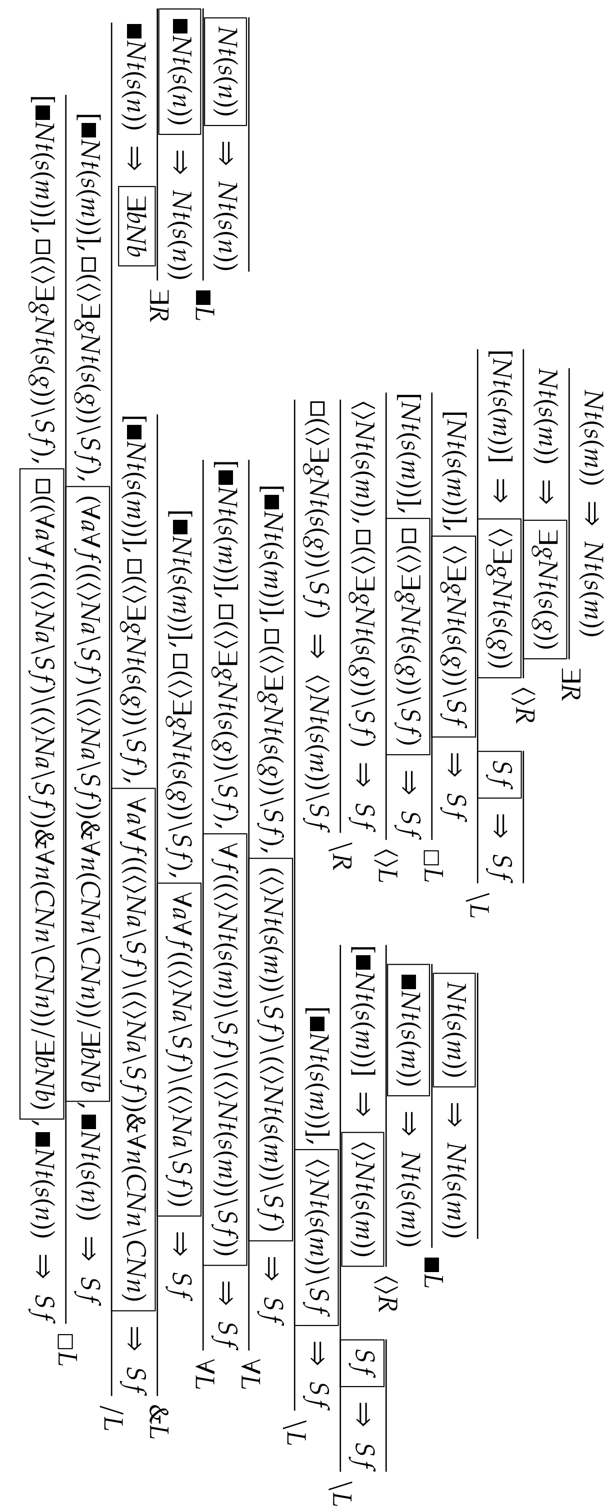


Figure 12.5: Derivation for **John walks from Edinburgh**

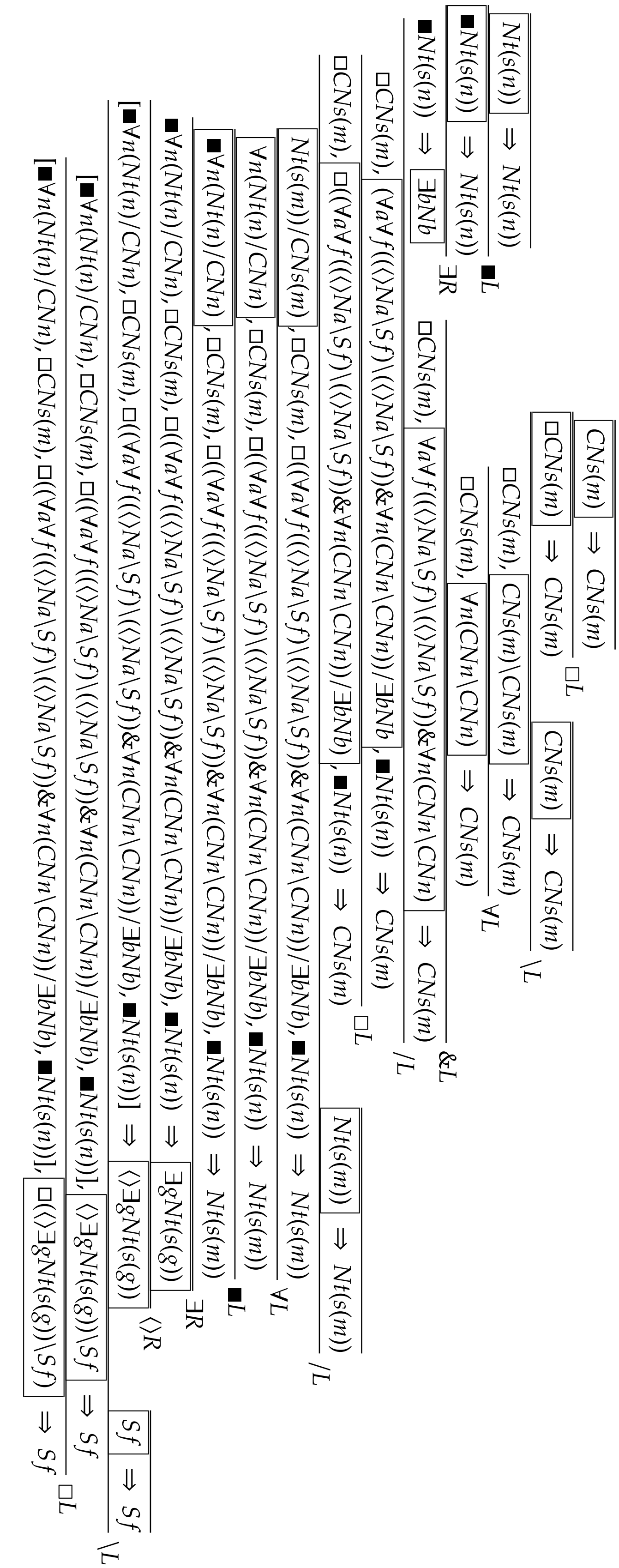


Figure 12.6: Derivation for **The man from Edinburgh walks**

$$\dfrac{Nt(s(A)) \Rightarrow Nt(s(A))}{\forall g Nt(s(g)) \Rightarrow Nt(s(A))}\ \forall L$$

$$\dfrac{\forall g Nt(s(g)) \Rightarrow Nt(s(A))}{\blacksquare\forall g Nt(s(g)) \Rightarrow Nt(s(A))}\ \blacksquare L$$

$$\dfrac{\blacksquare\forall g Nt(s(g)) \Rightarrow Nt(s(A))}{\blacksquare\forall g Nt(s(g)) \Rightarrow \exists a Na}\ \exists R$$

$$\dfrac{\blacksquare\forall g Nt(s(g)) \Rightarrow \exists a Na}{\blacksquare\forall g Nt(s(g)) \Rightarrow \exists a Na \oplus (\exists g((CNg/CNg)\sqcup(CNg\backslash CNg)) - I)}\ \oplus R$$

$$\dfrac{Nt(s(m)) \Rightarrow Nt(s(m))}{\blacksquare Nt(s(m)) \Rightarrow Nt(s(m))}\ \blacksquare L$$

$$\dfrac{\blacksquare Nt(s(m)) \Rightarrow Nt(s(m))}{\blacksquare Nt(s(m)) \Rightarrow \exists g Nt(s(g))}\ \exists R$$

$$\dfrac{\blacksquare Nt(s(m)) \Rightarrow \exists g Nt(s(g))}{[\blacksquare Nt(s(m))] \Rightarrow \langle\rangle\exists g Nt(s(g))}\ \langle\rangle R$$

$$\dfrac{[\blacksquare Nt(s(m))] \Rightarrow \langle\rangle\exists g Nt(s(g)) \qquad Sf \Rightarrow Sf}{[\blacksquare Nt(s(m))], \langle\rangle\exists g Nt(s(g))\backslash Sf \Rightarrow Sf}\ \backslash L$$

$$\dfrac{\blacksquare\forall g Nt(s(g)) \Rightarrow \exists a Na \oplus (\exists g((CNg/CNg)\sqcup(CNg\backslash CNg)) - I) \qquad [\blacksquare Nt(s(m))], \langle\rangle\exists g Nt(s(g))\backslash Sf \Rightarrow Sf}{[\blacksquare Nt(s(m))], (\langle\rangle\exists g Nt(s(g))\backslash Sf)/(\exists a Na \oplus (\exists g((CNg/CNg)\sqcup(CNg\backslash CNg)) - I)), \blacksquare\forall g Nt(s(g)) \Rightarrow Sf}\ /L$$

$$\dfrac{[\blacksquare Nt(s(m))], (\langle\rangle\exists g Nt(s(g))\backslash Sf)/(\exists a Na \oplus (\exists g((CNg/CNg)\sqcup(CNg\backslash CNg)) - I)), \blacksquare\forall g Nt(s(g)) \Rightarrow Sf}{[\blacksquare Nt(s(m))], \blacksquare((\langle\rangle\exists g Nt(s(g))\backslash Sf)/(\exists a Na \oplus (\exists g((CNg/CNg)\sqcup(CNg\backslash CNg)) - I))), \blacksquare\forall g Nt(s(g)) \Rightarrow Sf}\ \blacksquare L$$

Figure 12.7: Derivation for **Bond is 007**

(153) $(Pres\ (\check{}walk\ (\iota\ ((\check{}fromadn\ e)\ \check{}man))))$

The last two initial examples involve the copula with nominal and (intersective) adjectival complementation respectively. We consider first the nominal case:

(154) **[bond]+is+007** : Sf

Lexical lookup inserts a single argument-polymorphic copula type, which uses both semantically active and semantically inactive additive disjunction:[3]

(155) $[\blacksquare Nt(s(m)) : b], \blacksquare((\langle\rangle\exists g Nt(s(g))\backslash Sf)/(\exists a Na \oplus (\exists g((CNg/CNg)\sqcup(CNg\backslash CNg)) - I))) : \lambda A\lambda B(Pres\ (A{-}{>}C.[B = C]; D.((D\ \lambda E[E = B])\ B))), \blacksquare\forall g Nt(s(g)) : 007 \Rightarrow Sf$

There is the derivation given in Figure 12.7. After elimination of the outer copula modality the copula is applied to its nominal complement. Additive disjunction right selects the first, nominal, disjunct. The derivation delivers semantics:

(156) $(Pres\ [b = 007])$

The (intersective) adjectival case is:

(157) **[bond]+is+teetotal** : Sf

Lexical lookup yields:

(158) $[\blacksquare Nt(s(m)) : b], \blacksquare((\langle\rangle\exists g Nt(s(g))\backslash Sf)/(\exists a Na \oplus (\exists g((CNg/CNg)\sqcup(CNg\backslash CNg)) - I))) : \lambda A\lambda B(Pres\ (A{-}{>}C.[B = C]; D.((D\ \lambda E[E = B])\ B))), \Box\forall n(CNn/CNn) : \hat{}\lambda F\lambda G[(F\ G) \wedge (\check{}teetotal\ G)] \Rightarrow Sf$

There is the derivation given in Figure 12.8. After elimination of its outer modality, the copula is applied to its adjectival complement. Semantically active additive disjunction right selects the second disjunct. The difference right rule checks that the antecedent is not empty, but this is not displayed. Exists right substitutes the existentially quantified variable for a metavariable A and semantically inactive additive disjunction right then selects the adjectival disjunct. The following semantics is delivered:

(159) $(Pres\ (\check{}teetotal\ b))$

[3]The difference operator (Morrill and Valentín 2014[71]) for linguistic exceptions is also used. It involves negation as failure, which cannot easily be displayed. We do not dwell on this operator here.

$$\dfrac{CNA \Rightarrow CNA \qquad \boxed{CNA} \Rightarrow CNA}{\boxed{CNA/CNA}, CNA \Rightarrow CNA}\,/L$$

$$\dfrac{\boxed{CNA/CNA}, CNA \Rightarrow CNA}{\boxed{\forall n(CNn/CNn)}, CNA \Rightarrow CNA}\,\forall L$$

$$\dfrac{\boxed{\forall n(CNn/CNn)}, CNA \Rightarrow CNA}{\boxed{\Box\forall n(CNn/CNn)}, CNA \Rightarrow CNA}\,\Box L$$

$$\dfrac{\boxed{\Box\forall n(CNn/CNn)}, CNA \Rightarrow CNA}{\Box\forall n(CNn/CNn) \Rightarrow CNA/CNA}\,/R$$

$$\dfrac{\Box\forall n(CNn/CNn) \Rightarrow CNA/CNA}{\Box\forall n(CNn/CNn) \Rightarrow \boxed{(CNA/CNA)\sqcup(CNA\backslash CNA)}}\,\sqcup R$$

$$\dfrac{\Box\forall n(CNn/CNn) \Rightarrow \boxed{(CNA/CNA)\sqcup(CNA\backslash CNA)}}{\Box\forall n(CNn/CNn) \Rightarrow \boxed{\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))}}\,\exists R$$

$$\dfrac{\Box\forall n(CNn/CNn) \Rightarrow \boxed{\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))}}{\Box\forall n(CNn/CNn) \Rightarrow \boxed{\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I}}\,-R$$

$$\dfrac{\Box\forall n(CNn/CNn) \Rightarrow \boxed{\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I}}{\Box\forall n(CNn/CNn) \Rightarrow \boxed{\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I)}}\,\oplus R$$

$$\dfrac{\boxed{Nt(s(m))} \Rightarrow Nt(s(m))}{\boxed{\blacksquare Nt(s(m))} \Rightarrow Nt(s(m))}\,\blacksquare L$$

$$\dfrac{\boxed{\blacksquare Nt(s(m))} \Rightarrow Nt(s(m))}{\blacksquare Nt(s(m)) \Rightarrow \boxed{\exists gNt(s(g))}}\,\exists R$$

$$\dfrac{\blacksquare Nt(s(m)) \Rightarrow \boxed{\exists gNt(s(g))}}{[\blacksquare Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g))}}\,\langle\rangle R$$

$$\dfrac{[\blacksquare Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g))} \qquad \boxed{Sf} \Rightarrow Sf}{[\blacksquare Nt(s(m))], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} \Rightarrow Sf}\,\backslash L$$

$$\dfrac{\Box\forall n(CNn/CNn) \Rightarrow \boxed{\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I)} \qquad [\blacksquare Nt(s(m))], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} \Rightarrow Sf}{[\blacksquare Nt(s(m))], \boxed{(\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I))}, \Box\forall n(CNn/CNn) \Rightarrow Sf}\,/L$$

$$\dfrac{[\blacksquare Nt(s(m))], \boxed{(\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I))}, \Box\forall n(CNn/CNn) \Rightarrow Sf}{[\blacksquare Nt(s(m))], \boxed{\blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I)))}, \Box\forall n(CNn/CNn) \Rightarrow Sf}\,\blacksquare L$$

Figure 12.8: Derivation for **Bond is teetotal**

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Chapter 13

The Montague Fragment

In this chapter we give derivations of the Montague grammar fragment examples analysed in Chapter 7 of Dowty, Wall and Peters (1981[18]), DWP. This 'Montague test' was performed on the 9th of October 2015 at the Colloque de Sintaxe et Sémantique à Paris. The example sentences are shown in Figure 13.1. The lexicon is shown in Figure 13.2. The first example is as follows, the same as example (122) of Chapter 12. (We continue to include the indexation of CatLog2, which contains the numeration of the source, within the display.)

(160) (dwp((7-7))) [**john**]+**walks** : Sf

Recall that in our syntactical forms the subjects are bracketed domains — implementing that subjects are weak islands. Lookup in our lexicon yields the following semantically labelled sequent:

(161) $[\blacksquare Nt(s(m)) : j], \square(\langle\rangle\exists g Nt(s(g))\backslash Sf) : \hat{}\lambda A(Pres\ (\check{}walk\ A)) \ \Rightarrow\ Sf$

As always the lexical types are semantically modalized outermost — implementing that word meanings are intensions/senses; the modality of the proper name subject is semantically inactive (proper names are rigid designators), while the modality of the tensed verb is semantically active (the interpretation of tensed verbs depends on the temporal reference points). The verb projects a finite sentence (feature f) when it combines with a third person singular (bracketed) subject of any gender; the actual subject is masculine (feature m).

The derivation is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(m))} \Rightarrow Nt(s(m))}}{\boxed{\blacksquare Nt(s(m))} \Rightarrow Nt(s(m))}\blacksquare L}{\blacksquare Nt(s(m)) \Rightarrow \boxed{\exists g Nt(s(g))}}\exists R}{[\blacksquare Nt(s(m))] \Rightarrow \boxed{\langle\rangle \exists g Nt(s(g))}}\langle\rangle R \qquad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{[\blacksquare Nt(s(m))], \boxed{\langle\rangle\exists g Nt(s(g))\backslash Sf} \Rightarrow Sf}\backslash L}{[\blacksquare Nt(s(m))], \boxed{\square(\langle\rangle\exists g Nt(s(g))\backslash Sf)} \Rightarrow Sf}\square L$$

The semantics delivered by the derivation of this example is:

(162) $(Pres\ (\check{}walk\ j))$

The next example involves a quantifier phrase in subject position:

(163) (dwp((7-16))) [**every**+**man**]+**talks** : Sf

Lookup yields the following semantically labelled sequent:

str(dwp('(7-7)'), [b([john]), walks], s(f)).
str(dwp('(7-16)'), [b([every, man]), talks], s(f)).
str(dwp('(7-19)'), [b([the, fish]), walks], s(f)).
str(dwp('(7-32)'), [b([every, man]), b([b([walks, or, talks])])], s(f)).
str(dwp('(7-34)'), [b([b([b([every, man]), walks, or, b([every, man]), talks])])], s(f)).
str(dwp('(7-39)'), [b([b([b([a, woman]), walks, and, b([she]), talks])])], s(f)).
str(dwp('(7-43, 45)'), [b([john]), believes, that, b([a, fish]), walks], s(f)).
str(dwp('(7-48, 49, 52)'), [b([every, man]), believes, that, b([a, fish]), walks], s(f)).
str(dwp('(7-57)'), [b([every, fish, such, that, b([it]), walks]), talks], s(f)).
str(dwp('(7-60, 62)'), [b([john]), seeks, a, unicorn], s(f)).
str(dwp('(7-73)'), [b([john]), is, bill], s(f)).
str(dwp('(7-76)'), [b([john]), is, a, man], s(f)).
str(dwp('(7-83)'), [necessarily, b([john]), walks], s(f)).
str(dwp('(7-86)'), [b([john]), walks, slowly], s(f)).
str(dwp('(7-91)'), [b([john]), tries, to, walk], s(f)).
str(dwp('(7-94)'), [b([john]), tries, to, b([b([catch, a, fish, and, eat, it])])], s(f)).
str(dwp('(7-98)'), [b([john]), finds, a, unicorn], s(f)).
str(dwp('(7-105)'), [b([every, man, such, that, b([he]), loves, a, woman]), loses, her], s(f)).
str(dwp('(7-110)'), [b([john]), walks, in, a, park], s(f)).
str(dwp('(7-116, 118)'), [b([every, man]), doesnt, walk], s(f)).

Figure 13.1: The mini-corpus of the Montague test

(164) $[\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)) : \lambda A\lambda B\forall C[(A\ C) \to (B\ C)], \Box CNs(m) : man],$
$\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda D(Pres\ (\check{}talk\ D)) \ \Rightarrow\ Sf$

The semantic modality of the quantifier is inactive since its semantics is purely logical. Within its modality the type for the quantifier is a functor seeking a count noun to its right; the feature variable g transmits gender from the count noun argument to the value of the functor. The functor yields a generalised quantifier type which will infix at the result of extracting a nominal in a sentence, simulating Montague's rule of term insertion, or quantifying in, S14. The derivation is given below:

$$\dfrac{\overline{CNs(m) \Rightarrow CNs(m)}}{\Box CNs(m) \Rightarrow CNs(m)}\ \Box L$$

$$\dfrac{\dfrac{\overline{Nt(s(m)) \Rightarrow Nt(s(m))}}{Nt(s(m)) \Rightarrow \exists gNt(s(g))}\ \exists R}{[Nt(s(m))] \Rightarrow \langle\rangle\exists gNt(s(g))}\ \langle\rangle R \qquad \overline{Sf \Rightarrow Sf}$$

$$\dfrac{[Nt(s(m))] \Rightarrow \langle\rangle\exists gNt(s(g)) \qquad Sf \Rightarrow Sf}{[Nt(s(m))], \langle\rangle\exists gNt(s(g))\backslash Sf \Rightarrow Sf}\ \backslash L$$

$$\dfrac{[Nt(s(m))], \langle\rangle\exists gNt(s(g))\backslash Sf \Rightarrow Sf}{[Nt(s(m))], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}\ \Box L$$

$$\dfrac{[Nt(s(m))], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}{[1], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf{\uparrow}Nt(s(m))}\ {\uparrow}R$$

$$\dfrac{[1], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf{\uparrow}Nt(s(m)) \qquad \overline{Sf \Rightarrow Sf}}{[(Sf{\uparrow}Nt(s(m))){\downarrow}Sf], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}\ {\downarrow}L$$

$$\dfrac{[(Sf{\uparrow}Nt(s(m))){\downarrow}Sf], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}{[\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}\ \forall L$$

$$\dfrac{\Box CNs(m) \Rightarrow CNs(m) \qquad [\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}{[\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)/CNs(m), \Box CNs(m)], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}\ /L$$

$$\dfrac{[\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)/CNs(m), \Box CNs(m)], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}{[\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \Box CNs(m)], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}\ \forall L$$

$$\dfrac{[\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \Box CNs(m)], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}{[\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \Box CNs(m)], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}\ \blacksquare L$$

This delivers the required semantics:

(165) $\forall C[(\check{}man\ C) \to (Pres\ (\check{}talk\ C))]$

The next example of DWP is:

a: $\blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda A\lambda B\exists C[(A\ C)\wedge(B\ C)]$

and: $\blacksquare\forall f((?\blacksquare Sf\backslash[\,]^{-1}[\,]^{-1}Sf)/\blacksquare Sf) : (\mathbf{\Phi^{n+}}\ 0\ \mathit{and})$

and: $\blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)) : (\mathbf{\Phi^{n+}}\ (s\ 0)\ \mathit{and})$

believes: $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/(CPthat\sqcup\square Sf)) : \hat{}\lambda A\lambda B(Pres\ ((\check{}believe\ A)\ B))$

bill: $\blacksquare Nt(s(m)) : b$

catch: $\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa) : \hat{}\lambda A\lambda B((\check{}catch\ A)\ B)$

doesnt: $\blacksquare\forall g\forall a((Sg\uparrow((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)))\downarrow Sg) : \lambda A\neg(A\ \lambda B\lambda C(B\ C))$

eat: $\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa) : \hat{}\lambda A\lambda B((\check{}eat\ A)\ B)$

every: $\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda A\lambda B\forall C[(A\ C)\rightarrow(B\ C)]$

finds: $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}find\ A)\ B))$

fish: $\square CNs(n) : fish$

he: $\blacksquare[\,]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg)) : \lambda AA$

her: $\blacksquare\forall g\forall a(((\langle\rangle Na\backslash Sg)\uparrow\blacksquare Nt(s(f)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f)))) : \lambda AA$

in: $\square(\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))/\exists aNa) : \hat{}\lambda A\lambda B\lambda C((\check{}in\ A)\ (B\ C))$

is: $\blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I))) : \lambda A\lambda B(Pres\ (A\rightarrow C.[B=C];D.((D\ \lambda E[E=B])\ B)))$

it: $\blacksquare\forall f\forall a(((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n)))) : \lambda AA$

it: $\blacksquare[\,]^{-1}\forall f((\blacksquare Sf|\blacksquare Nt(s(n)))/(\langle\rangle Nt(s(n))\backslash Sf)) : \lambda AA$

john: $\blacksquare Nt(s(m)) : j$

loses: $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}lose\ A)\ B))$

loves: $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}love\ A)\ B))$

man: $\square CNs(m) : man$

necessarily: $\blacksquare(SA/\square SA) : Nec$

or: $\blacksquare\forall f((?\blacksquare Sf\backslash[\,]^{-1}[\,]^{-1}Sf)/\blacksquare Sf) : (\mathbf{\Phi^{n+}}\ 0\ or)$

or: $\blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)) : (\mathbf{\Phi^{n+}}\ (s\ 0)\ or)$

or: $\blacksquare\forall f((?\blacksquare(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf))\backslash[\,]^{-1}[\,]^{-1}(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf)))/\blacksquare(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf))) : (\mathbf{\Phi^{n+}}\ (s\ 0)\ or)$

park: $\square CNs(n) : park$

seeks: $\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\square\forall a\forall f(((Na\backslash Sf)/\exists bNb)\backslash(Na\backslash Sf))) : \hat{}\lambda A\lambda B((\check{}tries\ \hat{}((\check{}A\ \check{}find)\ B))\ B)$

she: $\blacksquare[\,]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(f)))/(\langle\rangle Nt(s(f))\backslash Sg)) : \lambda AA$

slowly: $\square\forall a\forall f(\square(\langle\rangle Na\backslash Sf)\backslash(\langle\rangle\square Na\backslash Sf)) : \hat{}\lambda A\lambda B(\check{}slowly\ \hat{}(\check{}A\ \check{}B))$

such+that: $\blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))) : \lambda A\lambda B\lambda C[(B\ C)\wedge(A\ C)]$

talks: $\square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda A(Pres\ (\check{}talk\ A))$

that: $\blacksquare(CPthat/\square Sf) : \lambda AA$

the: $\blacksquare\forall n(Nt(n)/CNn) : \iota$

to: $\blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))) : \lambda AA$

unicorn: $\square CNs(n) : unicorn$

walks: $\square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda A(Pres\ (\check{}walk\ A))$

woman: $\square CNs(f) : woman$

Figure 13.2: Lexicon for the Montague test

(166) (dwp((7-19))) **[the+fish]+walks** : Sf

Montague analysed the definite article Russelian style as universal quantification with unicity, but this does not reflect its presuppositional character. In our grammar we assume that the presupposition of unicity is somehow otherwise given, and the article maps Hilbert style to a simple nominal with a higher-order iota logical constant. Lexical lookup yields the following semantically labelled sequent:

(167) $[\blacksquare\forall n(Nt(n)/CNn) : \iota, \Box CNs(n) : \mathit{fish}], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda A(\mathit{Pres}\ (\check{}\mathit{walk}\ A)) \Rightarrow Sf$

This has the derivation:

$$\boxed{CNs(n)} \Rightarrow CNs(n)$$
$$\boxed{\Box CNs(n)} \Rightarrow CNs(n) \quad \Box L \qquad \boxed{Nt(s(n))} \Rightarrow Nt(s(n))$$
$$\boxed{Nt(s(n))/CNs(n)}, \Box CNs(n) \Rightarrow Nt(s(n)) \quad /L$$
$$\boxed{\forall n(Nt(n)/CNn)}, \Box CNs(n) \Rightarrow Nt(s(n)) \quad \forall L$$
$$\boxed{\blacksquare\forall n(Nt(n)/CNn)}, \Box CNs(n) \Rightarrow Nt(s(n)) \quad \blacksquare L$$
$$\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow \boxed{\exists gNt(s(g))} \quad \exists R$$
$$[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n)] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g))} \quad \langle\rangle R \qquad \boxed{Sf} \Rightarrow Sf$$
$$[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n)], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} \Rightarrow Sf \quad \backslash L$$
$$[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n)], \boxed{\Box(\langle\rangle\exists gNt(s(g))\backslash Sf)} \Rightarrow Sf \quad \Box L$$

The derivation delivers semantics:

(168) $(\mathit{Pres}\ (\check{}\mathit{walk}\ (\iota\ \check{}\mathit{fish})))$

The next example involves subject quantification and verb phrase coordination:

(169) (dwp((7-32))) **[every+man]+[[walks+or+talks]]** : Sf

Lexical lookup inserting the disjunctive verb phrase coordinator yields the semantically labelled sequent:

(170) $[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda A\lambda B\forall C[(A\ C) \rightarrow (B\ C)], \Box CNs(m) : \mathit{man}],$
$[[\Box(\langle\rangle\exists gNt(s(g))\backslash\ Sf) : \hat{}\lambda D(\mathit{Pres}\ (\check{}\mathit{walk}\ D)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/$
$\blacksquare(\langle\rangle Na\backslash Sf)) : (\mathbf{\Phi^{n+}}\ (s\ 0)\ or), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda E(\mathit{Pres}\ (\check{}\mathit{talk}\ E))]] \Rightarrow Sf$

The combinator lexical semantics $(\mathbf{\Phi^{n+}}\ (s\ 0)\ or)$ of the disjunctive coordinator is such that:

$$((((\Phi^{n+}\ (s\ 0)\ or)\ x)\ [y])\ z) = (((\mathbf{\Phi^{n+}}\ 0\ or)\ (x\ z))\ (\alpha^{+}\ [y]\ z)) = (((\mathbf{\Phi^{n+}}\ 0\ or)\ (x\ z))\ [(y\ z)]) = [(y\ z) \vee (x\ z)]$$

See Chapter 14. Syntactically, the value of the coordinator type projects double brackets, making the coordinate structure a string island; the disjuncts are semantically inactive modal domains, making the coordinate structure a scope island. The derivation is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}}{Nt(s(m)) \Rightarrow \exists gNt(s(g))}\exists R}{[Nt(s(m))] \Rightarrow \langle\rangle\exists gNt(s(g))}\langle\rangle R \quad \dfrac{}{Sf \Rightarrow Sf}}{[Nt(s(m))], \langle\rangle\exists gNt(s(g))\backslash Sf \Rightarrow Sf}\backslash L}{[Nt(s(m))], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}\Box L}{\langle\rangle Nt(s(m)), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow Sf}\langle\rangle L}{\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}\backslash R}{\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow \blacksquare(\langle\rangle Nt(s(m))\backslash Sf)}\blacksquare R$$

$$\dfrac{\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow \blacksquare(\langle\rangle Nt(s(m))\backslash Sf)}{\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow ?\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)}?R$$

(the premise of $?R$ is derived exactly as above, by $\exists R$, $\langle\rangle R$, $\backslash L$, $\Box L$, $\langle\rangle L$, $\backslash R$, $\blacksquare R$)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}}{[Nt(s(m))] \Rightarrow \langle\rangle Nt(s(m))}\langle\rangle R \quad \dfrac{}{Sf \Rightarrow Sf}}{[Nt(s(m))], \langle\rangle Nt(s(m))\backslash Sf \Rightarrow Sf}\backslash L}{[Nt(s(m))], [[]^{-1}(\langle\rangle Nt(s(m))\backslash Sf)] \Rightarrow Sf}[]^{-1}L}{[Nt(s(m))], [[[]^{-1}[]^{-1}(\langle\rangle Nt(s(m))\backslash Sf)]] \Rightarrow Sf}[]^{-1}L$$

$$\dfrac{\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow ?\blacksquare(\langle\rangle Nt(s(m))\backslash Sf) \quad [Nt(s(m))], [[[]^{-1}[]^{-1}(\langle\rangle Nt(s(m))\backslash Sf)]] \Rightarrow Sf}{[Nt(s(m))], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), ?\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Nt(s(m))\backslash Sf)]] \Rightarrow Sf}\backslash L$$

$$\dfrac{\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) \Rightarrow \blacksquare(\langle\rangle Nt(s(m))\backslash Sf) \quad [Nt(s(m))], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), ?\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Nt(s(m))\backslash Sf)]] \Rightarrow Sf}{[Nt(s(m))], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), (?\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Nt(s(m))\backslash Sf))/\blacksquare(\langle\rangle Nt(s(m))\backslash Sf), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf}/L$$

$$\dfrac{[Nt(s(m))], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), (?\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Nt(s(m))\backslash Sf))/\blacksquare(\langle\rangle Nt(s(m))\backslash Sf), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf}{[Nt(s(m))], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), \forall f((?\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Nt(s(m))\backslash Sf))/\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf}\forall L$$

$$\dfrac{\dfrac{\dfrac{[Nt(s(m))], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), \forall f((?\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Nt(s(m))\backslash Sf))/\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf}{[Nt(s(m))], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), \forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf}\forall L}{[Nt(s(m))], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf}\blacksquare L}{[1], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf\uparrow Nt(s(m))}\uparrow R$$

$$\dfrac{\dfrac{[1], [[\ldots]] \Rightarrow Sf\uparrow Nt(s(m)) \quad \dfrac{}{Sf \Rightarrow Sf}}{[(Sf\uparrow Nt(s(m)))\downarrow Sf], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf}\downarrow L}{[\forall f((Sf\uparrow Nt(s(m)))\downarrow Sf)], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf}\forall L$$

$$\dfrac{\dfrac{\dfrac{}{CNs(m) \Rightarrow CNs(m)}}{\Box CNs(m) \Rightarrow CNs(m)}\Box L \quad [\forall f((Sf\uparrow Nt(s(m)))\downarrow Sf)], [[\ldots]] \Rightarrow Sf}{[\forall f((Sf\uparrow Nt(s(m)))\downarrow Sf)/CNs(m), \Box CNs(m)], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf}/L$$

$$\dfrac{\dfrac{[\forall f((Sf\uparrow Nt(s(m)))\downarrow Sf)/CNs(m), \Box CNs(m)], [[\ldots]] \Rightarrow Sf}{[\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)), \Box CNs(m)], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf}\forall L}{[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)), \Box CNs(m)], [[\Box(\langle\rangle\exists gNt(s(g))\backslash Sf), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box(\langle\rangle\exists gNt(s(g))\backslash Sf)]] \Rightarrow Sf}\blacksquare L$$

This delivers semantics:

(171) $\forall C[(\check{~}man\ C) \rightarrow [(Pres\ (\check{~}walk\ C)) \vee (Pres\ (\check{~}talk\ C))]]$

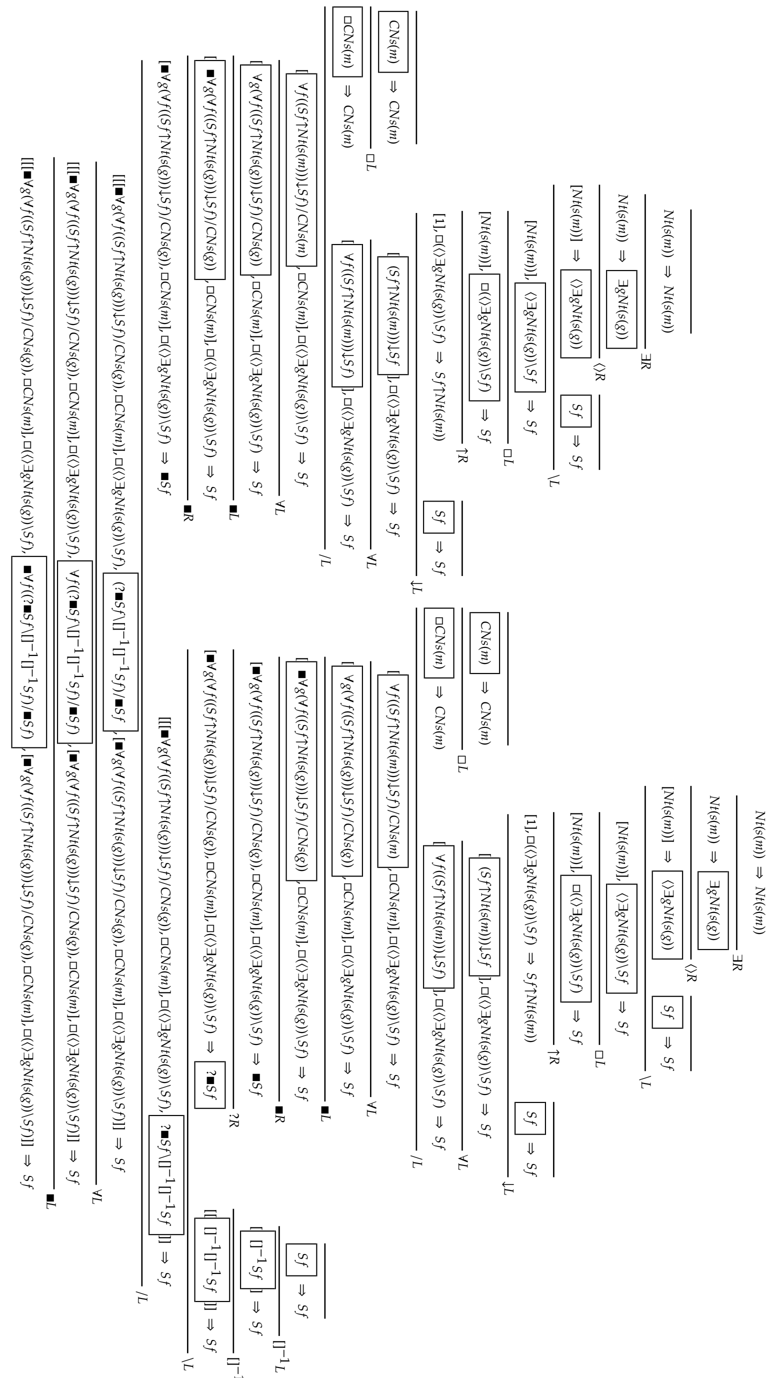
The next example is:

(172) (dwp((7-34))) [[[**every**+**man**]+**walks**+**or**+[**every**+**man**]+**talks**]] : Sf

This involves sentential coordination:

(173) $[[[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda A\lambda B\forall C[(A\ C) \rightarrow (B\ C)], \Box CNs(m) : man],$
$\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{~}\lambda D(Pres\ (\check{~}walk\ D)), \blacksquare\forall f((?\blacksquare Sf\backslash[]^{-1}[]^{-1}Sf)/\blacksquare Sf) : (\mathbf{\Phi^{n+}}\ 0\ or),$
$[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda E\lambda F\forall G[(E\ G) \rightarrow (F\ G)], \Box CNs(m) : man],$
$\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{~}\lambda H(Pres\ (\check{~}talk\ H))]] \Rightarrow Sf$

Because the disjuncts are (semantically inactive) modal domains the quantifier phrases can only take scope within their disjuncts. The derivation is as follows:



The semantics assigned is:

(174) $[\forall H[(\check{}man\ H) \rightarrow (Pres\ (\check{}walk\ H))] \vee \forall C[(\check{}man\ C) \rightarrow (Pres\ (\check{}talk\ C))]]$

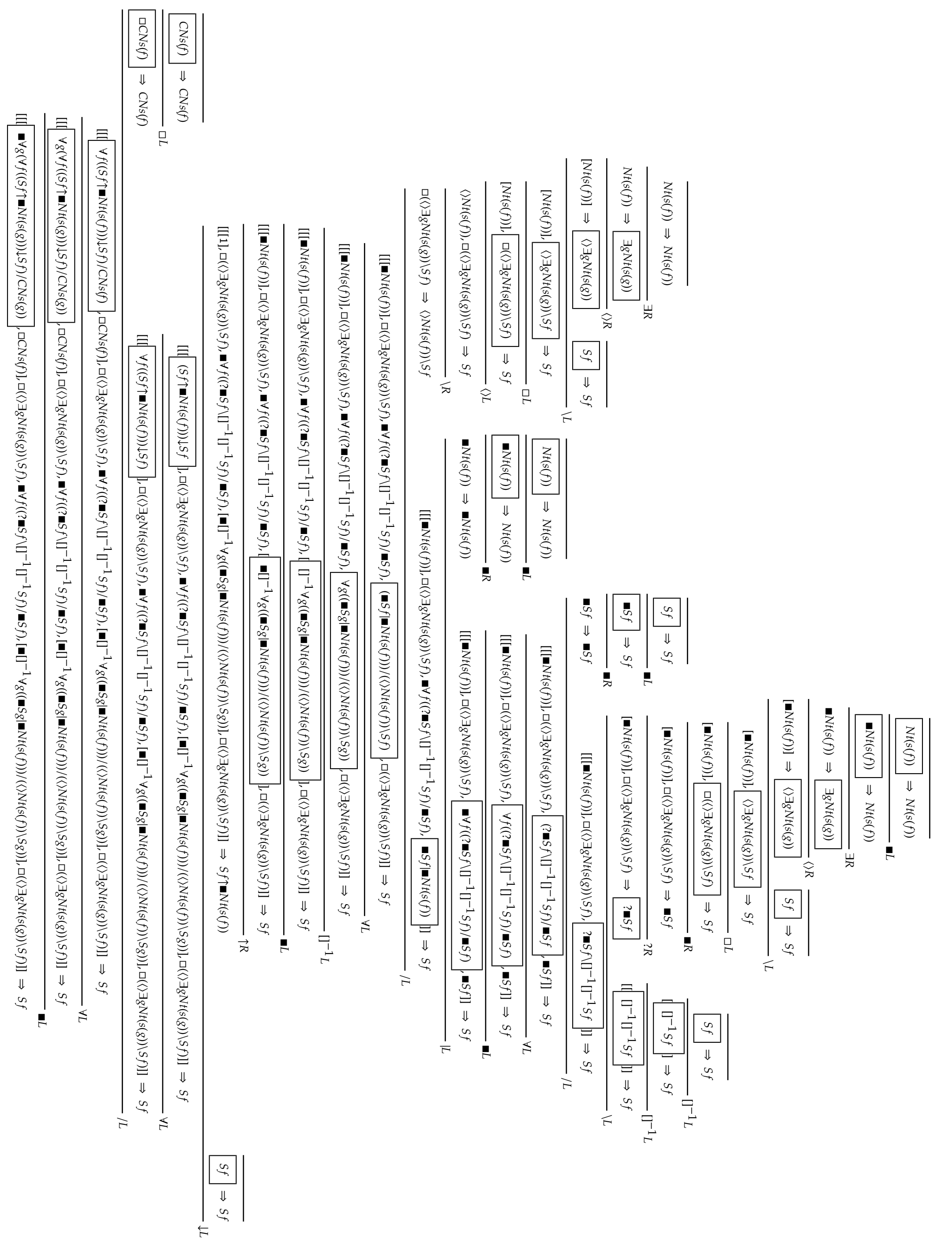
The next example involves anaphora and sentential conjunction:

(175) (dwp((7-39))) [[[**a**+**woman**]+**walks**+**and**+[**she**]+**talks**]] : Sf

Our lexical type assignment to the nominative pronoun ensures that it combines to the right with a verb phrase, i.e. that it occupies subject position:

(176) $[[[\blacksquare\forall g(\forall f((Sf{\uparrow}\blacksquare Nt(s(g))){\downarrow}Sf)/CNs(g)) : \lambda A\lambda B\exists C[(A\ C) \wedge (B\ C)], \Box CNs(f) : woman],$
$\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : {}^\wedge\lambda D(Pres\ ({}^\vee walk\ D)), \blacksquare\forall f((?\blacksquare Sf\backslash[\,]^{-1}[\,]^{-1}Sf)/\blacksquare Sf) : (\mathbf{\Phi^{n+}}\ 0\ and),$
$[\blacksquare[\,]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(f)))/(\langle\rangle Nt(s(f))\backslash Sg)) : \lambda EE], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : {}^\wedge\lambda F(Pres\ ({}^\vee talk\ F))]]$
$\Rightarrow\ Sf$

The limited contraction for anaphora type constructor is used to create anaphoric dependency. There is the following derivation where the treatment of quantification and anaphora (and coordination) interact in such a way that the pronoun is bound by the quantificational noun phrase antecedent.



This assigns the correct semantics:

(177) $\exists C[(\check{}woman\ C) \wedge [(Pres\ (\check{}walk\ C)) \wedge (Pres\ (\check{}talk\ C))]]$

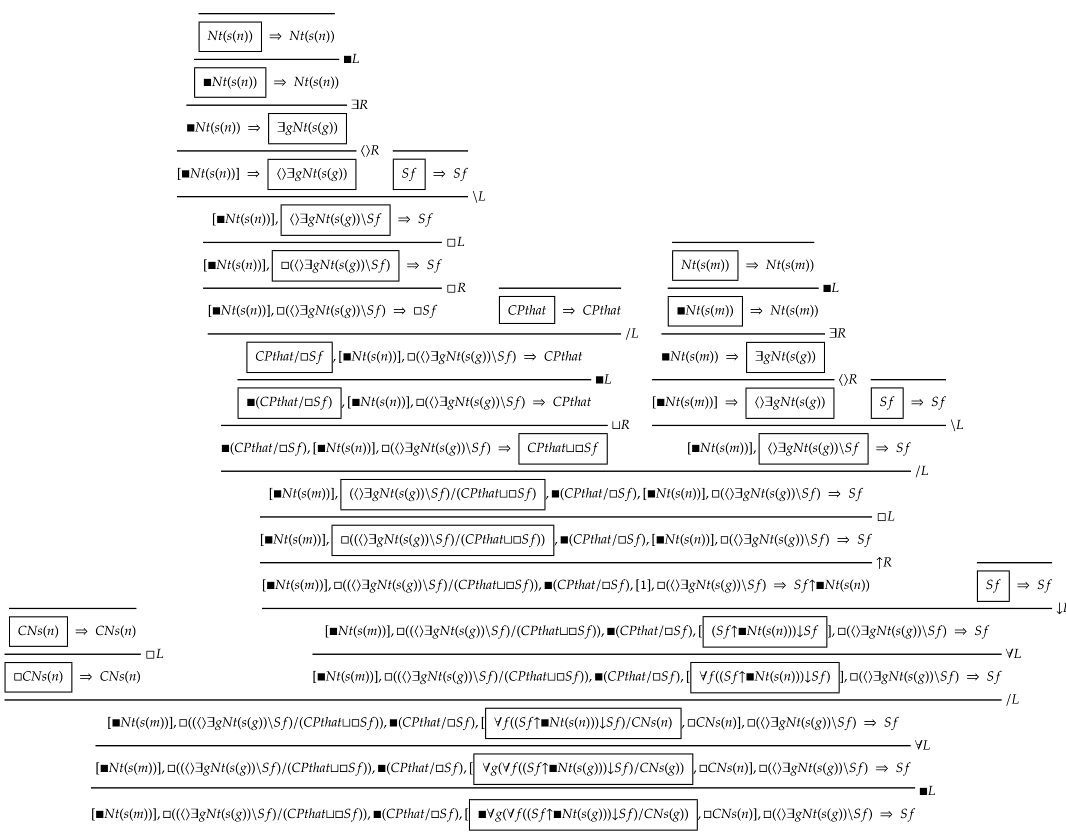
The next analyses of DWP are for the de re or specific versus de dicto or non-specific ambiguity of:

(178) (dwp((7-43, 45))) **[john]+believes+that+[a+fish]+walks** : Sf

On Montague's account the indefinite quantifier phrase can quantity in at the matrix level yielding the de re reading in which the propositional attitude verb is within the scope of the existential quantification (John's belief is directed towards a specific fish) or it can quantify in at the level of the subordinate clause in which case the existential quantification is within the scope of the propositional attitude verb (John has no specific fish in mind). Our grammar conserves this account. Lexical lookup yields the semantically labelled sequent:

(179) $[\blacksquare Nt(s(m)) : j], \square((\langle\rangle\exists g Nt(s(g))\backslash Sf)/(CPthat \sqcup \square Sf)) : \hat{}\lambda A\lambda B(Pres\ ((\check{}believe\ A)\ B)), \blacksquare(CPthat/\square Sf) : \lambda CC, [\blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda D\lambda E\exists F[(D\ F) \wedge (E\ F)], \square CNs(n) : fish], \square(\langle\rangle\exists g Nt(s(g))\backslash Sf) : \hat{}\lambda G(Pres\ (\check{}walk\ G)) \Rightarrow Sf$

In Montague grammar all term phrases have the same semantic type, and this requirement necessitates the type raising of proper names so that they have the same type as quantifier phrases; a meaning postulate is then required to make them rigid designators (i.e. denote the same individual in all worlds). Note how in our type logical grammar proper names are assigned their lower type, and our semantically inactive modality lexical semantics encodes that they are intensional, but rigid designators.



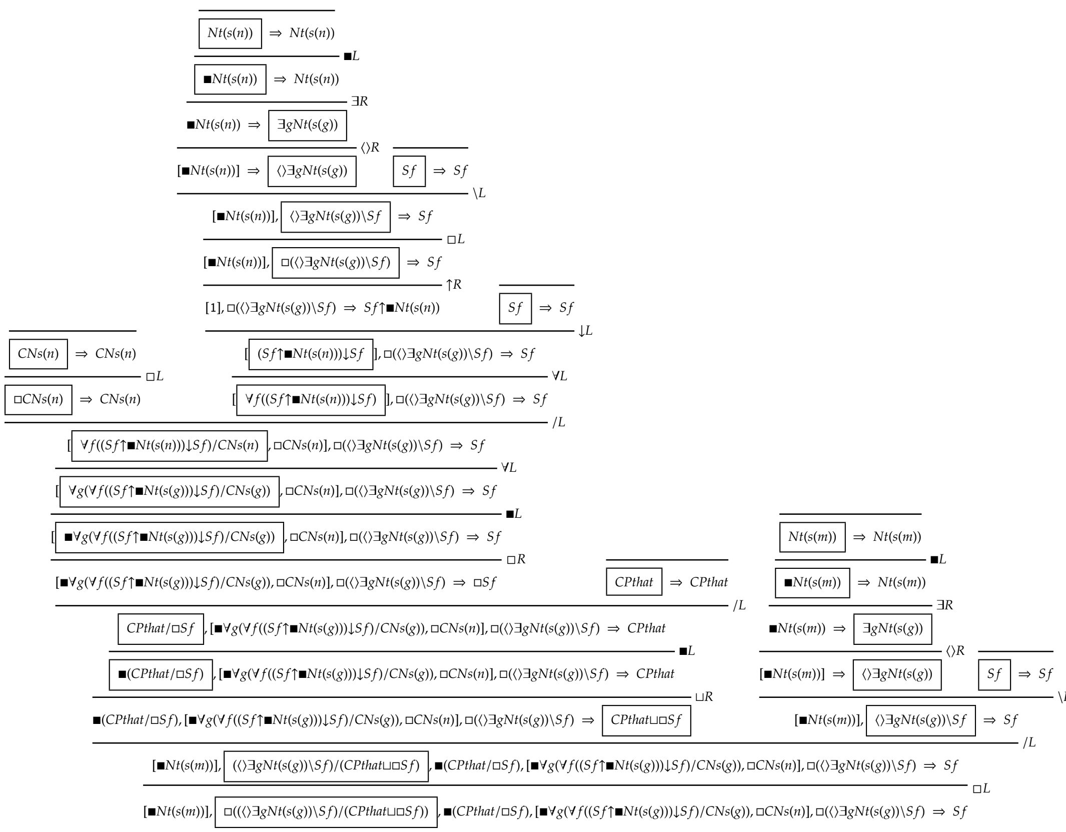
This de re derivation delivers semantics:

(180) $\exists C[(\check{}fish\ C) \wedge (Pres\ ((\check{}believe\ \hat{}(Pres\ (\check{}walk\ C)))\ j))]$

This has existential commitment with respect to fish: it cannot be true without some fish existing in the actual world.

The modalization of the complement sentence corresponds to translation into an intensional formula, i.e. a proposition, as argument to the propositional attitude verb. The $\Box R$ inference in the de re derivation depends on the hypothetical subtype introduced by the indefinite article being modal. Observe that by contrast in our type logical grammar the corresponding subtype of *every* is not modal, capturing that its quantification is limited to local scope (something Montague did not capture).

The de dicto derivation is:



This delivers semantics:

(181) $(Pres\ ((\check{}believe\ \hat{}\exists F[(\check{}fish\ F) \wedge (Pres\ (\check{}walk\ F))])\ j))$

This does not have existential commitment with respect to fish: it can be true without any fish existing in the actual world. The next example combines the previous ambiguity with alternative quantifier scopings at the matrix level resulting in a total of three readings:

(182) (dwp((7-48, 49, 52))) **[every+man]+believes+that+[a+fish]+walks** : Sf

Lexical lookup yields the semantically labelled sequent:

(183) $[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda A\lambda B\forall C[(A\ C) \rightarrow (B\ C)], \Box CNs(m) : man],$
$\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/(CPthat\sqcup\Box Sf)) : \hat{}\lambda D\lambda E(Pres\ ((\check{}believe\ D)\ E)), \blacksquare(CPthat/\Box Sf) : \lambda FF,$
$[\blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda G\lambda H\exists I[(G\ I) \wedge (H\ I)], \Box CNs(n) : fish],$
$\Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda J(Pres\ (\check{}walk\ J)) \Rightarrow Sf$

A first derivation, in which the existential quantifies in at the widest level, is thus:

This delivers semantics:

(184) $\exists C[(\check{}\textit{fish}\ C) \wedge \forall G[(\check{}\textit{man}\ G) \rightarrow (\textit{Pres}\ ((\check{}\textit{believe}\ \hat{}(\textit{Pres}\ (\check{}\textit{walk}\ C)))\ G))]]$

A second derivation, in which the existential quantifies in outside of the propositional attitude verb but within the universal, is thus:

This delivers semantics:

(185) $\forall C[(\check{}\textit{man}\ C) \rightarrow \exists G[(\check{}\textit{fish}\ G) \wedge (\textit{Pres}\ ((\check{}\textit{believe}\ \hat{}(\textit{Pres}\ (\check{}\textit{walk}\ G)))\ C))]]$

The third derivation, in which the existential takes narrowest scope, is thus:

This delivers semantics:

(186) $\forall C[(\check{}man\ C) \to (Pres\ ((\check{}believe\ \hat{}\exists J[(\check{}fish\ J) \wedge (Pres\ (\check{}walk\ J))]])\ C))]$

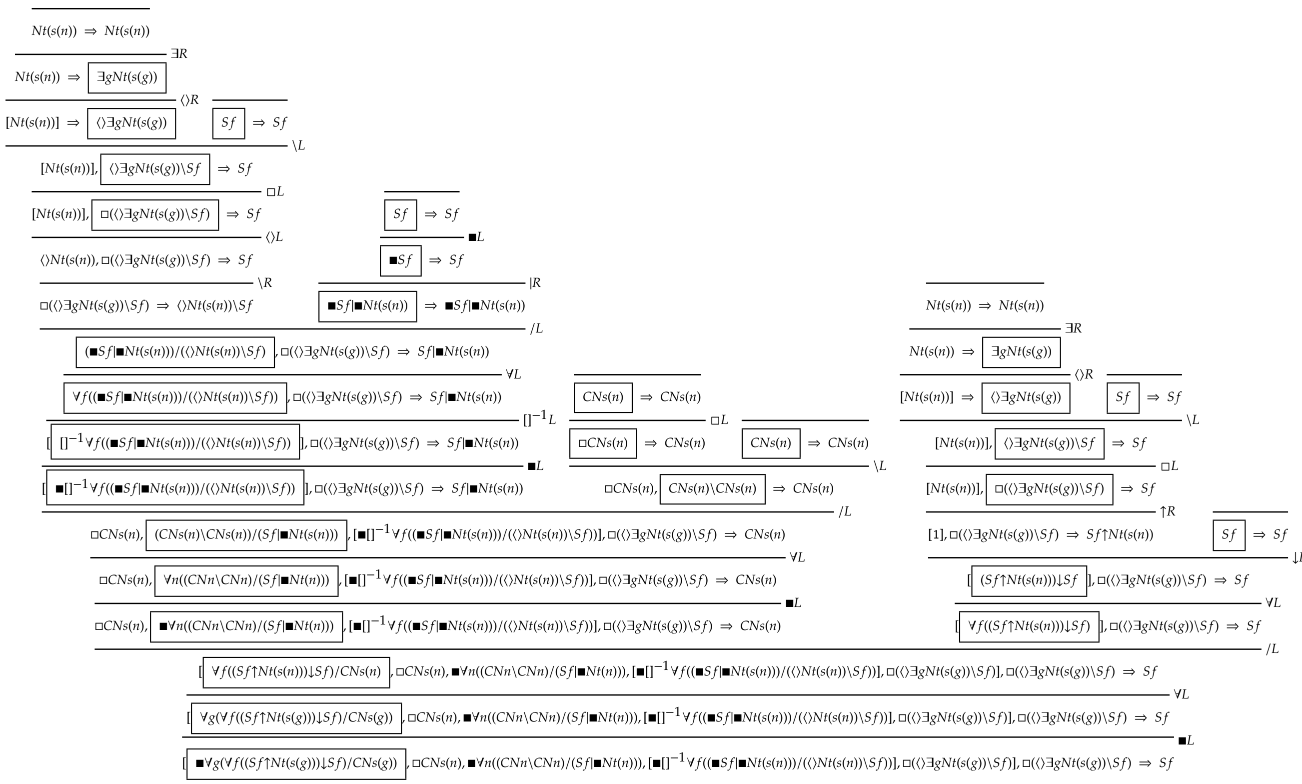
The next example involves the *such that* relativization of Montague's fragment:

(187) (dwp((7-57))) [**every**+**fish**+**such**+**that**+[**it**]+**walks**]+**talks** : Sf

This is relativization in which the clause subordinate to *such that* must contain a pronoun bound by the head which is modified by the relative clause. Semantically, the meaning of the body of the relative clause abstracted over the pronoun meaning restricts the head modified. The relative pronoun *such that* is categorized accordingly using the limited contraction for anaphora type constructor. Lexical lookup produces the following semantically labelled sequent:

(188) $[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda A\lambda B\forall C[(A\ C) \to (B\ C)], \square CNs(n) : fish,$
$\blacksquare\forall n((CNn\backslash CNn)/\ (Sf|\blacksquare Nt(n))) : \lambda D\lambda E\lambda F[(E\ F) \wedge (D\ F)], [\blacksquare[]^{-1}\forall f((\blacksquare Sf|\blacksquare Nt(s(n)))/$
$(\langle\rangle Nt(s(n))\backslash Sf)) : \lambda GG], \square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda H(Pres\ (\check{}walk\ H))], \square(\langle\rangle\exists gNt(s(g))\backslash Sf) :$
$\hat{}\lambda I(Pres\ (\check{}talk\ I)) \Rightarrow Sf$

This sequent has the derivation:



This delivers semantics:

(189) $\forall C[[(\check{}\mathit{fish}\ C) \wedge (\mathit{Pres}\ (\check{}\mathit{walk}\ C))] \rightarrow (\mathit{Pres}\ (\check{}\mathit{talk}\ C))]$

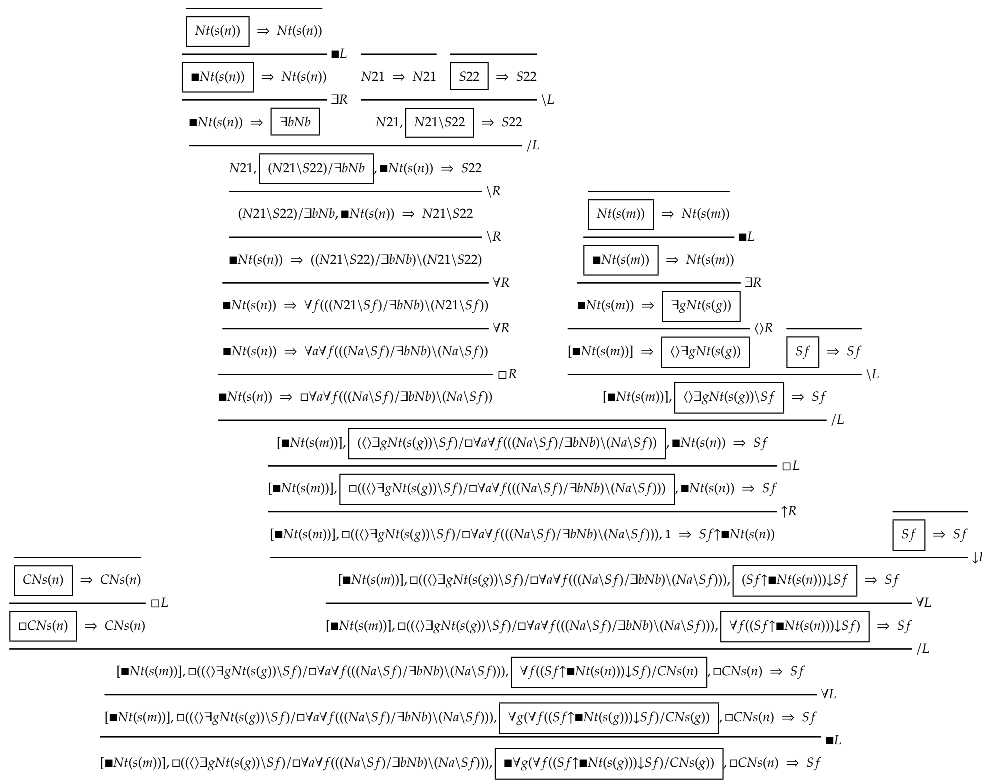
The next example of DWP involves *seek* which is an intensional object verb synonymous with *try to find*:

(190) (dwp((7-60, 62))) **[john]+seeks+a+unicorn** : Sf

The sentence has a specific reading in which there is a particular unicorn which John is trying to find, and a non-specific reading in which John is just trying to bring it about that he finds some, any, unicorn. The former reading has existential commitment with respect to unicorns, the latter does not. Lexical lookup gives the following semantically labelled sequent:

(191) $[\blacksquare Nt(s(m)) : j], \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\square\forall a\forall f(((Na\backslash Sf)/\exists bNb)\backslash(Na\backslash Sf))) :$
$\hat{}\lambda A\lambda B((\check{}\mathit{tries}\ \hat{}((\check{}A\ \check{}\mathit{find})\ B))\ B), \blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda C\lambda D\exists E[(C\ E)\wedge(D\ E)],$
$\square CNs(n) : \mathit{unicorn}\ \Rightarrow\ Sf$

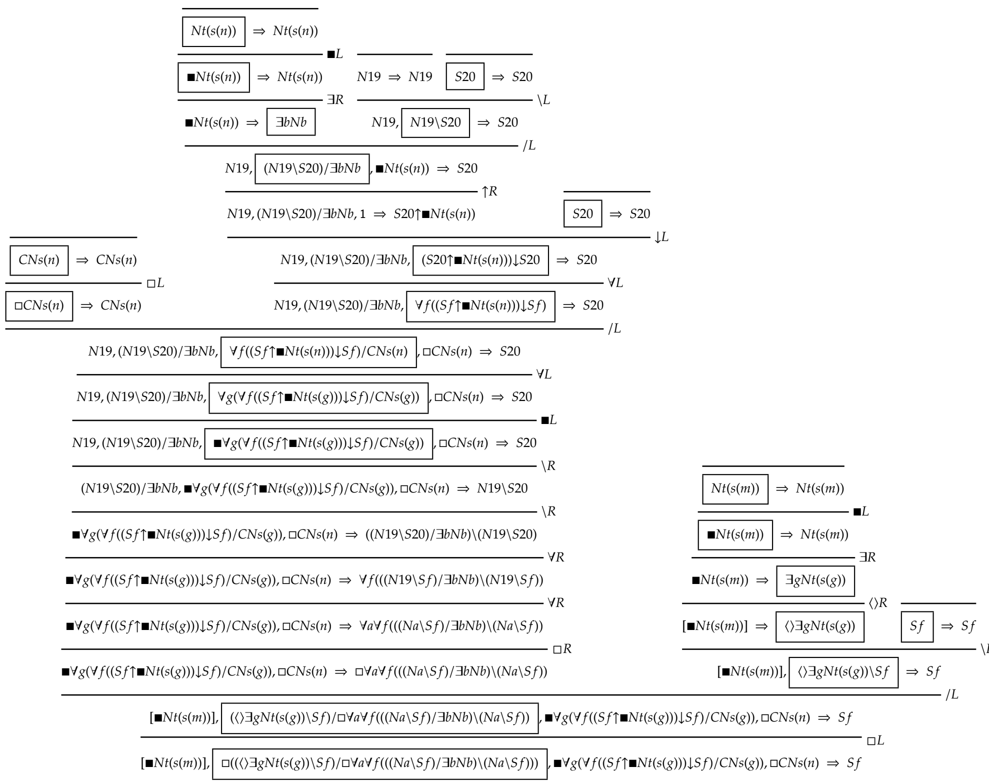
For the specific reading there is the derivation:



This delivers the semantics with existential commitment:

(192) $\exists C[(\check{}unicorn\ C) \wedge ((\check{}tries\ \hat{}((\check{}find\ C)\ j))\ j)]$

For the non-specific reading there is the derivation:



This delivers the semantics without existential commitment:

(193) $((\check{}tries\ \hat{}\exists G[(\check{}unicorn\ G) \wedge ((\check{}find\ G)\ j)])\ j)$

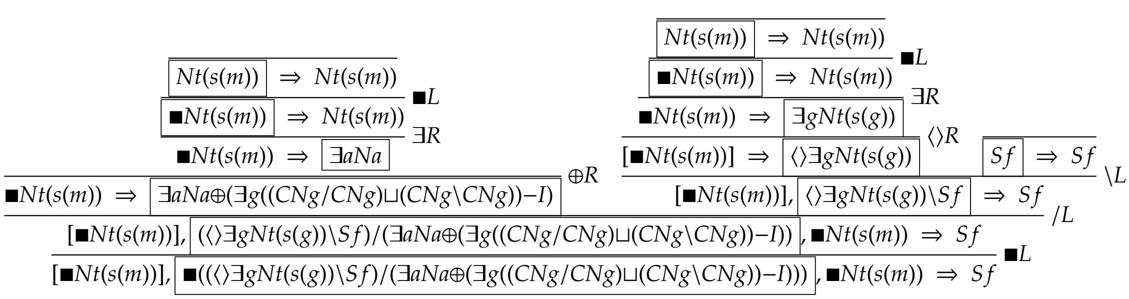
The next examples involve the copula of identity. First, and minimally:

(194) (dwp((7-73))) [**john**]+**is**+**bill** : Sf

For this there is the semantically labelled sequent:

(195) $[\blacksquare Nt(s(m)) : j], \blacksquare(((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I))) :$
$\lambda A\lambda B(Pres\ (A{-}{>}C.[B = C]; D.((D\ \lambda E[E = B])\ B))), \blacksquare Nt(s(m)) : b\ \Rightarrow\ Sf$

This has the derivation:



It delivers semantics:

(196) $(Pres\ [j = b])$

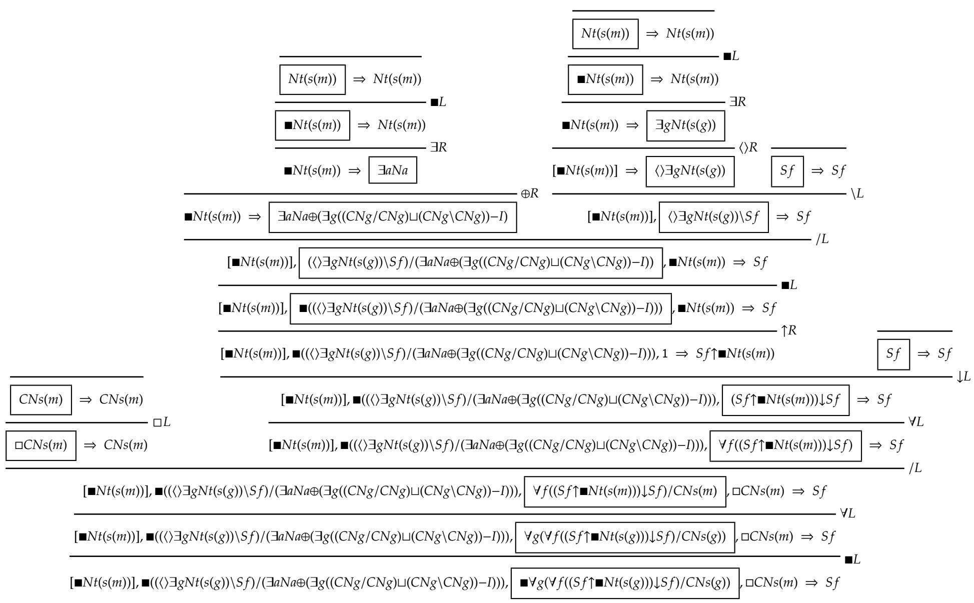
Second, and more subtly:

(197) (dwp((7-76))) **[john]+is+a+man** : Sf

Lexical lookup (with the same lexical entry for the copula) yields the semantically annotated sequent:

(198) $[\blacksquare Nt(s(m)) : j], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa \oplus (\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I))) : \lambda A\lambda B(Pres\ (A->C.[B = C]; D.((D\ \lambda E[E = B])\ B))), \blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda F\lambda G\exists H[(F\ H)\wedge (G\ H)], \square CNs(m) : man \Rightarrow Sf$

This has the derivation:



The derivation delivers the semantics:

(199) $\exists C[(\check{}man\ C) \wedge (Pres\ [j = C])]$

Leaving aside tense, this is logically equivalent to ($\check{}man\ j$), as required. This correct interaction of the copula of identity with an indefinitely quantified complement is a nice prediction of Montague grammar. It is preserved in type logical grammar, and simplified by the lower type of the copula nominal complement.

The next example involves an intensional adsentential modifier:

(200) (dwp((7-83))) **necessarily+[john]+walks** : Sf

Lexical lookup yields the following semantically labelled sequent:

(201) $\blacksquare(SA/\square SA) : Nec, [\blacksquare Nt(s(m)) : j], \square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda B(Pres\ (\check{}walk\ B)) \Rightarrow Sf$

This has the derivation:

$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$\blacksquare Nt(s(m)) \Rightarrow Nt(s(m)) \quad \blacksquare L$$
$$\blacksquare Nt(s(m)) \Rightarrow \exists g Nt(s(g)) \quad \exists R$$
$$[\blacksquare Nt(s(m))] \Rightarrow \langle\rangle \exists g Nt(s(g)) \quad \langle\rangle R \qquad Sf \Rightarrow Sf$$
$$[\blacksquare Nt(s(m))], \langle\rangle \exists g Nt(s(g)) \backslash Sf \Rightarrow Sf \quad \backslash L$$
$$[\blacksquare Nt(s(m))], \Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf) \Rightarrow Sf \quad \Box L$$
$$[\blacksquare Nt(s(m))], \Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf) \Rightarrow \Box Sf \quad \Box R \qquad Sf \Rightarrow Sf$$
$$Sf/\Box Sf, [\blacksquare Nt(s(m))], \Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf) \Rightarrow Sf \quad /L$$
$$\blacksquare(Sf/\Box Sf), [\blacksquare Nt(s(m))], \Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf) \Rightarrow Sf \quad \blacksquare L$$

The derivation delivers semantics:

(202) $(Nec\ \hat{}(Pres\ (\check{}walk\ j)))$

The following example involves an adverb:

(203) (dwp((7-86))) **[john]+walks+slowly** : Sf

This is also assumed to create an intensional context.[1] Lexical lookup yields:

(204) $[\blacksquare Nt(s(m)) : j], \Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf) : \hat{}\lambda A(Pres\ (\check{}walk\ A)), \Box\forall a \forall f(\Box(\langle\rangle Na \backslash Sf) \backslash (\langle\rangle \Box Na \backslash Sf)) : \hat{}\lambda B \lambda C(\check{}slowly\ \hat{}(\check{}B\ \check{}C)) \Rightarrow Sf$

This has the derivation:

$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$Nt(s(m)) \Rightarrow \exists g Nt(s(g)) \quad \exists R$$
$$[Nt(s(m))] \Rightarrow \langle\rangle \exists g Nt(s(g)) \quad \langle\rangle R \qquad Sf \Rightarrow Sf$$
$$[Nt(s(m))], \langle\rangle \exists g Nt(s(g)) \backslash Sf \Rightarrow Sf \quad \backslash L$$
$$[Nt(s(m))], \Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf) \Rightarrow Sf \quad \Box L$$
$$\langle\rangle Nt(s(m)), \Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf) \Rightarrow Sf \quad \langle\rangle L$$
$$\Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf) \Rightarrow \langle\rangle Nt(s(m)) \backslash Sf \quad \backslash R$$
$$\Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf) \Rightarrow \Box(\langle\rangle Nt(s(m)) \backslash Sf) \quad \Box R$$

$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$\blacksquare Nt(s(m)) \Rightarrow Nt(s(m)) \quad \blacksquare L$$
$$\blacksquare Nt(s(m)) \Rightarrow \Box Nt(s(m)) \quad \Box R$$
$$[\blacksquare Nt(s(m))] \Rightarrow \langle\rangle \Box Nt(s(m)) \quad \langle\rangle R \qquad Sf \Rightarrow Sf$$
$$[\blacksquare Nt(s(m))], \langle\rangle \Box Nt(s(m)) \backslash Sf \Rightarrow Sf \quad \backslash L$$

$$[\blacksquare Nt(s(m))], \Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf), \Box(\langle\rangle Nt(s(m)) \backslash Sf) \backslash (\langle\rangle \Box Nt(s(m)) \backslash Sf) \Rightarrow Sf \quad \backslash L$$
$$[\blacksquare Nt(s(m))], \Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf), \forall f(\Box(\langle\rangle Nt(s(m)) \backslash Sf) \backslash (\langle\rangle \Box Nt(s(m)) \backslash Sf)) \Rightarrow Sf \quad \forall L$$
$$[\blacksquare Nt(s(m))], \Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf), \forall a \forall f(\Box(\langle\rangle Na \backslash Sf) \backslash (\langle\rangle \Box Na \backslash Sf)) \Rightarrow Sf \quad \forall L$$
$$[\blacksquare Nt(s(m))], \Box(\langle\rangle \exists g Nt(s(g)) \backslash Sf), \Box\forall a \forall f(\Box(\langle\rangle Na \backslash Sf) \backslash (\langle\rangle \Box Na \backslash Sf)) \Rightarrow Sf \quad \Box L$$

It delivers semantics:

(205) $(\check{}slowly\ \hat{}(Pres\ (\check{}walk\ j)))$

The next example involves an equi control verb:

[1] The subject argument of 'slowly' is intensionalized in order for it to be modally closed to undergo β-conversion in the semantics. But then this does not give the usual verb phrase type for 'walks slowly'; the ramifications of this remain to be explored.

(206) (dwp((7-91))) **[john]+tries+to+walk** : Sf

We lexically analyse the equi semantics a a relation of trying between the subject and a proposition of which the subject is agent (something Montague did not do). Lexical lookup yields:

(207) $[\blacksquare Nt(s(m)) : j], \Box(((\langle\rangle \exists gNt(s(g))\backslash Sf)/\Box(\langle\rangle \exists gNt(s(g))\backslash Si)) : \hat{}\lambda A\lambda B((\check{}tries\ \hat{}(\check{}A\ B))\ B),$
$\blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))) : \lambda CC, \Box(\langle\rangle \exists aNa\backslash Sb) : \hat{}\lambda D(\check{}walk\ D) \Rightarrow Sf$

This has the derivation:

It delivers the semantics:

(208) $((\check{}tries\ \hat{}(\check{}walk\ j))\ j)$

I.e. John tries to bring it about that he (John) walks.

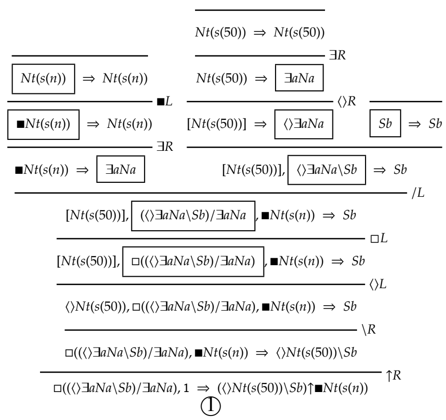
The next example involves control, quantification, coordination and anaphora:

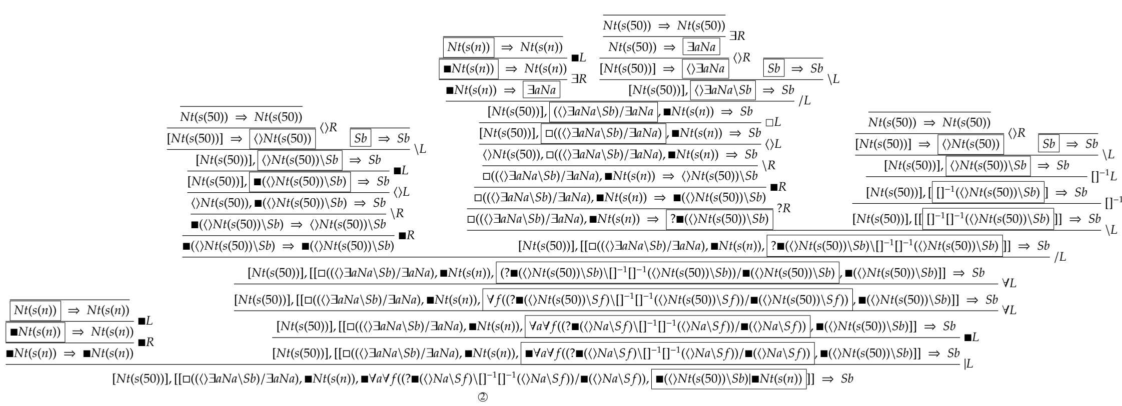
(209) (dwp((7-94))) **[john]+tries+to+[[catch+a+fish+and+eat+it]]** : Sf

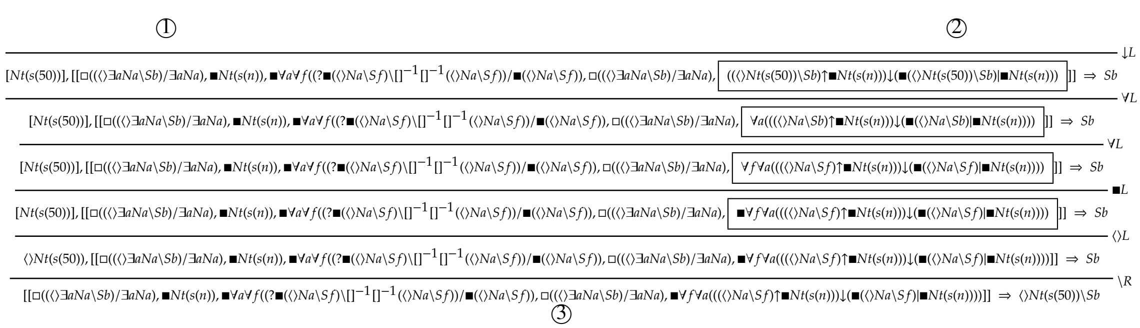
The sentence is ambiguous as to whether *a fish* is wide scope (with existential commitment) or narrow scope (without existential commitment) with respect to *tries*, but in both cases it must be the antecedent of *it*. Lexical lookup inserting the verb phrase coordinator and the pronoun assignment yields the semantically labelled sequent:

(210) $[\blacksquare Nt(s(m)) : j], \Box(((\langle\rangle \exists gNt(s(g))\backslash Sf)/\Box(\langle\rangle \exists gNt(s(g))\backslash Si)) : \hat{}\lambda A\lambda B((\check{}tries\ \hat{}(\check{}A\ B))\ B),$
$\blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))) : \lambda CC, [[\Box((\langle\rangle \exists aNa\backslash Sb)/\exists aNa) :$
$\hat{}\lambda D\lambda E((\check{}catch\ D)\ E), \blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda F\lambda G\exists H[(F\ H)\wedge(G\ H)],$
$\Box CNs(n) : fish, \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)) : (\mathbf{\Phi^{n+}}\ (s\ 0)\ and),$
$\Box((\langle\rangle \exists aNa\backslash Sb)/\exists aNa) : \hat{}\lambda I\lambda J((\check{}eat\ I)\ J), \blacksquare\forall f\forall a(((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow$
$(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n)))) : \lambda KK]] \Rightarrow Sf$

The wide scope existential derivation is:







$$Nt(s(50)) \Rightarrow Nt(s(50))$$

$$[Nt(s(50))] \Rightarrow \boxed{\langle\rangle Nt(s(50))} \quad \langle\rangle R \qquad \boxed{Si} \Rightarrow Si$$

$$[Nt(s(50))], \boxed{\langle\rangle Nt(s(50))\backslash Si} \Rightarrow Si \quad \backslash L$$

③

$$[Nt(s(50))], \boxed{(\langle\rangle Nt(s(50))\backslash Si)/(\langle\rangle Nt(s(50))\backslash Sb)}, [[\square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare\forall f\forall a((((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n)))))]] \Rightarrow Si \quad /L$$

$$[Nt(s(50))], \boxed{\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))}, [[\square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare\forall f\forall a((((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n)))))]] \Rightarrow Si \quad \forall L$$

$$[Nt(s(50))], \boxed{(PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))}, [[\square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare\forall f\forall a((((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n)))))]] \Rightarrow Si \quad \sqcap L$$

$$[Nt(s(50))], \boxed{\blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb)))}, [[\square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare\forall f\forall a((((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n)))))]] \Rightarrow Si \quad \blacksquare L$$

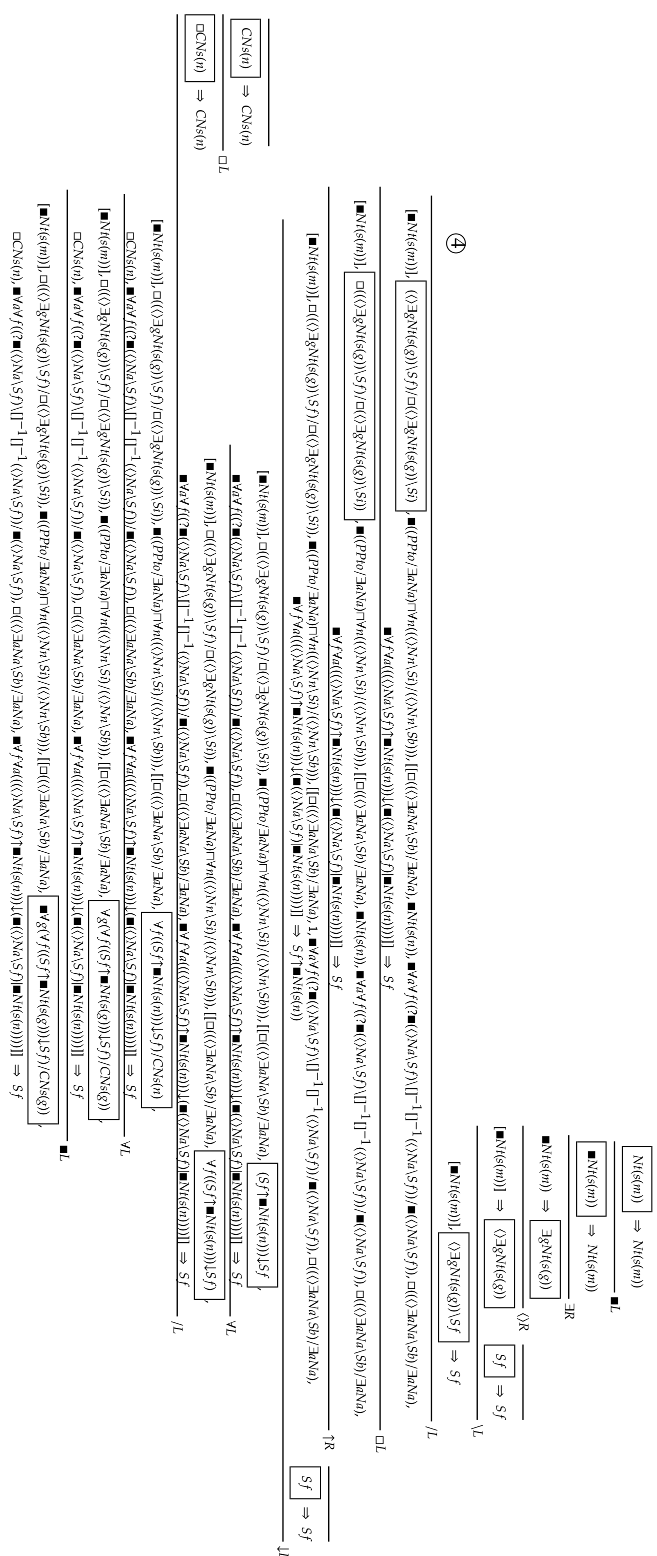
$$[\exists gNt(s(g))], \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), [[\square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare\forall f\forall a((((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n)))))]] \Rightarrow Si \quad \exists L$$

$$\langle\rangle\exists gNt(s(g)), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), [[\square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare\forall f\forall a((((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n)))))]] \Rightarrow Si \quad \langle\rangle L$$

$$\blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), [[\square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare\forall f\forall a((((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n)))))]] \Rightarrow \langle\rangle\exists gNt(s(g))\backslash Si \quad \backslash R$$

$$\blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), [[\square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \square(((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare\forall f\forall a((((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n)))))]] \Rightarrow \square(\langle\rangle\exists gNt(s(g))\backslash Si) \quad \square R$$

④



This delivers semantics with existential commitment:

(211) $\exists C[(\breve{}\textit{fish}\ C) \wedge ((\breve{}\textit{tries}\ \hat{}[((\breve{}\textit{catch}\ C)\ j) \wedge ((\breve{}\textit{eat}\ C)\ j)])\ j)]$

The existential narrow scope derivation is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(n))} \Rightarrow Nt(s(n))}}{\boxed{\blacksquare Nt(s(n))} \Rightarrow Nt(s(n))}\blacksquare L}{\blacksquare Nt(s(n)) \Rightarrow \boxed{\exists aNa}}\exists R \quad \dfrac{\dfrac{\dfrac{\dfrac{}{Nt(s(50)) \Rightarrow Nt(s(50))}}{Nt(s(50)) \Rightarrow \boxed{\exists aNa}}\exists R}{[Nt(s(50))] \Rightarrow \boxed{\langle\rangle\exists aNa}}\langle\rangle R \quad \dfrac{}{\boxed{Sb} \Rightarrow Sb}}{[Nt(s(50))], \boxed{\langle\rangle\exists aNa\backslash Sb} \Rightarrow Sb}\backslash L}{[Nt(s(50))], \boxed{(\langle\rangle\exists aNa\backslash Sb)/\exists aNa}, \blacksquare Nt(s(n)) \Rightarrow Sb}/L}{[Nt(s(50))], \boxed{\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa)}, \blacksquare Nt(s(n)) \Rightarrow Sb}\square L}{\langle\rangle Nt(s(50)), \square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)) \Rightarrow Sb}\langle\rangle L}{\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)) \Rightarrow \langle\rangle Nt(s(50))\backslash Sb}\backslash R}{\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), 1 \Rightarrow (\langle\rangle Nt(s(50))\backslash Sb)\uparrow\blacksquare Nt(s(n))}\uparrow R$$

①

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(n))} \Rightarrow Nt(s(n))}}{\boxed{\blacksquare Nt(s(n))} \Rightarrow Nt(s(n))}\blacksquare L}{\blacksquare Nt(s(n)) \Rightarrow \boxed{\exists aNa}}\exists R \quad \dfrac{\dfrac{\dfrac{\dfrac{}{Nt(s(50)) \Rightarrow Nt(s(50))}}{Nt(s(50)) \Rightarrow \boxed{\exists aNa}}\exists R}{[Nt(s(50))] \Rightarrow \boxed{\langle\rangle\exists aNa}}\langle\rangle R \quad \dfrac{}{\boxed{Sb} \Rightarrow Sb}}{[Nt(s(50))], \boxed{\langle\rangle\exists aNa\backslash Sb} \Rightarrow Sb}\backslash L}{[Nt(s(50))], \boxed{(\langle\rangle\exists aNa\backslash Sb)/\exists aNa}, \blacksquare Nt(s(n)) \Rightarrow Sb}/L}{[Nt(s(50))], \boxed{\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa)}, \blacksquare Nt(s(n)) \Rightarrow Sb}\square L}{\langle\rangle Nt(s(50)), \square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)) \Rightarrow Sb}\langle\rangle L}{\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)) \Rightarrow \langle\rangle Nt(s(50))\backslash Sb}\backslash R}{\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)) \Rightarrow \blacksquare(\langle\rangle Nt(s(50))\backslash Sb)}\blacksquare R}{\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)) \Rightarrow \boxed{?\blacksquare(\langle\rangle Nt(s(50))\backslash Sb)}}?R$$

②

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Nt(s(50)) \Rightarrow Nt(s(50))}{[Nt(s(50))] \Rightarrow \boxed{\langle\rangle Nt(s(50))}}\langle\rangle R \qquad \boxed{Sb} \Rightarrow Sb}{[Nt(s(50))], \boxed{\langle\rangle Nt(s(50))\backslash Sb} \Rightarrow Sb}\backslash L}{[Nt(s(50))], \boxed{\blacksquare(\langle\rangle Nt(s(50))\backslash Sb)} \Rightarrow Sb}\blacksquare L}{\langle\rangle Nt(s(50)), \blacksquare(\langle\rangle Nt(s(50))\backslash Sb) \Rightarrow Sb}\langle\rangle L}{\blacksquare(\langle\rangle Nt(s(50))\backslash Sb) \Rightarrow \langle\rangle Nt(s(50))\backslash Sb}\backslash R}{\blacksquare(\langle\rangle Nt(s(50))\backslash Sb) \Rightarrow \blacksquare(\langle\rangle Nt(s(50))\backslash Sb)}\blacksquare R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{Nt(s(50)) \Rightarrow Nt(s(50))}{[Nt(s(50))] \Rightarrow \boxed{\langle\rangle Nt(s(50))}}\langle\rangle R \qquad \boxed{Sb} \Rightarrow Sb}{[Nt(s(50))], \boxed{\langle\rangle Nt(s(50))\backslash Sb} \Rightarrow Sb}\backslash L}{[Nt(s(50))], [\boxed{[\,]^{-1}(\langle\rangle Nt(s(50))\backslash Sb)}] \Rightarrow Sb}[\,]^{-1}L}{[Nt(s(50))], [[\boxed{[\,]^{-1}[\,]^{-1}(\langle\rangle Nt(s(50))\backslash Sb)}]] \Rightarrow Sb}[\,]^{-1}L$$

②

$$\dfrac{②}{[Nt(s(50))], [[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \boxed{?\blacksquare(\langle\rangle Nt(s(50))\backslash Sb)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Nt(s(50))\backslash Sb)}]] \Rightarrow Sb}\backslash L$$

$$\dfrac{}{[Nt(s(50))], [[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \boxed{(?\blacksquare(\langle\rangle Nt(s(50))\backslash Sb)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Nt(s(50))\backslash Sb))/\blacksquare(\langle\rangle Nt(s(50))\backslash Sb)}, \blacksquare(\langle\rangle Nt(s(50))\backslash Sb)]] \Rightarrow Sb}/L$$

$$\dfrac{}{[Nt(s(50))], [[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \boxed{\forall f((?\blacksquare(\langle\rangle Nt(s(50))\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Nt(s(50))\backslash Sf))/\blacksquare(\langle\rangle Nt(s(50))\backslash Sf))}, \blacksquare(\langle\rangle Nt(s(50))\backslash Sb)]] \Rightarrow Sb}\forall L$$

$$\dfrac{}{[Nt(s(50))], [[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \boxed{\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf))}, \blacksquare(\langle\rangle Nt(s(50))\backslash Sb)]] \Rightarrow Sb}\forall L$$

$$\dfrac{}{[Nt(s(50))], [[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \boxed{\blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf))}, \blacksquare(\langle\rangle Nt(s(50))\backslash Sb)]] \Rightarrow Sb}\blacksquare L$$

$$\dfrac{\dfrac{\boxed{Nt(s(n))} \Rightarrow Nt(s(n))}{\boxed{\blacksquare Nt(s(n))} \Rightarrow Nt(s(n))}\blacksquare L}{\blacksquare Nt(s(n)) \Rightarrow \blacksquare Nt(s(n))}\blacksquare R$$

①

$$\dfrac{}{[Nt(s(50))], [[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \boxed{\blacksquare(\langle\rangle Nt(s(50))\backslash Sb)|\blacksquare Nt(s(n))}]] \Rightarrow Sb}|L$$

$$\dfrac{①}{[Nt(s(50))], [[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \boxed{((\langle\rangle Nt(s(50))\backslash Sb)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Nt(s(50))\backslash Sb)|\blacksquare Nt(s(n)))}]] \Rightarrow Sb}\downarrow L$$

$$\dfrac{}{[Nt(s(50))], [[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \boxed{\forall a(((\langle\rangle Na\backslash Sb)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sb)|\blacksquare Nt(s(n))))}]] \Rightarrow Sb}\forall L$$

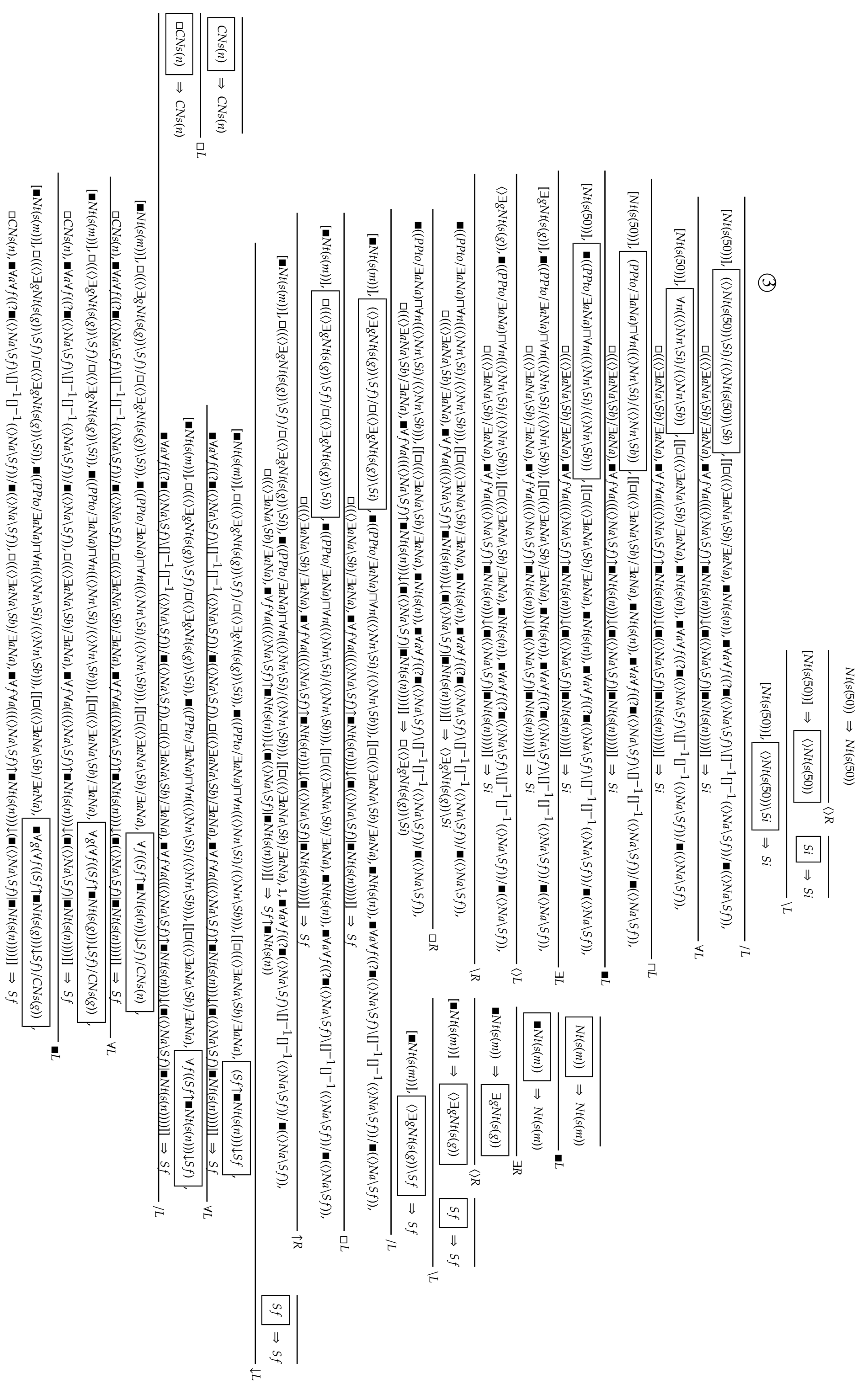
$$\dfrac{}{[Nt(s(50))], [[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \boxed{\forall f\forall a(((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n))))}]] \Rightarrow Sb}\forall L$$

$$\dfrac{}{[Nt(s(50))], [[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \boxed{\blacksquare\forall f\forall a(((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n))))}]] \Rightarrow Sb}\blacksquare L$$

$$\dfrac{}{\langle\rangle Nt(s(50)), [[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare\forall f\forall a(((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n))))]] \Rightarrow Sb}\langle\rangle L$$

$$\dfrac{}{[[\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare Nt(s(n)), \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[\,]^{-1}[\,]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)), \Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa), \blacksquare\forall f\forall a(((\langle\rangle Na\backslash Sf)\uparrow\blacksquare Nt(s(n)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(n))))]] \Rightarrow \langle\rangle Nt(s(50))\backslash Sb}\backslash R$$

③



This delivers semantics without existential commitment:

(212) $((\check{}tries\ \hat{}\exists F[(\check{}fish\ F) \wedge [((\check{}catch\ F)\ j) \wedge ((\check{}eat\ F)\ j)]])\ j)$

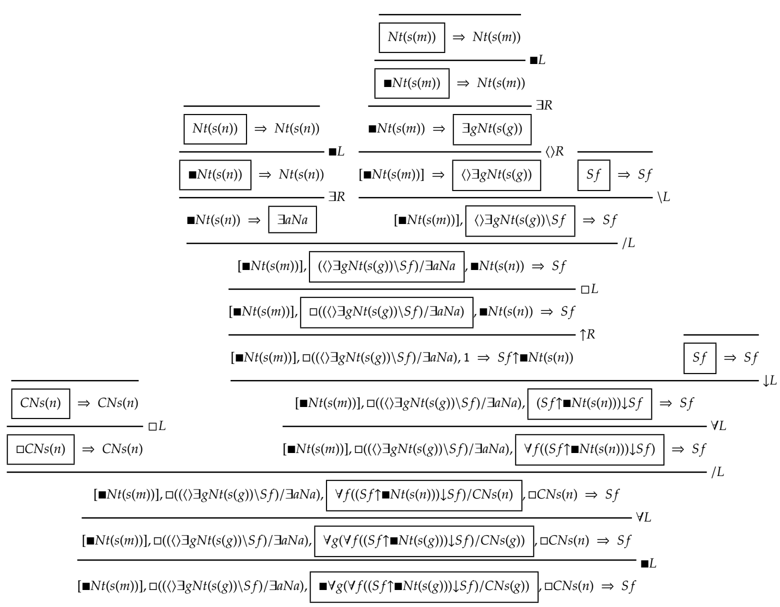
The next example involves an extensional transitive verb:

(213) (dwp((7-98))) **[john]+finds+a+unicorn** : Sf

The sentence cannot be true unless a unicorn exists. Montague treated extensional and intensional verbs uniformly syntactically by raising the type of extensional verbs to accommodate intensional verbs ('raising to the worst case"), and then using meaning postulates to capture the existential commitment of the former. Our type logical treatment allows assignment of the lower type to the existential which, as well as being simpler, captures the existential commitment automatically. Lexical lookup yields:

(214) $[\blacksquare Nt(s(m)) : j], \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}find\ A)\ B)), \blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda C\lambda D\exists E[(C\ E) \wedge (D\ E)], \square CNs(n) : unicorn \ \Rightarrow\ Sf$

This has the derivation:



It yields a semantics with existential commitment:

(215) $\exists C[(\check{}unicorn\ C) \wedge (Pres\ ((\check{}find\ C)\ j))]$

DWP continue with a donkey sentence, for which Montague grammar does not make the right prediction, and nor our cover grammar:

(216) (dwp((7-105))) **[every+man+such+that+[he]+loves+a+woman]+loses+her** : Sf

Lexical lookup yields:

(217) $[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda A\lambda B\forall C[(A\ C) \rightarrow (B\ C)], \square CNs(m) : man,$
$\blacksquare\forall n((CNn\backslash CNn)/\ (Sf|\blacksquare Nt(n))) : \lambda D\lambda E\lambda F[(E\ F) \wedge (D\ F)], [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/$
$(\langle\rangle Nt(s(m))\backslash Sg)) : \lambda GG], \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda H\lambda I(Pres\ ((\check{}love\ H)\ I)),$
$\blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda J\lambda K\exists L[(J\ L) \wedge (K\ L)], \square CNs(f) : woman],$
$\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda M\lambda N(Pres\ ((\check{}lose\ M)\ N)),$
$\blacksquare\forall g\forall a((((\langle\rangle Na\backslash Sg)\uparrow\blacksquare Nt(s(f)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f)))) : \lambda OO \ \Rightarrow\ Sf$

There is a dominant reading in which *a woman*, which is the donkey anaphora antecedent, is understood as quantified universally, but Montague grammar obtains only a subordinate reading in which *a woman* is quantified existentially at the matrix level. There is the derivation:

$$
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\dfrac{}{\boxed{Nt(s(f))} \Rightarrow Nt(s(f))}
}{\boxed{\blacksquare Nt(s(f))} \Rightarrow Nt(s(f))}\,\blacksquare L
}{\blacksquare Nt(s(f)) \Rightarrow \boxed{\exists aNa}}\,\exists R
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}
}{Nt(s(m)) \Rightarrow \boxed{\exists gNt(s(g))}}\,\exists R
}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g))}}\,\langle\rangle R
\quad
\dfrac{}{\boxed{Sf} \Rightarrow Sf}
}{[Nt(s(m))], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} \Rightarrow Sf}\,\backslash L
}{[Nt(s(m))], \boxed{(\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa}, \blacksquare Nt(s(f)) \Rightarrow Sf}\,/L
}{[Nt(s(m))], \boxed{\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)}, \blacksquare Nt(s(f)) \Rightarrow Sf}\,\square L
}{\langle\rangle Nt(s(m)), \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare Nt(s(f)) \Rightarrow Sf}\,\langle\rangle L
}{\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare Nt(s(f)) \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}\,\backslash R
}{\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), 1 \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)\uparrow\blacksquare Nt(s(f))}\,\uparrow R
$$

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\dfrac{}{\boxed{Nt(s(f))} \Rightarrow Nt(s(f))}
}{\boxed{\blacksquare Nt(s(f))} \Rightarrow Nt(s(f))}\,\blacksquare L
}{\blacksquare Nt(s(f)) \Rightarrow \boxed{\exists aNa}}\,\exists R
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}{Nt(s(m)) \Rightarrow \boxed{\exists gNt(s(g))}}\,\exists R
}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g))}}\,\langle\rangle R
\quad
\dfrac{}{\boxed{Sf} \Rightarrow Sf}
}{[Nt(s(m))], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} \Rightarrow Sf}\,\backslash L
}{[Nt(s(m))], \boxed{(\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa}, \blacksquare Nt(s(f)) \Rightarrow Sf}\,/L
}{[Nt(s(m))], \boxed{\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)}, \blacksquare Nt(s(f)) \Rightarrow Sf}\,\square L
}{\langle\rangle Nt(s(m)), \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare Nt(s(f)) \Rightarrow Sf}\,\langle\rangle L
}{\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare Nt(s(f)) \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}\,\backslash R
\quad
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\dfrac{}{\boxed{Sf} \Rightarrow Sf}
}{\boxed{\blacksquare Sf} \Rightarrow Sf}\,\blacksquare L
}{\boxed{\blacksquare Sf|\blacksquare Nt(s(m))} \Rightarrow \blacksquare Sf|\blacksquare Nt(s(m))}\,|R
}{\boxed{(\blacksquare Sf|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sf)}, \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare Nt(s(f)) \Rightarrow Sf|\blacksquare Nt(s(m))}\,/L
}{\boxed{\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))}, \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare Nt(s(f)) \Rightarrow Sf|\blacksquare Nt(s(m))}\,\forall L
}{[\boxed{[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))}], \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare Nt(s(f)) \Rightarrow Sf|\blacksquare Nt(s(m))}\,[]^{-1}L
}{[\boxed{\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))}], \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare Nt(s(f)) \Rightarrow Sf|\blacksquare Nt(s(m))}\,\blacksquare L
$$

②

$$Nt(s(m)) \Rightarrow Nt(s(m))$$

$$[Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))} \;\; \langle\rangle R \qquad \boxed{Sf} \Rightarrow Sf$$

$$[Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf \;\; \backslash L$$

$$[Nt(s(m))], \boxed{\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)} \Rightarrow Sf \;\; \blacksquare L$$

$$[1], \blacksquare(\langle\rangle Nt(s(m))\backslash Sf) \Rightarrow Sf{\uparrow}Nt(s(m)) \;\; \uparrow R \qquad \boxed{Sf} \Rightarrow Sf$$

$$[\boxed{(Sf{\uparrow}Nt(s(m))){\downarrow}Sf}], \blacksquare(\langle\rangle Nt(s(m))\backslash Sf) \Rightarrow Sf \;\; \downarrow L$$

$$[\boxed{\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)}], \blacksquare(\langle\rangle Nt(s(m))\backslash Sf) \Rightarrow Sf \;\; \forall L$$

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$$\boxed{CNs(m)} \Rightarrow CNs(m)$$

$$\boxed{\square CNs(m)} \Rightarrow CNs(m) \;\; \square L \qquad \boxed{CNs(m)} \Rightarrow CNs(m)$$

$$\square CNs(m), \boxed{CNs(m)\backslash CNs(m)} \Rightarrow CNs(m) \;\; \backslash L$$

$$\square CNs(m), \boxed{(CNs(m)\backslash CNs(m))/(Sf|\blacksquare Nt(s(m)))}, [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \blacksquare Nt(s(f)) \Rightarrow CNs(m) \;\; /L$$

$$\square CNs(m), \boxed{\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n)))}, [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \blacksquare Nt(s(f)) \Rightarrow CNs(m) \;\; \forall L$$

$$\square CNs(m), \boxed{\blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n)))}, [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \blacksquare Nt(s(f)) \Rightarrow CNs(m) \;\; \blacksquare L$$

$$[\boxed{\forall f((Sf{\uparrow}Nt(s(m))){\downarrow}Sf)/CNs(m)}, \square CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \blacksquare Nt(s(f))], \blacksquare(\langle\rangle Nt(s(m))\backslash Sf) \Rightarrow Sf \;\; /L$$

$$[\boxed{\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g))}, \square CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \blacksquare Nt(s(f))], \blacksquare(\langle\rangle Nt(s(m))\backslash Sf) \Rightarrow Sf \;\; \forall L$$

$$[\boxed{\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g))}, \square CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \blacksquare Nt(s(f))], \blacksquare(\langle\rangle Nt(s(m))\backslash Sf) \Rightarrow Sf \;\; \blacksquare L$$

$$\boxed{Nt(s(f))} \Rightarrow Nt(s(f))$$

$$\boxed{\blacksquare Nt(s(f))} \Rightarrow Nt(s(f)) \;\; \blacksquare L$$

$$\blacksquare Nt(s(f)) \Rightarrow \blacksquare Nt(s(f)) \;\; \blacksquare R$$

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$$[\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \square CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \blacksquare Nt(s(f))], \boxed{\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)|\blacksquare Nt(s(f))} \Rightarrow Sf \;\; |L$$

$$[\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \square CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \blacksquare Nt(s(f))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \boxed{((\langle\rangle Nt(s(m))\backslash Sf){\uparrow}\blacksquare Nt(s(f))){\downarrow}(\blacksquare(\langle\rangle Nt(s(m))\backslash Sf)|\blacksquare Nt(s(f)))} \Rightarrow Sf \;\; \downarrow L$$

$$[\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \square CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \blacksquare Nt(s(f))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \boxed{\forall a(((\langle\rangle Na\backslash Sf){\uparrow}\blacksquare Nt(s(f))){\downarrow}(\blacksquare(\langle\rangle Na\backslash Sf)|\blacksquare Nt(s(f))))} \Rightarrow Sf \;\; \forall L$$

$$[\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \square CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \blacksquare Nt(s(f))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \boxed{\forall g\forall a(((\langle\rangle Na\backslash Sg){\uparrow}\blacksquare Nt(s(f))){\downarrow}(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f))))} \Rightarrow Sf \;\; \forall L$$

$$[\blacksquare\forall g(\forall f((Sf{\uparrow}Nt(s(g))){\downarrow}Sf)/CNs(g)), \square CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \blacksquare Nt(s(f))], \square(((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na), \boxed{\blacksquare\forall g\forall a(((\langle\rangle Na\backslash Sg){\uparrow}\blacksquare Nt(s(f))){\downarrow}(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f))))} \Rightarrow Sf \;\; \blacksquare L$$

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$$\frac{\boxed{CNs(f)} \Rightarrow CNs(f)}{\boxed{\Box CNs(f)} \Rightarrow CNs(f)}\ \Box L$$

$$\frac{[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)), \Box CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), 1], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare\forall g\forall a((((\langle\rangle Na\backslash Sg)\uparrow\blacksquare Nt(s(f)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f)))) \Rightarrow Sf\uparrow\blacksquare Nt(s(f))}{}\ \uparrow R \qquad \boxed{Sf} \Rightarrow Sf$$

$$[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)), \Box CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \boxed{(Sf\uparrow\blacksquare Nt(s(f)))\downarrow Sf}], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare\forall g\forall a((((\langle\rangle Na\backslash Sg)\uparrow\blacksquare Nt(s(f)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f)))) \Rightarrow Sf \quad \downarrow L$$

$$[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)), \Box CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \boxed{\forall f((Sf\uparrow\blacksquare Nt(s(f)))\downarrow Sf)}], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare\forall g\forall a((((\langle\rangle Na\backslash Sg)\uparrow\blacksquare Nt(s(f)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f)))) \Rightarrow Sf \quad \forall L$$

$$[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)), \Box CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \boxed{\forall f((Sf\uparrow\blacksquare Nt(s(f)))\downarrow Sf)/CNs(f)}, \Box CNs(f)], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare\forall g\forall a((((\langle\rangle Na\backslash Sg)\uparrow\blacksquare Nt(s(f)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f)))) \Rightarrow Sf \quad /L$$

$$[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)), \Box CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \boxed{\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g))}, \Box CNs(f)], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare\forall g\forall a((((\langle\rangle Na\backslash Sg)\uparrow\blacksquare Nt(s(f)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f)))) \Rightarrow Sf \quad \forall L$$

$$[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)), \Box CNs(m), \blacksquare\forall n((CNn\backslash CNn)/(Sf|\blacksquare Nt(n))), [\blacksquare[]^{-1}\forall g((\blacksquare Sg|\blacksquare Nt(s(m)))/(\langle\rangle Nt(s(m))\backslash Sg))], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \boxed{\blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g))}, \Box CNs(f)], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \blacksquare\forall g\forall a((((\langle\rangle Na\backslash Sg)\uparrow\blacksquare Nt(s(f)))\downarrow(\blacksquare(\langle\rangle Na\backslash Sg)|\blacksquare Nt(s(f)))) \Rightarrow Sf \quad \blacksquare L$$

This delivers semantics:

(218) $\exists C[(\check{}woman\ C) \wedge \forall G[[(\check{}man\ G) \wedge (Pres\ ((\check{}love\ C)\ G))] \rightarrow (Pres\ ((\check{}lose\ C)\ G))]]$

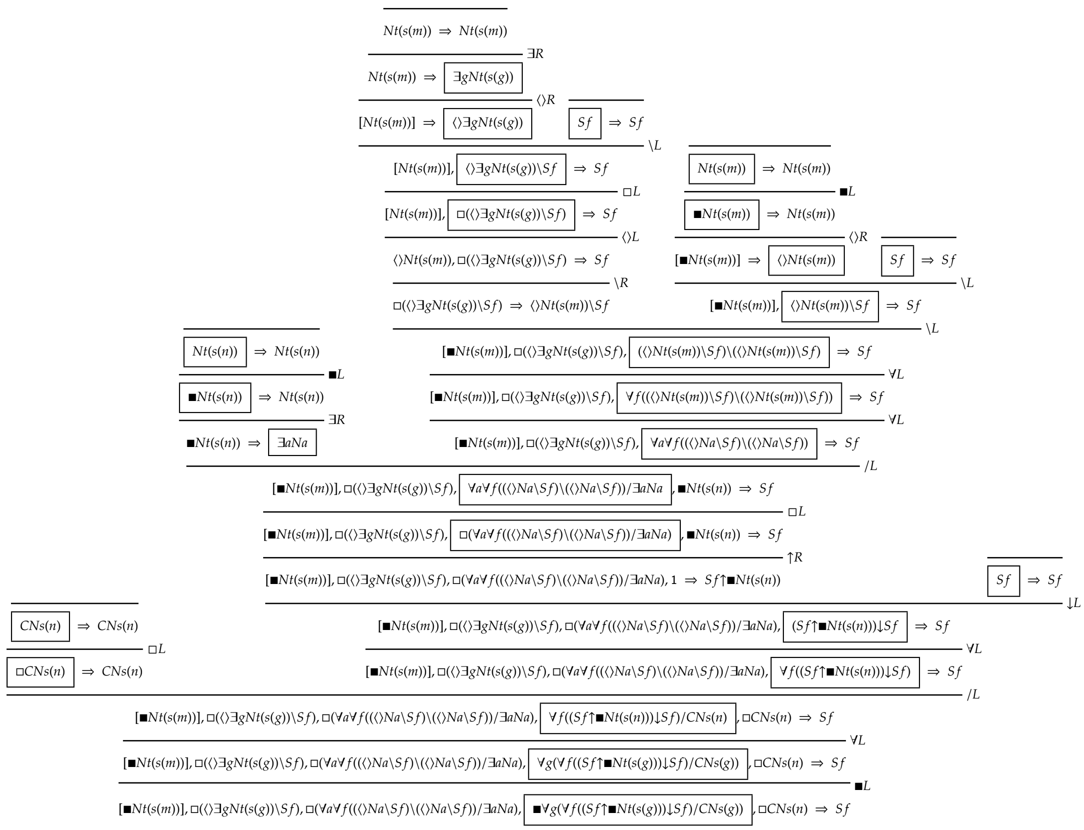
The assignment of lowest type in type logical grammar also means that existential commitment of a preposition comes without the need for devices such as the meaning postulates of Montague grammar:

(219) (dwp((7-110))) [**john**]+**walks**+**in**+**a**+**park** : Sf

Lexical lookup for this example yields the semantically annotated sequent:

(220) $[\blacksquare Nt(s(m)) : j], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda A(Pres\ (\check{}walk\ A)), \Box(\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))/$
$\exists aNa) : \hat{}\lambda B\lambda C\lambda D((\check{}in\ B)\ (C\ D)), \blacksquare\forall g(\forall f((Sf\uparrow\blacksquare Nt(s(g)))\downarrow Sf)/CNs(g)) :$
$\lambda E\lambda F\exists G[(E\ G) \wedge (F\ G)], \Box CNs(n) : park\ \Rightarrow\ Sf$

This sequent has the proof:



It delivers the semantics (with existential comittment):

(221) $\exists C[(\check{}park\ C) \wedge ((\check{}in\ C)\ (Pres\ (\check{}walk\ j)))]$

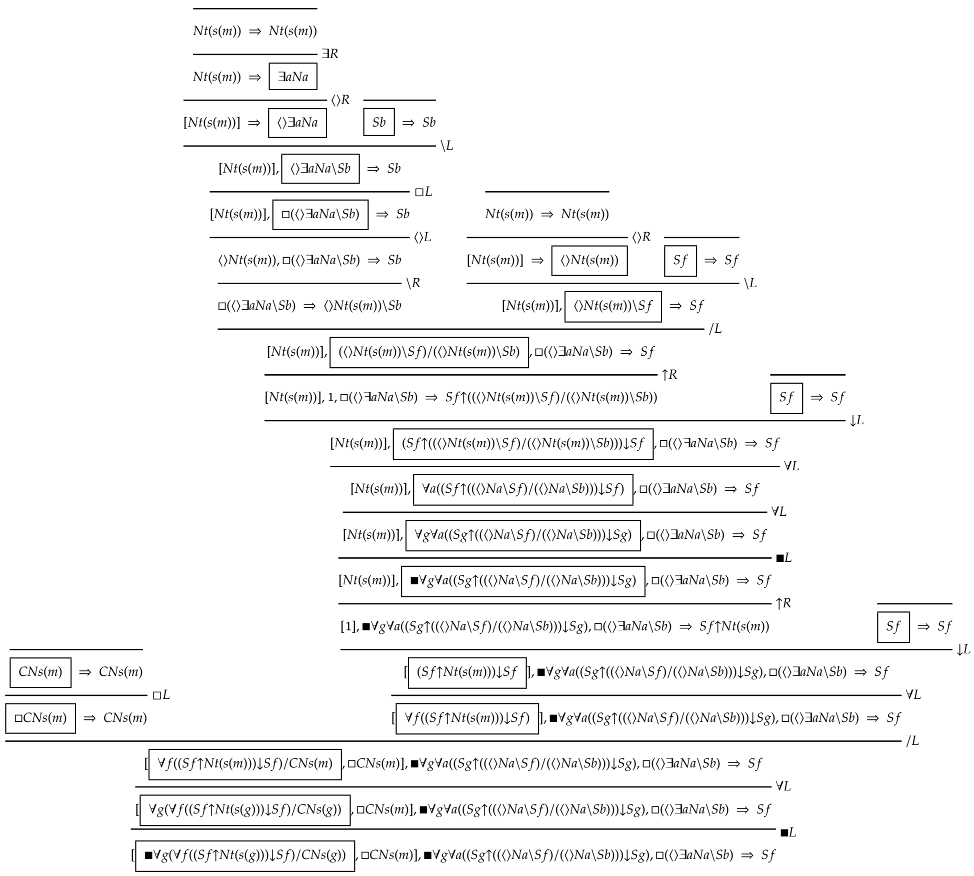
Finally DWP analyse the ambiguous example:

(222) (dwp((7-116, 118))) **[every+man]+doesnt+walk** : Sf

This has a dominant reading in which the universal has narrow scope with respect to the negation, and a subordinate reading in which the universal has wide scope with respect to the negation. Lexical lookup yields:

(223) $[\blacksquare\forall g(\forall f((Sf\uparrow Nt(s(g)))\downarrow Sf)/CNs(g)) : \lambda A\lambda B\forall C[(A\ C) \rightarrow (B\ C)], \square CNs(m) : man],$
$\blacksquare\forall g\forall a((Sg\uparrow((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)))\downarrow Sg) : \lambda D\neg(D\ \lambda E\lambda F(E\ F)), \square(\langle\rangle\exists aNa\backslash Sb) : \hat{}\lambda G(\check{}walk\ G)$
$\Rightarrow\ Sf$

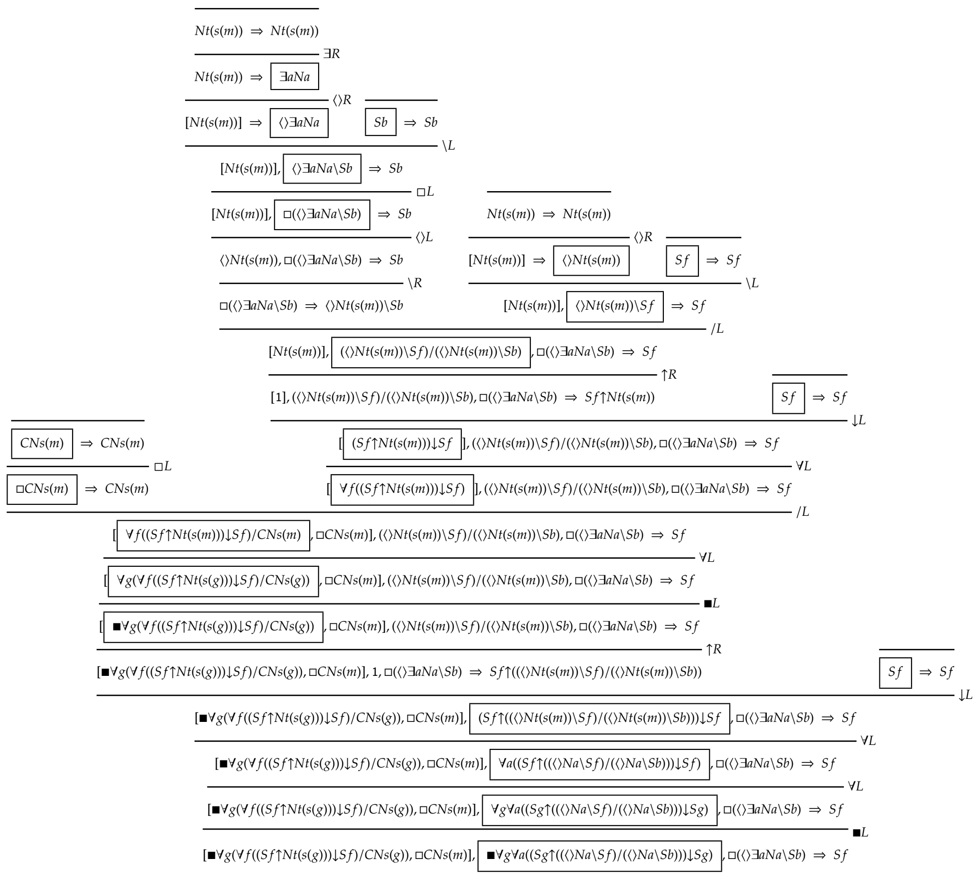
This has a first derivation:



This delivers the subordinate reading:

(224) $\forall C[(\check{}\mathit{man}\ C) \rightarrow \neg(\check{}\mathit{walk}\ C)]$

There is also the derivation:



This delivers the dominant reading:

(225) $\neg\forall G[(\check{}man\ G) \to (\check{}walk\ G)]$

Chapter 14

Coordination

In this chapter we analyse examples of coordination, cf. Morrill (2011[81], Chapter 3, Section 10).

14.1 Constituent and 'non-constituent' coordination

To express the lexical semantics of coordination, including iterated coordination (e.g. **Bill, Mary, Suzy and Fred**) and various arities (zeroary e.g. sentence, unary e.g. verb phrase, binary e.g. transitive verb, . . .), we use combinators: a non-empty list map apply α^+ and a non-empty list map $\mathbf{\Phi^n}$ combinator $\mathbf{\Phi^{n+}}$.

The combinator $\mathbf{\Phi}$ is such that $\mathbf{\Phi}\ x\ y\ z\ w = x\ (y\ w)\ (z\ w)$ (Curry and Feys 1958[16]). The non-empty list map apply combinator α^+ is as follows:

(226)
$$\begin{array}{rcl} (\alpha^+\ [x]\ y) & = & [(x\ y)] \\ (\alpha^+\ [x, y|z]\ w) & = & [(x\ w)|(\alpha^+\ [y|z]\ w)] \end{array}$$

The non-empty list map $\mathbf{\Phi^n}$ combinator $\mathbf{\Phi^{n+}}$ is thus:

(227)
$$\begin{array}{rcl} (((\mathbf{\Phi^{n+}}\ 0\ \mathit{and})\ x)\ [y]) & = & [y \wedge x] \\ (((\mathbf{\Phi^{n+}}\ 0\ \mathit{or})\ x)\ [y]) & = & [y \vee x] \\ (((\mathbf{\Phi^{n+}}\ 0\ \mathit{and})\ x)\ [y, z|w]) & = & [y \wedge (((\mathbf{\Phi^{n+}}\ 0\ \mathit{and})\ x)\ [z|w])] \\ (((\mathbf{\Phi^{n+}}\ 0\ \mathit{or})\ x)\ [y, z|w]) & = & [y \vee (((\mathbf{\Phi^{n+}}\ 0\ \mathit{or})\ x)\ [z|w])] \\ ((((\mathbf{\Phi^{n+}}\ (s\ n)\ c)\ x)\ y)\ z) & = & (((\mathbf{\Phi^{n+}}\ n\ c)\ (x\ z))\ (\alpha^+\ y\ z)) \end{array}$$

These equations mean that in semantic evaluation any subterm of the form on the left is to be replaced by that on the right, successively.

14.1.1 Sentence coordination

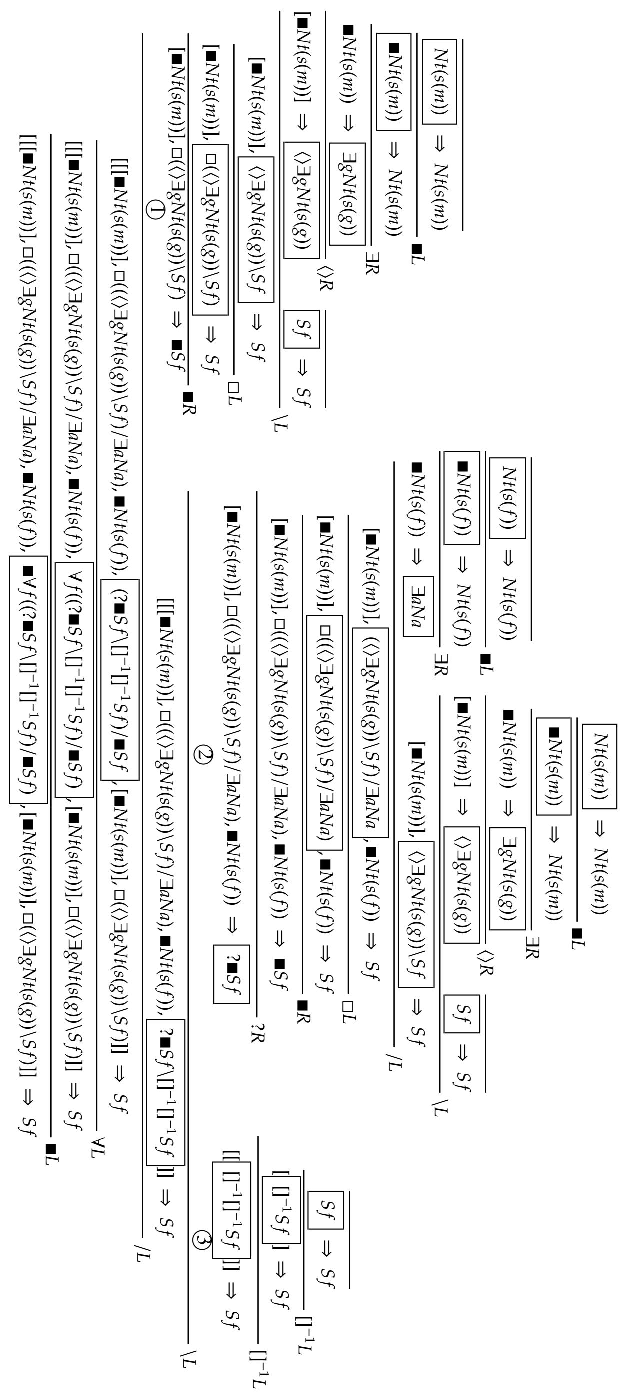
The first example is simple sentential conjunction; as we have said subjects are bracketed since they are (weak) islands and coordinate structures are doubly bracketed corresponding to the fact that they are strong islands; these brackets are given in the input:

(228) [[[**john**]+**praises**+**mary**+**and**+[**john**]+**laughs**]] : Sf

Lexical lookup yields the following where the coordinator type is essentially $(?X\backslash[\,]^{-1}[\,]^{-1}X)/X$ with $X = S$.

(229)
$$[[[\blacksquare Nt(s(m)) : j], \square((\langle\rangle\exists g Nt(s(g))\backslash Sf)/\exists a Na) : \hat{}\lambda A\lambda B(\mathit{Pres}\ ((\check{}\mathit{praise}\ A)\ B)),$$
$$\blacksquare Nt(s(f)) : m, \blacksquare\forall f((?\blacksquare Sf\backslash[]^{-1}[]^{-1}Sf)/\blacksquare Sf) : (\mathbf{\Phi^{n+}}\ \mathit{0\ and}), [\blacksquare Nt(s(m)) : j],$$
$$\square(\langle\rangle\exists g Nt(s(g))\backslash Sf) : \hat{}\lambda C(\mathit{Pres}\ (\check{}\mathit{laugh}\ C))]] \Rightarrow Sf$$

The left conjunt is marked with the existential exponential to allow iterated coordination, which we will illustrate later; the conjunts are marked with semantically inactive normal modalities to make coordinate structures scope islands to quantifiers other than indefinites. There is the derivation:



In the conclusions of rules boxes mark the active type of the rule application, i.e. the type which is decomposed in the premises; as we have seen the boxes mark *focused* types (Andreoli 1992[5]), which are the active types in the conclusions of noninvertible rules, subject to focusing rule application; the active types in conclusions of invertible rules are not marked; whenever a type is boxed it is the active type — this improves readability. Reading from the root, the type of the coordinator projecting the construction is decomposed, eliminating the semantically inactive outermost modality and instantiating the tense feature to finite (f). The result then applies forwards to the righthand conjunct, the analysis of which is shown in the subtree marked ①: after removal of the semantically inactive modality in the succedent (which is licensed by the fact that the antecedents are modalised), left box elimination is applied to the intransitive verb; this then applies to its subject and in the minor premise the subject bracket modality is removed, then the agreement instantiated to third person singular masculine ($t(s(m))$), and then the semantically inactive subject modality is removed and the axiom is matched. The coordinator then applies to the lefthand conjunct, in the subtree marked ②: the existential exponential is removed directly from the succedent, since the coordination is not iterative, and then the semantically inactive succedent modality is eliminated (as required the antecedents are modalised), and the outer modality of the transitive verb is removed; the transitive verb then applies to its object, where the agreement is instantiated to third person singular feminine ($t(s(f))$), and the analysis of the resulting verb phrase is the same as in ①. Finally, in the subderivation ③ the coordinator checks off the doubly bracketed context which it requires/projects. All this delivers the correct reading under:

(230) $[(Pres\ ((\check{}praise\ m)\ j)) \wedge (Pres\ (\check{}laugh\ j))]$

14.1.2 Verb phrase coordination

The next example is one of verb phrase conjunction:

(231) [**john**]+[[**praises**+**mary**+**and**+**laughs**]] : Sf

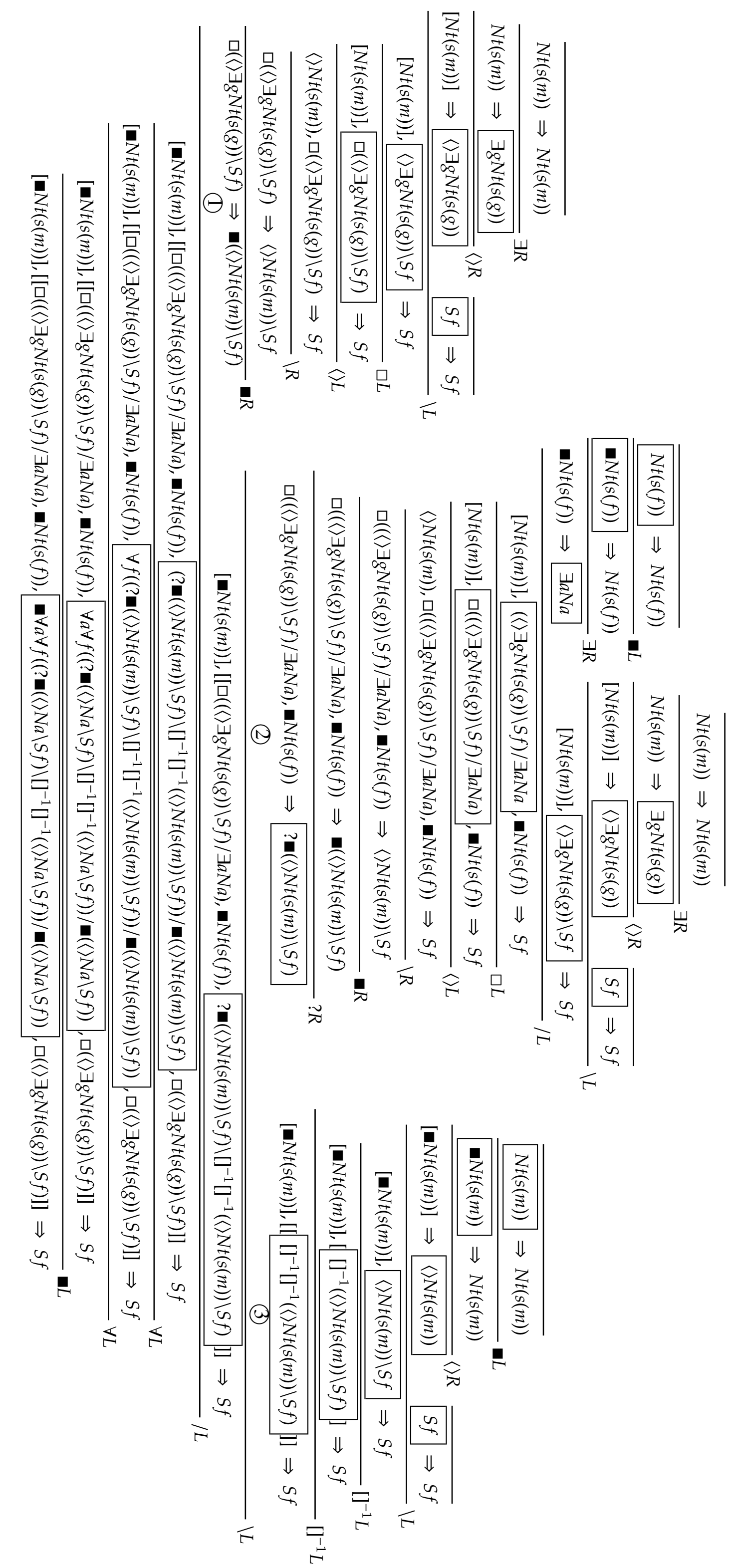
Lexical lookup yields the coordinator type basically $(?X\backslash[\,]^{-1}[\,]^{-1}X)/X$ with $X = N\backslash S$.

(232) $[\blacksquare Nt(s(m)) : j], [[\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}praise\ A)\ B)),$
$\blacksquare Nt(s(f)) : m, \blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)) :$
$(\mathbf{\Phi^{n+}}\ (s\ 0)\ and), \square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda C(Pres\ (\check{}laugh\ C))]] \Rightarrow Sf$

The coordination combinator semantics ($\mathbf{\Phi^{n+}}$ (s 0) *and*) is such that:

(233) $((((\mathbf{\Phi^{n+}}\ (s\ 0)\ and)\ x)\ [y])\ z) =$
$(((\mathbf{\Phi^{n+}}\ 0\ and)\ (x\ z))\ (\alpha^{+}\ [y]\ z)) =$
$(((\mathbf{\Phi^{n+}}\ 0\ and)\ (x\ z))\ [(y\ z)]) =$
$[(y\ z) \wedge (x\ z)]$

There is the derivation:



Reading from the root, first the outermost semantically inactive modality of the coordinator is removed and the agreement features of the subject, $t(s(m))$, and the value of the tense feature, f, are instantiated. The coordinator then applies to the righthand conjunct, for which the subderivation is ①: the succedent semantically inactive modality is removed (the antecedent is modalised) and the argument modally bracketed nominal is lowered into the antecedent where its bracket modality is unfolded; the modality of the intransitive verb is removed, and the subderivation finishes applying to the subject with bracket modality right and existential quantifier right rules. The coordinator then applies to the transitive verb plus object lefthand conjunct in the subderivation rooted at ②: after existential exponential right, semantically inactive modality right (the antecedents are modalised), and under right lowering the subject subtype into the antecedent, the bracket modality of the subject is unfolded; the transitive verb type is then selected, its modality removed, and applied with modality rules and instantiation of existentially quantified features to the object (left subsubderivation) and subject (middle subsubderivation). In the subderivation rooted at ③ the context of the coordinate structure is recognised: double bracketing, and a (bracketed) proper name subject. All this delivers semantics:

(234) $[(Pres\ ((\check{\ }praise\ m)\ j)) \wedge (Pres\ (\check{\ }laugh\ j))]$

14.1.3 Transitive verb coordination

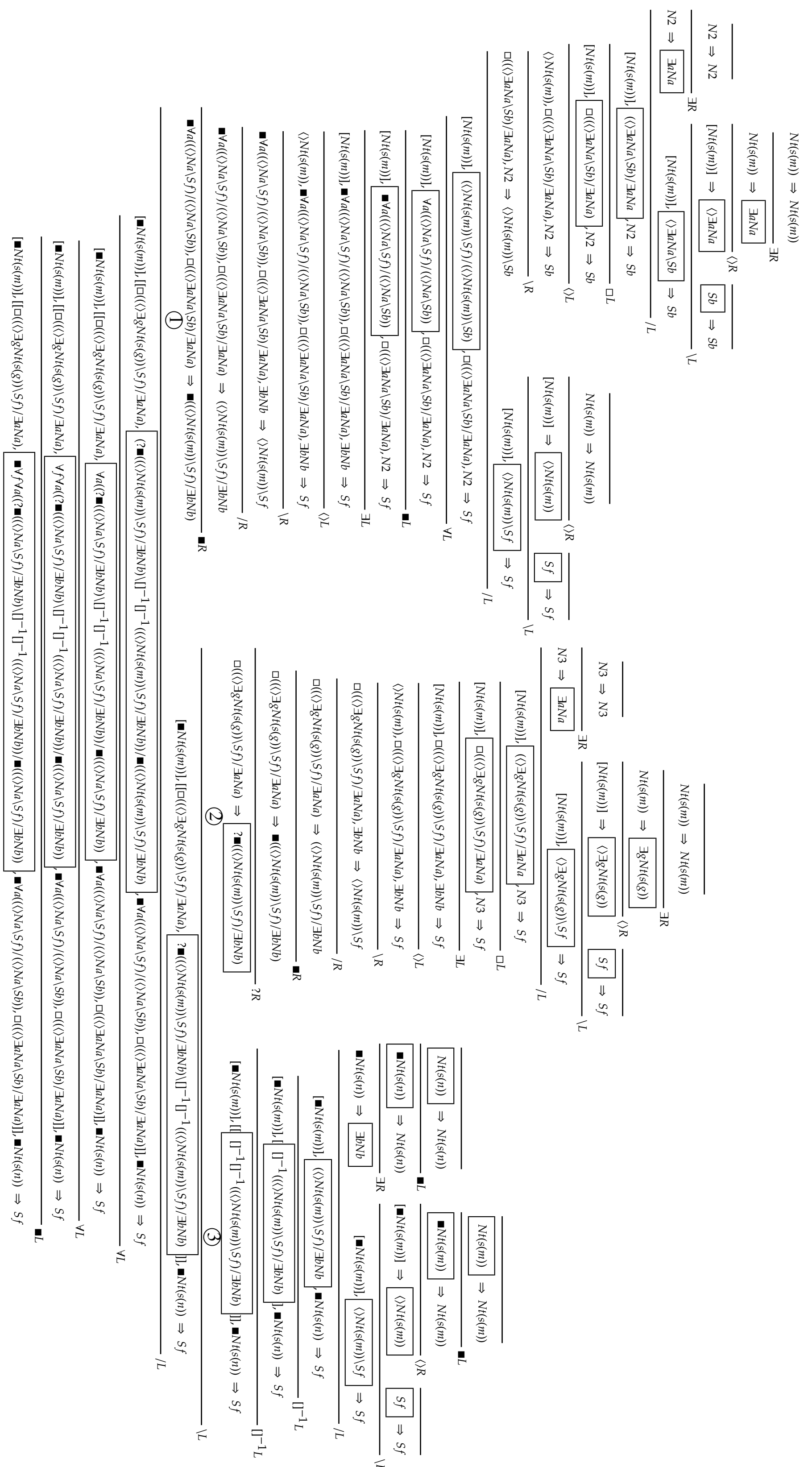
The next example is of transitive verb coordination, with a complex non-standard constituent transitive verb in the right-hand conjunt.

(235) **[john]+[[likes+and+will+love]]+london** : Sf

Appropriate lexical lookup yields the following where the coordinator type is essentially $(?X\backslash [\,]^{-1}[\,]^{-1}X)/X$ with $X = (N\backslash S)/N$.

(236) $[\blacksquare Nt(s(m)) : j], [[\square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{\ }\lambda A\lambda B(Pres\ ((\check{\ }like\ A)\ B)),$
$\blacksquare\forall f\forall a((?\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash[]^{-1}[]^{-1}((\langle\rangle Na\backslash Sf)/\exists bNb))/\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb)) :$
$(\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and), \blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)) : \lambda C\lambda D(Fut\ (C\ D)),$
$\square((\langle\rangle\exists aNa\backslash Sb)/\exists aNa) : \hat{\ }\lambda E\lambda F((\check{\ }love\ E)\ F)]], \blacksquare Nt(s(n)) : l \Rightarrow Sf$

There is the following derivation. Again, after eliminating the modality of the coordinator, and instantiating the left universal quantifiers for tense and subject agreement, at the root, there are three main subderivations marked ①, ② and ③ deriving the right conjunct, the left conjunct and the coordinate structure context respectively. In ①, after removing the modality on the right and lowering the object and subject arguments into the antecedent we have to analyse a sequence which is essentially S-Aux-TV-O. This involves analysing TV-O as VP (left subsubderivation), and S-VP as S (right subsubderivation). Subtree ② involves essentially derivation of the identity TV yields TV. Subderivation ③ checks the double bracketing of the coordinate structure and recognises the right object context and left subject context.



The coordination combinator semantics is such that

(237) $(((((\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and)\ x)\ [y])\ z)\ w) =$
$((((\mathbf{\Phi^{n+}}\ (s\ 0)\ and)\ (x\ z))\ (\alpha^{+}\ [y]\ z))\ w) =$
$((((\mathbf{\Phi^{n+}}\ (s\ 0)\ and)\ (x\ z))\ [(y\ z)])\ w) =$
$(((\mathbf{\Phi^{n+}}\ 0\ and)\ ((x\ z)\ w))\ (\alpha^{+}\ [(y\ z)]\ w)) =$
$(((\mathbf{\Phi^{n+}}\ 0\ and)\ ((x\ z)\ w))\ [((y\ z)\ w)]) =$
$[((y\ z)\ w) \wedge ((x\ z)\ w)]$

All this delivers semantics:

(238) $[(Pres\ ((\check{}like\ l)\ j)) \wedge (Fut\ ((\check{}love\ l)\ j))]$

14.1.4 Ditransitive verb coordination

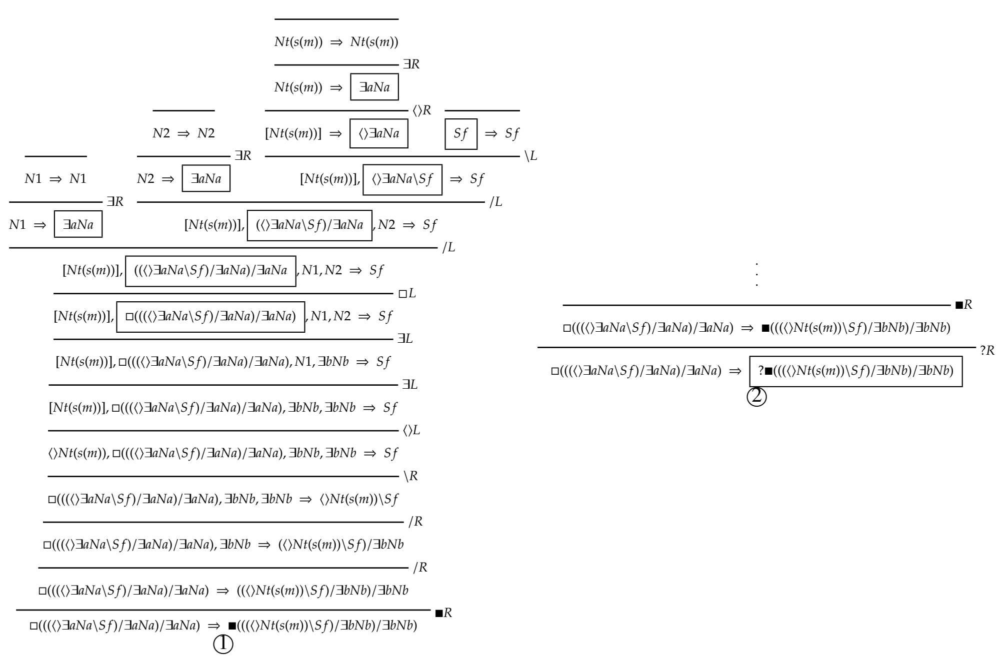
The next example is of ditransitive verb disjunction:

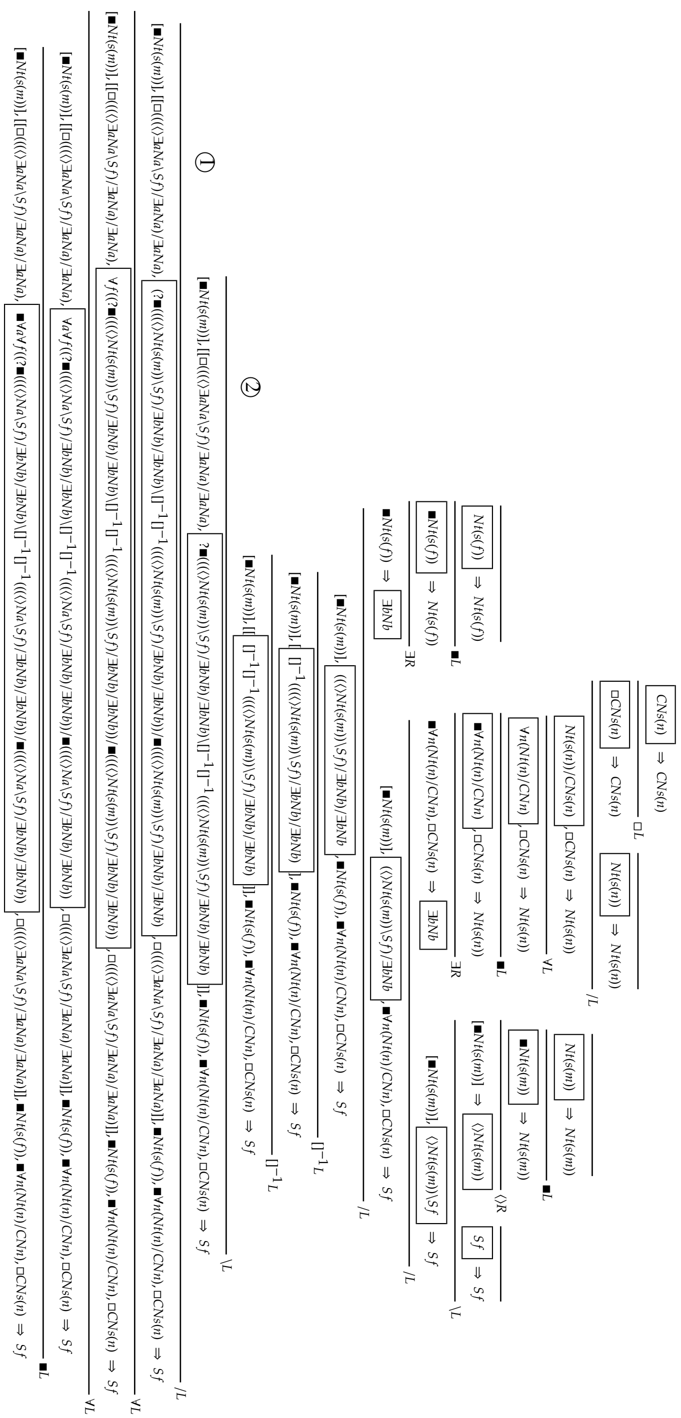
(239) [**john**]+[[**gave**+**or**+**sent**]]+**mary**+**the**+**book** : Sf

Appropriate lexical lookup yields the semantically labelled sequent where the coordinator type is essentially $(?X\backslash[\,]^{-1}[\,]^{-1}X)/X$ with $X = ((N\backslash S)/N)/N$ (using the curried ditransitive verb type):

(240) $[\blacksquare Nt(s(m)) : j], [[\square(((\langle\rangle\exists aNa\backslash Sf)/\exists aNa)/\exists aNa) : \hat{}\lambda A\lambda B\lambda C(Past\ (((\check{}give\ A)\ B)\ C)),$
$\blacksquare\forall a\forall f((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)/\exists bNb)\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)/\exists bNb))/$
$\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)/\exists bNb)) : (\mathbf{\Phi^{n+}}\ (s\ (s\ (s\ 0)))\ or), \square(((\langle\rangle\exists aNa\backslash Sf)/\exists aNa)/\exists aNa) :$
$\hat{}\lambda D\lambda E\lambda F(Past\ (((\check{}send\ D)\ E)\ F))]], \blacksquare Nt(s(f)) : m, \blacksquare\forall n(Nt(n)/CNn) : \iota,$
$\square CNs(n) : book \ \Rightarrow\ Sf$

There is the derivation:





As usual the coordinator modality is removed and the subject agreement and tense feature are instantiated at the root. Then the subderivations ① and ② derive the right and left conjuncts respectively by essentially deriving the identity TTV yields TTV on the ditransitive verb type TTV: they are the same except for the initial $?R$ in ②, hence the latter is elided. The remaining context derivation checks the double bracketing as usual and for the rest amounts to analysis of a sequence S-TTV-O1-O2 where S and O1 are proper names and O2 is a definite noun phrase.

The example is interesting in that it illustrates coordination of arity three. The coordination combinator semantics is such that:

(241) $((((((\mathbf{\Phi^{n+}}\ (s\ (s\ (s\ 0)))\ or)\ x)\ [y])\ z)\ w)\ u) =$
$(((((\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ or)\ (x\ z))\ (\alpha^+\ [y]\ z))\ w)\ u) =$
$(((((\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ or)\ (x\ z))\ [(y\ z)])\ w)\ u) =$
$((((\mathbf{\Phi^{n+}}\ (s\ 0)\ or)\ ((x\ z)\ w))\ (\alpha^+\ [(y\ z)]\ w))\ u) =$
$((((\mathbf{\Phi^{n+}}\ (s\ 0)\ or)\ ((x\ z)\ w))\ [((y\ z)\ w)])\ u) =$
$(((\mathbf{\Phi^{n+}}\ 0\ or)\ (((x\ z)\ w)\ u))\ (\alpha^+\ [((y\ z)\ w)]\ u)) =$
$(((\mathbf{\Phi^{n+}}\ 0\ or)\ (((x\ z)\ w)\ u))\ [(((y\ z)\ w)\ u)]) =$
$[(((y\ z)\ w)\ u) \wedge (((x\ z)\ w)\ u)]$

All this delivers semantics:

(242) $[(Past\ (((\check{~}give\ m)\ (\iota\ \check{~}book))\ j)) \vee (Past\ (((\check{~}send\ m)\ (\iota\ \check{~}book))\ j))]$

14.1.5 Subject coordination

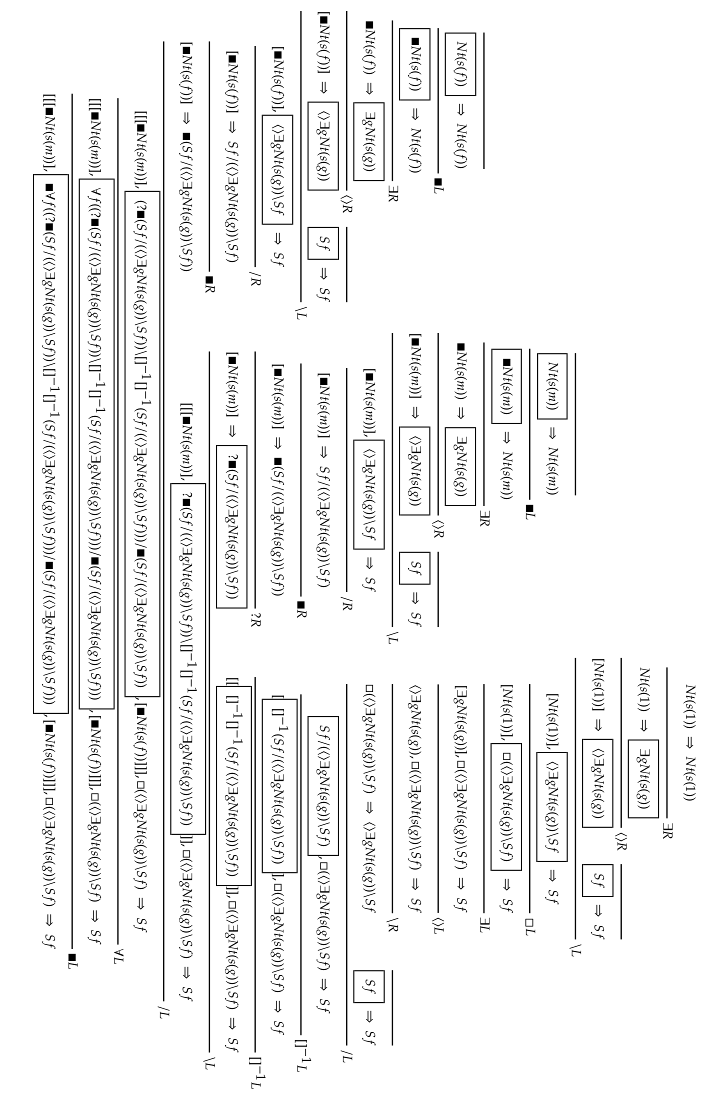
We continue with subject disjunction:

(243) **[[[john]+or+[mary]]]+sings** : Sf

Appropriate lexical lookup inserts a coordinator over lifted subject noun phrases (cf. Montague 1973[51]), essentially $(?X\backslash[\,]^{-1}[\,]^{-1}X)/X$ with $X = S/(N\backslash S)$.

(244) $[[[\blacksquare Nt(s(m)) : j], \blacksquare\forall f((?\blacksquare(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf))\backslash[]^{-1}[]^{-1}(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf)))/$
$\blacksquare(Sf/(\langle\rangle\exists gNt(s(g))\backslash Sf))) : (\mathbf{\Phi^{n+}}\ (s\ 0)\ or), [\blacksquare Nt(s(f)) : m]]],$
$\square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{~}\lambda A(Pres\ (\check{~}sing\ A)) \Rightarrow Sf$

There is the derivation:



In the same manner that we have seen before, in the left subderivation the righthand conjunct is analysed as of the lifted type, and in the middle subderivation the lefthand conjunct is analysed as of the lifted type; these are the same except for the eventual $?R$ of the latter, and they centre on the over right lowering of the higher order verb phrase argument into the antecedent where it subsequently applies as a functor. In the righthand derivation the brackets are checked and then application of the coordinate structure to its verb phrase context involves essentially derivation of the identity VP yields VP. This delivers the semantics:

(245) $[(Pres\ (\check{}sing\ j)) \vee (Pres\ (\check{}sing\ m))]$

14.1.6 Object coordination

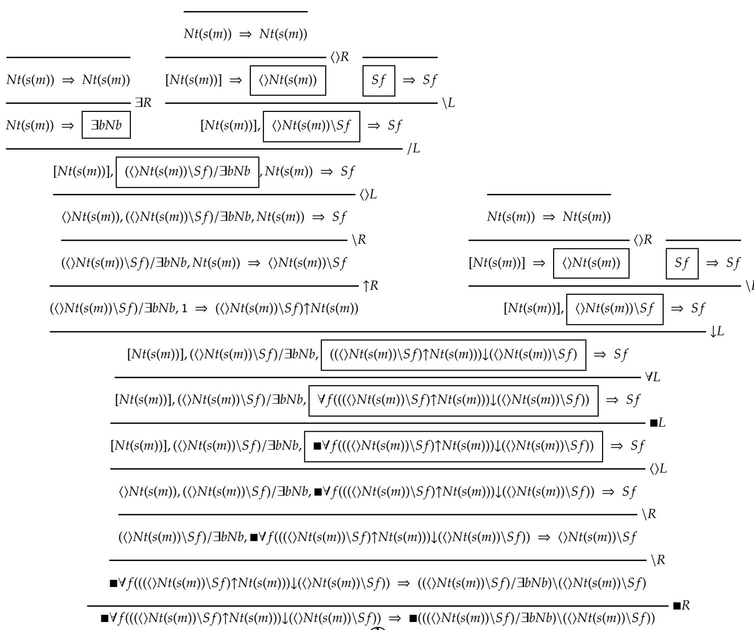
Object conjunction, including an object reflexive, is illustrated by:

(246) **[john]+loves+[[mary+and+himself]]** : Sf

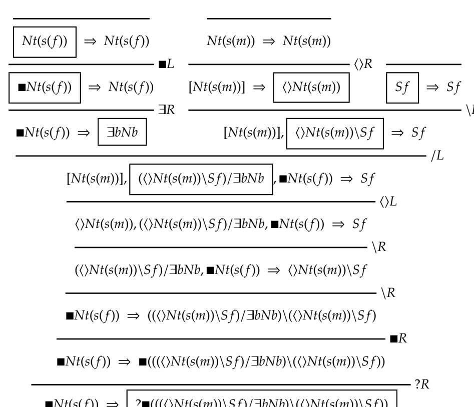
The coordination is basically $(?X\backslash[\,]^{-1}[\,]^{-1}X)/X$ with $X = ((N\backslash S)/N)\backslash(N\backslash S)$.

(247) $[\blacksquare Nt(s(m)) : j], \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}love\ A)\ B)), [[\blacksquare Nt(s(f)) : m,$
$\blacksquare\forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/$
$\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))) : (\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and),$
$\blacksquare\forall f(((\langle\rangle Nt(s(m))\backslash Sf)\uparrow Nt(s(m)))\downarrow(\langle\rangle Nt(s(m))\backslash Sf)) : \lambda C\lambda D((C\ D)\ D)]] \Rightarrow Sf$

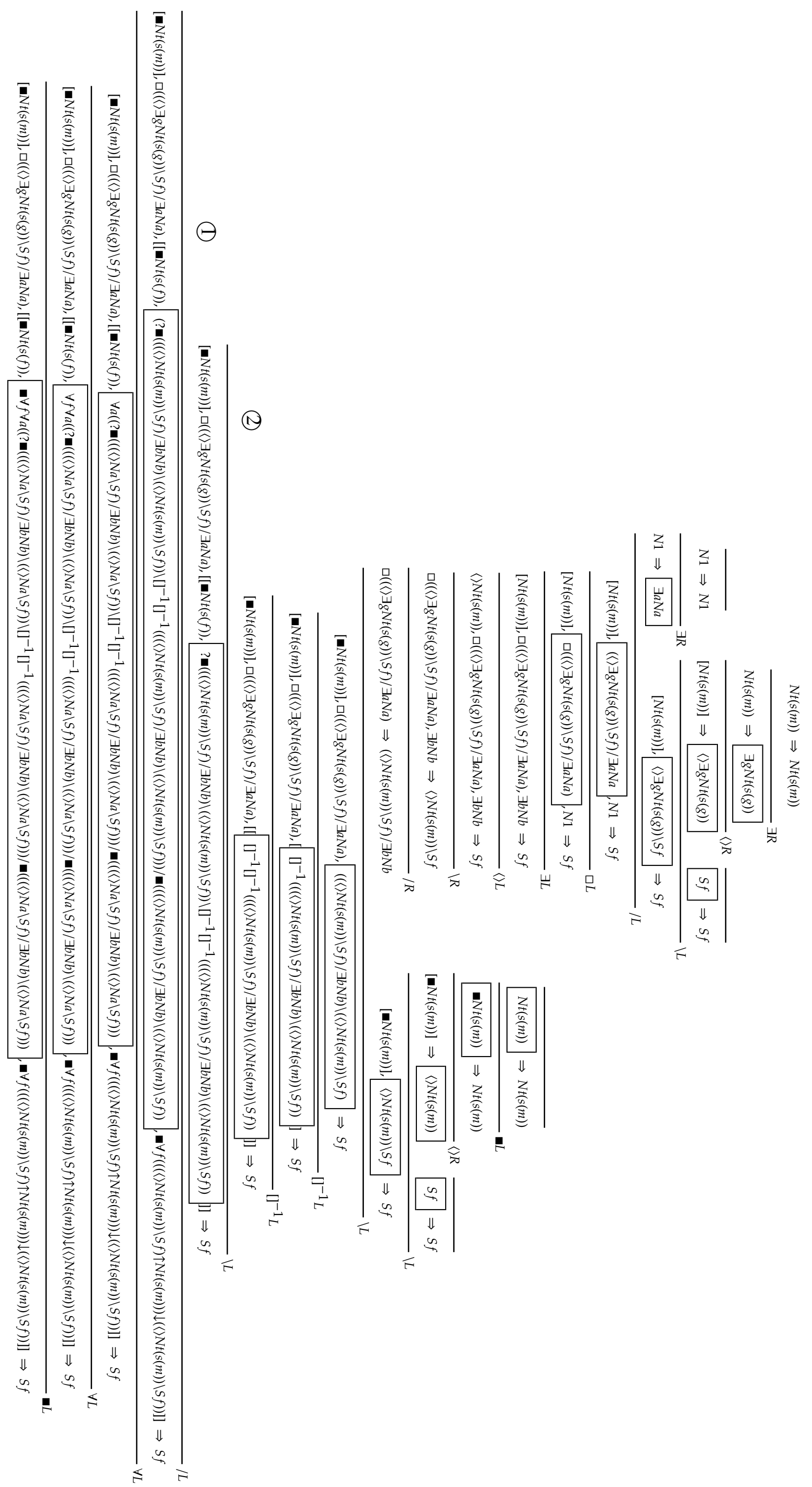
There is the derivation:



①



②



Again the main left (①), main middle (②) and main right subderivations are for the right conjunt, left conjunct, and context respectively. The subtree ① essentially derives that a subject-oriented reflexive $((N\backslash S)\uparrow N)\downarrow(N\backslash S)$ yields a lifted object type $((N\backslash S)/N)\backslash(N\backslash S)$:

(248)
$$\dfrac{\dfrac{\dfrac{\dfrac{(N\backslash S)/N, N \Rightarrow N\backslash S}{(N\backslash S)/N, 1 \Rightarrow (N\backslash S)\uparrow N}\uparrow R \qquad N, N\backslash S \Rightarrow S}{N, (N\backslash S)/N, ((N\backslash S)\uparrow N)\downarrow(N\backslash S) \Rightarrow S}\downarrow L}{(N\backslash S)/N, ((N\backslash S)\uparrow N)\downarrow(N\backslash S) \Rightarrow N\backslash S}\backslash R}{((N\backslash S)\uparrow N)\downarrow(N\backslash S) \Rightarrow ((N\backslash S)/N)\backslash(N\backslash S)}\backslash R$$

The derivation centres on the successive $\downarrow L$ and $\uparrow R$ half way up. The subtree ② essentially derives that a nominal N yields a lifted object type $((N\backslash S)/N)\backslash(N\backslash S)$:

(249) $N \Rightarrow ((N\backslash S)/N)\backslash(N\backslash S)$

The remaining main subtree checks the brackets and applies the coordinate structure to the transitive verb (deriving TV ⇒ TV) and subject (deriving N ⇒ N) contexts. All this delivers the correct semantics:

(250) $[(Pres\ ((\check{}love\ m)\ j)) \wedge (Pres\ ((\check{}love\ j)\ j))]$

14.1.7 Right node raising coordination

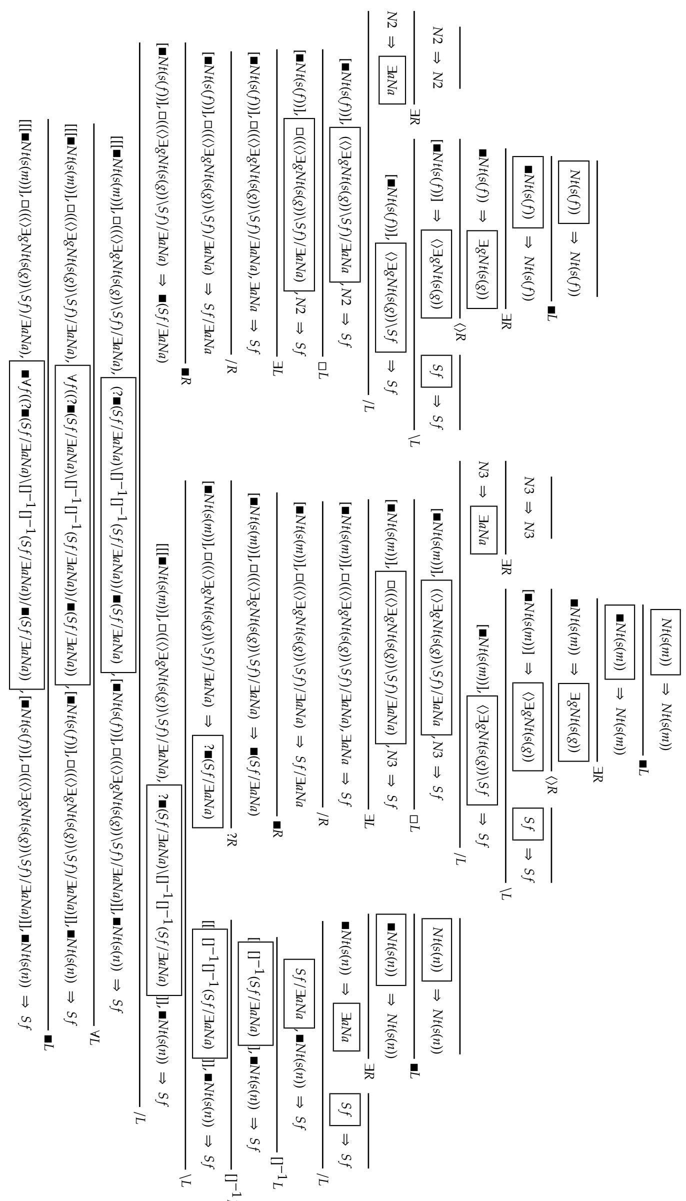
The next example is an instance of right node raising:

(251) **[[[john]+likes+and+[mary]+loves]]+london** : Sf

Appropriate lexical lookup yields the following where the coordinator type is essentially $(?X\backslash [\,]^{-1}[\,]^{-1}X)/X$ with $X = S/N$.

(252) $[[[\blacksquare Nt(s(m)) : j], \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}like\ A)\ B)),$
$\blacksquare\forall f((?\blacksquare(Sf/\exists aNa)\backslash[]^{-1}[]^{-1}(Sf/\exists aNa))/\blacksquare(Sf/\exists aNa)) : (\mathbf{\Phi^{n+}}\ (s\ 0)\ and),$
$[\blacksquare Nt(s(f)) : m], \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda C\lambda D(Pres\ ((\check{}love\ C)\ D))]],$
$\blacksquare Nt(s(n)) : l\ \Rightarrow\ Sf$

There is the derivation below.



As ever the left and middle subderivations are for the right and left conjuncts respectively, and involve in this case essentially the derivation of $N, TV \Rightarrow S/N$ which proceeds:

(253) $$\frac{N, TV, N \Rightarrow S}{N, TV \Rightarrow S/N}\ /R$$

The right, context, subderivation, applies the coordinate structure of type, basically, S/N to its right node raised N. All this assigns semantics:

(254) $[(Pres\ ((\check{}like\ l)\ j)) \wedge (Pres\ ((\check{}love\ l)\ m))]$

14.1.8 Argument-cluster left node raising coordination

The following example is of non-standard constituent argument-cluster coordination, or, left node raising:

(255) **[john]+gave+[[the+book+to+mary+and+the+cd+to+suzy]]** : Sf

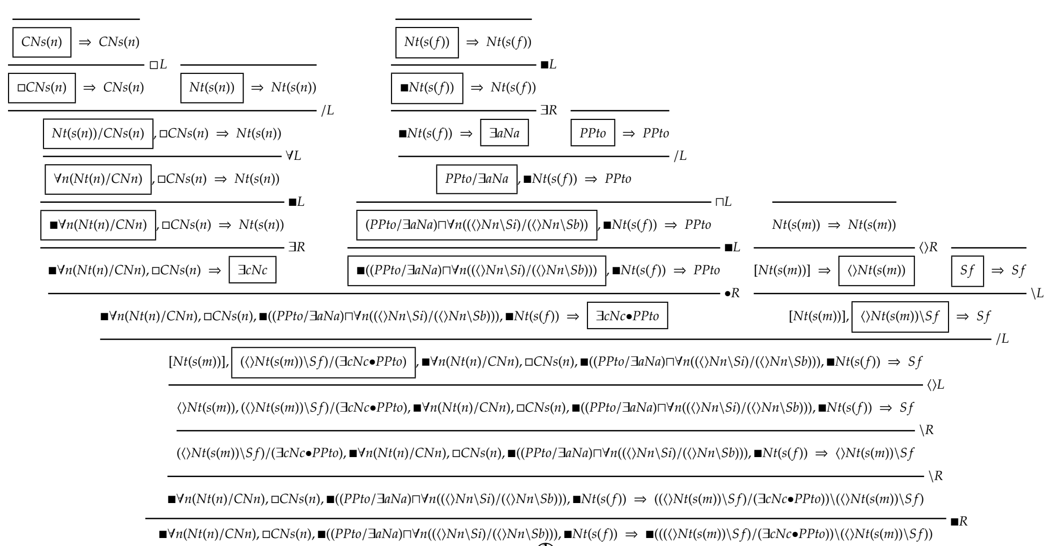
Appropriate lexical lookup yields the following where the coordinator type is essentially $(?X\backslash[\,]^{-1}[\,]^{-1}X)/X$ with $X = ((N\backslash S)/(N\bullet PP))\backslash(N\backslash S)$ (using the uncurried prepositional ditransitive verb type):

(256)
$$[\blacksquare Nt(s(m)) : j], \square(((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)) : \hat{}\lambda A\lambda B(Past\ (((\check{}give\ \pi_2 A)\ \pi_1 A)\ B)),$$
$$[[\blacksquare\forall n(Nt(n)/CNn) : \iota, \square CNs(n) : book, \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))) :$$
$$\lambda CC, \blacksquare Nt(s(f)) : m, \blacksquare\forall a\forall b\forall f((?\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf))\backslash$$
$$[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf))) :$$
$$(\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and), \blacksquare\forall n(Nt(n)/CNn) : \iota, \square CNs(n) : cd,$$
$$\blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))) : \lambda DD, \blacksquare Nt(s(f)) : s]] \ \Rightarrow\ Sf$$

This has the following derivation. In ① the righthand conjunct is analysed as essentially of the shape $((N\backslash S)/(N\bullet PP))\backslash(N\backslash S)$. The main action is in the initial unfolding of this succedent to yield a 'canonical' sequent:

(257)
$$\dfrac{\dfrac{N, (N\backslash S)/(N\bullet PP), N/CN, CN, PP, N \Rightarrow S}{(N\backslash S)/(N\bullet PP), N/CN, CN, PP, N \Rightarrow N\backslash S}\ \backslash R}{N/CN, CN, PP, N \Rightarrow ((N\backslash S)/(N\bullet PP))\backslash(N\backslash S)}\ \backslash R$$

Subderivation ② is exactly the same — except for the additional bottommost existential exponential right rule.Reading upwards in the main derivation, after modality elimination and instantiation of features on the coordinator type and application to the two conjuncts, the brackets are checked and there is application of the whole coordinate structure to the left node raised verb $(N\backslash S)/(N\bullet PP)$ (left subsubderivation) and to the subject N (right subsubderivation).



①

$$CNs(n) \Rightarrow CNs(n)$$
$$\Box CNs(n) \Rightarrow CNs(n) \quad \Box L \qquad Nt(s(n)) \Rightarrow Nt(s(n))$$
$$Nt(s(n))/CNs(n), \Box CNs(n) \Rightarrow Nt(s(n)) \quad /L$$
$$\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow Nt(s(n)) \quad \forall L$$
$$\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow Nt(s(n)) \quad \blacksquare L$$
$$\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow \exists cNc \quad \exists R$$

$$Nt(s(f)) \Rightarrow Nt(s(f))$$
$$\blacksquare Nt(s(f)) \Rightarrow Nt(s(f)) \quad \blacksquare L$$
$$\blacksquare Nt(s(f)) \Rightarrow \exists aNa \quad \exists R \qquad PPto \Rightarrow PPto$$
$$PPto/\exists aNa, \blacksquare Nt(s(f)) \Rightarrow PPto \quad /L$$
$$(PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb)), \blacksquare Nt(s(f)) \Rightarrow PPto \quad \sqcap L$$
$$\blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)) \Rightarrow PPto \quad \blacksquare L$$

$$\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)) \Rightarrow \exists cNc\bullet PPto \quad \bullet R$$

$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$[Nt(s(m))] \Rightarrow \langle\rangle Nt(s(m)) \quad \langle\rangle R \qquad Sf \Rightarrow Sf$$
$$[Nt(s(m))], \langle\rangle Nt(s(m))\backslash Sf \Rightarrow Sf \quad \backslash L$$

$$[Nt(s(m))], (\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)) \Rightarrow Sf \quad /L$$
$$\langle\rangle Nt(s(m)), (\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)) \Rightarrow Sf \quad \langle\rangle L$$
$$(\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)) \Rightarrow \langle\rangle Nt(s(m))\backslash Sf \quad \backslash R$$
$$\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)) \Rightarrow ((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf) \quad \backslash R$$
$$\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)) \Rightarrow \blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf)) \quad \blacksquare R$$
$$\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)) \Rightarrow ?\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf)) \quad ?R$$

②

$$N5 \Rightarrow N5$$
$$N5 \Rightarrow \exists bNb \quad \exists R \qquad PPto \Rightarrow PPto$$
$$N5, PPto \Rightarrow \exists bNb\bullet PPto \quad \bullet R$$

$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$Nt(s(m)) \Rightarrow \exists aNa \quad \exists R$$
$$[Nt(s(m))] \Rightarrow \langle\rangle\exists aNa \quad \langle\rangle R \qquad Sf \Rightarrow Sf$$
$$[Nt(s(m))], \langle\rangle\exists aNa\backslash Sf \Rightarrow Sf \quad \backslash L$$

$$[Nt(s(m))], (\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto), N5, PPto \Rightarrow Sf \quad /L$$
$$[Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), N5, PPto \Rightarrow Sf \quad \Box L$$
$$[Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), \exists cNc, PPto \Rightarrow Sf \quad \exists L$$
$$\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), \exists cNc, PPto \Rightarrow Sf \quad \langle\rangle L$$
$$\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), \exists cNc\bullet PPto \Rightarrow Sf \quad \bullet L$$
$$\Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), \exists cNc\bullet PPto \Rightarrow \langle\rangle Nt(s(m))\backslash Sf \quad \backslash R$$
$$\Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto) \quad /R$$

$$Nt(s(m)) \Rightarrow Nt(s(m))$$
$$\blacksquare Nt(s(m)) \Rightarrow Nt(s(m)) \quad \blacksquare L$$
$$[\blacksquare Nt(s(m))] \Rightarrow \langle\rangle Nt(s(m)) \quad \langle\rangle R \qquad Sf \Rightarrow Sf$$
$$[\blacksquare Nt(s(m))], \langle\rangle Nt(s(m))\backslash Sf \Rightarrow Sf \quad \backslash L$$

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), ((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf) \Rightarrow Sf \quad \backslash L$$
$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), [[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf))] \Rightarrow Sf \quad []^{-1}L$$
$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), [[[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf))]] \Rightarrow Sf \quad []^{-1}L$$

②

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)), ?\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf))]] \Rightarrow Sf \quad \backslash L$$

①

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)), (?\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf)))/\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf)), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f))]] \Rightarrow Sf \quad /L$$

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)), \forall f((?\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf)))/\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPto))\backslash(\langle\rangle Nt(s(m))\backslash Sf))), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f))]] \Rightarrow Sf \quad \forall L$$

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)), \forall b\forall f((?\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Nt(s(m))\backslash Sf)))/\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Nt(s(m))\backslash Sf))), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f))]] \Rightarrow Sf \quad \forall L$$

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)), \forall a\forall b\forall f((?\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf))), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f))]] \Rightarrow Sf \quad \forall L$$

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPto)), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f)), \blacksquare\forall a\forall b\forall f((?\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/(\exists cNc\bullet PPb))\backslash(\langle\rangle Na\backslash Sf))), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare((PPto/\exists aNa)\sqcap\forall n((\langle\rangle Nn\backslash Si)/(\langle\rangle Nn\backslash Sb))), \blacksquare Nt(s(f))]] \Rightarrow Sf \quad \blacksquare L$$

All this correctly assigns semantics:

(258) $[(Past\ (((\check{}give\ m)\ (\iota\ \check{}book))\ j)) \wedge (Past\ (((\check{}give\ s)\ (\iota\ \check{}cd))\ j))]$

14.1.9 Argument plus modifier left node raising coordination

The next example has LNR with arguments and adverbs in the conjuncts:

(259) **[john]+saw+[[mary+today+and+bill+yesterday]]** : Sf

Appropriate lexical lookup yields the following where the coordinator is essentially of the form $(?X\backslash[\,]^{-1}[\,]^{-1}X)/X$ where $X = ((N\backslash S)/N)\backslash(N\backslash S)$.[1]

(260) $[\blacksquare Nt(s(m)) : j], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)) :$
$\hat{}\lambda A\lambda B(Past\ ((A{-}{>}C.(\check{}seee\ C); D.(\check{}seet\ D))\ B)),$
$[[\blacksquare Nt(s(f)) : m, \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) : \hat{}\lambda E\lambda F(\check{}today\ (E\ F)),$
$\blacksquare\forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/$
$\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))) : (\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and), \blacksquare Nt(s(m)) : b,$
$\Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) : \hat{}\lambda G\lambda H(\check{}yesterday\ (G\ H))]] \ \Rightarrow\ Sf$

The example has the following derivation. In ① the righthand conjunct is analysed as essentially of the shape $((N\backslash S)/N)\backslash(N\backslash S)$. The main action is in the initial unfolding of this succedent to yield a ‘canonical’ sequent:

(261)
$$\dfrac{\dfrac{N, (N\backslash S)/N, N, (N\backslash S)\backslash(N\backslash S) \Rightarrow S}{(N\backslash S)/N, N, (N\backslash S)\backslash(N\backslash S) \Rightarrow N\backslash S}\backslash R}{N, (N\backslash S)\backslash(N\backslash S) \Rightarrow ((N\backslash S)/N)\backslash(N\backslash S)}\backslash R$$

The subtree ② for the lefthand conjunct is exactly the same — except for the bottommost existential exponential right rule and the gender of the object — hence it has been elided. The trunk of the main derivation and the checking of the bracket context are fairly standard by now. In the left and right subsubderivations the (polymorphic) verb type is shown to yield the left node raised transitive verb $(N\backslash S)/N$ and subject N coordinate structure arguments.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(m))} \Rightarrow Nt(s(m))}}{\boxed{\blacksquare Nt(s(m))} \Rightarrow Nt(s(m))}\blacksquare L}{\blacksquare Nt(s(m)) \Rightarrow \boxed{\exists bNb}}\exists R \quad \dfrac{\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))}}\langle\rangle R \quad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{[Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf}\backslash L}{[Nt(s(m))], \boxed{(\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb}, \blacksquare Nt(s(m)) \Rightarrow Sf}/L}{\langle\rangle Nt(s(m)), (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb, \blacksquare Nt(s(m)) \Rightarrow Sf}\langle\rangle L}{(\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb, \blacksquare Nt(s(m)) \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}\backslash R \quad \dfrac{\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))}}\langle\rangle R \quad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{[Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf}\backslash L}{[Nt(s(m))], (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb, \blacksquare Nt(s(m)), \boxed{(\langle\rangle Nt(s(m))\backslash Sf)\backslash(\langle\rangle Nt(s(m))\backslash Sf)} \Rightarrow Sf}\backslash L}{[Nt(s(m))], (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb, \blacksquare Nt(s(m)), \boxed{\forall f((\langle\rangle Nt(s(m))\backslash Sf)\backslash(\langle\rangle Nt(s(m))\backslash Sf))} \Rightarrow Sf}\forall L}{[Nt(s(m))], (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb, \blacksquare Nt(s(m)), \boxed{\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))} \Rightarrow Sf}\forall L}{[Nt(s(m))], (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb, \blacksquare Nt(s(m)), \boxed{\Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))} \Rightarrow Sf}\Box L}{\langle\rangle Nt(s(m)), (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb, \blacksquare Nt(s(m)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) \Rightarrow Sf}\langle\rangle L}{(\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb, \blacksquare Nt(s(m)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}\backslash R}{\blacksquare Nt(s(m)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) \Rightarrow ((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf)}\backslash R}{\blacksquare Nt(s(m)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) \Rightarrow \blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))}\blacksquare R$$

①

[1] Here ‘saw‘ is polymorphic between seeing an entity (*seee*) and seeing a proposition (*seet*). We shall see in Section 14.3 how this allows coordination over ‘unlike’ types. The coordination here is over regular transitive verbs, but we wish to show now the integration of node raising with the propensity for such other features.

$$\dfrac{\dfrac{\dfrac{N1 \Rightarrow N1}{N1 \Rightarrow \boxed{\exists aNa}}\exists R}{N1 \Rightarrow \boxed{\exists aNa \oplus CPthat}}\oplus R \qquad \dfrac{\dfrac{\dfrac{Nt(s(m)) \Rightarrow Nt(s(m))}{Nt(s(m)) \Rightarrow \boxed{\exists aNa}}\exists R}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists aNa}}\langle\rangle R \qquad \boxed{Sf} \Rightarrow Sf}{[Nt(s(m))], \boxed{\langle\rangle\exists aNa\backslash Sf} \Rightarrow Sf}\backslash L}{[Nt(s(m))], \boxed{(\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)}, N1 \Rightarrow Sf}/L$$

$$\dfrac{[Nt(s(m))], \boxed{(\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)}, N1 \Rightarrow Sf}{[Nt(s(m))], \boxed{\Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat))}, N1 \Rightarrow Sf}\Box L$$

$$\dfrac{}{[Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \exists bNb \Rightarrow Sf}\exists L$$

$$\dfrac{}{\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \exists bNb \Rightarrow Sf}\langle\rangle L$$

$$\dfrac{}{\Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \exists bNb \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}\backslash R$$

$$\dfrac{}{\Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb}/R$$

$$\dfrac{\dfrac{\dfrac{\boxed{Nt(s(m))} \Rightarrow Nt(s(m))}{\boxed{\blacksquare Nt(s(m))} \Rightarrow Nt(s(m))}\blacksquare L}{[\blacksquare Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))}}\langle\rangle R \qquad \boxed{Sf} \Rightarrow Sf}{[\blacksquare Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf}\backslash L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \boxed{((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf)} \Rightarrow Sf}\backslash L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), [\boxed{[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))}] \Rightarrow Sf}[]^{-1}L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), [[\boxed{[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))}]] \Rightarrow Sf}[]^{-1}L$$

$$\dfrac{\vdots}{\blacksquare Nt(s(f)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) \Rightarrow \boxed{?\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))}}?R$$

①

$$\dfrac{}{[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), [[\blacksquare Nt(s(f)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)), \boxed{?\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))}]] \Rightarrow Sf}\backslash L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), [[\blacksquare Nt(s(f)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)), \boxed{(?\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf)))/\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))}, \blacksquare Nt(s(m)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))]] \Rightarrow Sf}/L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), [[\blacksquare Nt(s(f)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)), \boxed{\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))}, \blacksquare Nt(s(m)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))]] \Rightarrow Sf}\forall L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), [[\blacksquare Nt(s(f)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)), \boxed{\forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))}, \blacksquare Nt(s(m)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))]] \Rightarrow Sf}\forall L$$

$$\dfrac{}{[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), [[\blacksquare Nt(s(f)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)), \boxed{\blacksquare\forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))}, \blacksquare Nt(s(m)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))]] \Rightarrow Sf}\blacksquare L$$

This delivers the semantics:

(262) $[(\check{}today\ (Past\ ((\check{}seee\ m)\ j))) \wedge (\check{}yesterday\ (Past\ ((\check{}seee\ b)\ j)))]$

14.1.10 Across-the-board extraction

This example is (medial) sentential across-the-board extraction:

(263) **man**+[[**that**+[[[**john**]+**saw**+**yesterday**+**and**+[**bill**]+**saw**+**today**]]]] : $CNs(m)$

Appropriate lexical lookup yields the semantically annotated sequent where the coordinator is essentially of the form $(?X\backslash[\,]^{-1}[\,]^{-1}X)/X$ where $X = S/!N$.

(264) $\square CNs(m) : man, [[\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf)) :$
$\lambda A\lambda B\lambda C[(B\ C) \wedge (A\ C)], [[[\blacksquare Nt(s(m)) : j], \square((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)) :$
$\hat{}\lambda D\lambda E(Past\ ((D\text{-->}F.(\check{}seee\ F); G.(\check{}seet\ G))\ E)), \square\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) :$
$\hat{}\lambda H\lambda I(\check{}yesterday\ (H\ I)), \blacksquare\forall a\forall f((?\blacksquare(Sf/!Na)\backslash[]^{-1}[]^{-1}(Sf/!Na))/\blacksquare(Sf/!Na)) :$
$(\mathbf{\Phi^{n+}}\ (s\ 0)\ and), [\blacksquare Nt(s(m)) : b], \square((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)) :$
$\hat{}\lambda J\lambda K(Past\ ((J\text{-->}L.(\check{}seee\ L); M.(\check{}seet\ M))\ K)),$
$\square\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) : \hat{}\lambda N\lambda O(\check{}today\ (N\ O))]]]] \ \Rightarrow\ CNs(m)$

The relative pronoun is essentially of the form $(CN\backslash CN)/((\langle\rangle N\sqcap !N)\backslash S)$ where the semantically inactively conjoined nominals $\langle\rangle N$ and $!N$ are for subject relativisation and object relativisation respectively. There is the following derivation. In ① the righthand conjunct is derived as a type of the shape $S/!N$. The universal exponential argument is lowered into the antecedent and into the stoup:

(265)
$$\dfrac{\dfrac{N; \ldots \Rightarrow S}{\ldots, !N \Rightarrow S}\ !L}{\ldots \Rightarrow S/!N}\ /R$$

The nominal percolates leftwards in the stoup into the minor premise at the application of the adverb to the verb phrase and leftwards again in the stoup into the minor premise at the application of the (polymorphic) transitive verb to its object: at the top leftmost subderivation $!P$ brings the nominal out of the stoup to fulfil the role of this object. Subtree ② which analyses the lefthand conjunct is exactly the same as ① — except for the bottommost existential exponential right rule — hence it has been elided. The principal new action in the main derivation occurs above ③ where the coordinate structure is supplied as the higher order relative pronoun argument. The succession has the form:

(266)
$$\dfrac{\dfrac{\dfrac{N; \ldots \Rightarrow S}{!N, \ldots \Rightarrow S}\ !L}{\langle\rangle N\sqcap !N, \ldots \Rightarrow S}\ \sqcap L}{\ldots \Rightarrow (\langle\rangle N\sqcap !N)\backslash S}\ \backslash R$$

Once the coordinate structure (here ‘. . . ’) is analysed as $S/!N$ the hypothetical nominal subtype which has entered the stoup yields the $!N$ in the subderivation at the top, which has the form:

(267)
$$\dfrac{\dfrac{\dfrac{N \Rightarrow N}{N; \Rightarrow N}\ !P}{N; \Rightarrow !N}\ !R \qquad S \Rightarrow S}{N; S/!N \Rightarrow S}\ /L$$

$$\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}$$

$$\dfrac{Nt(s(m)) \Rightarrow Nt(s(m))}{\boxed{Nt(s(m))}; \Rightarrow Nt(s(m))}\ !P$$

$$\dfrac{\boxed{Nt(s(m))}; \Rightarrow Nt(s(m))}{Nt(s(m)); \Rightarrow \boxed{\exists aNa}}\ \exists R$$

$$\dfrac{Nt(s(m)); \Rightarrow \boxed{\exists aNa}}{Nt(s(m)); \Rightarrow \boxed{\exists aNa \oplus CPthat}}\ \oplus R$$

$$\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}$$

$$\dfrac{Nt(s(m)) \Rightarrow Nt(s(m))}{Nt(s(m)) \Rightarrow \boxed{\exists aNa}}\ \exists R$$

$$\dfrac{Nt(s(m)) \Rightarrow \boxed{\exists aNa}}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle \exists aNa}}\ \langle\rangle R$$

$$\dfrac{}{\boxed{Sf} \Rightarrow Sf}$$

$$\dfrac{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle \exists aNa} \qquad \boxed{Sf} \Rightarrow Sf}{[Nt(s(m))], \boxed{\langle\rangle \exists aNa \backslash Sf} \Rightarrow Sf}\ \backslash L$$

$$\dfrac{Nt(s(m)); \Rightarrow \boxed{\exists aNa \oplus CPthat} \qquad [Nt(s(m))], \boxed{\langle\rangle \exists aNa \backslash Sf} \Rightarrow Sf}{Nt(s(m)); [Nt(s(m))], \boxed{(\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)} \Rightarrow Sf}\ /L$$

$$\dfrac{Nt(s(m)); [Nt(s(m))], \boxed{(\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)} \Rightarrow Sf}{Nt(s(m)); [Nt(s(m))], \boxed{\Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat))} \Rightarrow Sf}\ \Box L$$

$$\dfrac{Nt(s(m)); [Nt(s(m))], \boxed{\Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat))} \Rightarrow Sf}{Nt(s(m)); \langle\rangle Nt(s(m)), \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)) \Rightarrow Sf}\ \langle\rangle L$$

$$\dfrac{Nt(s(m)); \langle\rangle Nt(s(m)), \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)) \Rightarrow Sf}{Nt(s(m)); \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)) \Rightarrow \langle\rangle Nt(s(m)) \backslash Sf}\ \backslash R$$

$$\dfrac{}{\boxed{Nt(s(m))} \Rightarrow Nt(s(m))}$$

$$\dfrac{\boxed{Nt(s(m))} \Rightarrow Nt(s(m))}{\boxed{\blacksquare Nt(s(m))} \Rightarrow Nt(s(m))}\ \blacksquare L$$

$$\dfrac{\boxed{\blacksquare Nt(s(m))} \Rightarrow Nt(s(m))}{[\blacksquare Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))}}\ \langle\rangle R$$

$$\dfrac{}{\boxed{Sf} \Rightarrow Sf}$$

$$\dfrac{[\blacksquare Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))} \qquad \boxed{Sf} \Rightarrow Sf}{[\blacksquare Nt(s(m))], \boxed{\langle\rangle Nt(s(m)) \backslash Sf} \Rightarrow Sf}\ \backslash L$$

$$\dfrac{Nt(s(m)); \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)) \Rightarrow \langle\rangle Nt(s(m)) \backslash Sf \qquad [\blacksquare Nt(s(m))], \boxed{\langle\rangle Nt(s(m)) \backslash Sf} \Rightarrow Sf}{Nt(s(m)); [\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \boxed{(\langle\rangle Nt(s(m)) \backslash Sf) \backslash (\langle\rangle Nt(s(m)) \backslash Sf)} \Rightarrow Sf}\ \backslash L$$

$$\dfrac{Nt(s(m)); [\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \boxed{(\langle\rangle Nt(s(m)) \backslash Sf) \backslash (\langle\rangle Nt(s(m)) \backslash Sf)} \Rightarrow Sf}{Nt(s(m)); [\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \boxed{\forall f((\langle\rangle Nt(s(m)) \backslash Sf) \backslash (\langle\rangle Nt(s(m)) \backslash Sf))} \Rightarrow Sf}\ \forall L$$

$$\dfrac{Nt(s(m)); [\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \boxed{\forall f((\langle\rangle Nt(s(m)) \backslash Sf) \backslash (\langle\rangle Nt(s(m)) \backslash Sf))} \Rightarrow Sf}{Nt(s(m)); [\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \boxed{\forall a \forall f((\langle\rangle Na \backslash Sf) \backslash (\langle\rangle Na \backslash Sf))} \Rightarrow Sf}\ \forall L$$

$$\dfrac{Nt(s(m)); [\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \boxed{\forall a \forall f((\langle\rangle Na \backslash Sf) \backslash (\langle\rangle Na \backslash Sf))} \Rightarrow Sf}{Nt(s(m)); [\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \boxed{\Box \forall a \forall f((\langle\rangle Na \backslash Sf) \backslash (\langle\rangle Na \backslash Sf))} \Rightarrow Sf}\ \Box L$$

$$\dfrac{Nt(s(m)); [\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \boxed{\Box \forall a \forall f((\langle\rangle Na \backslash Sf) \backslash (\langle\rangle Na \backslash Sf))} \Rightarrow Sf}{[\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \Box \forall a \forall f((\langle\rangle Na \backslash Sf) \backslash (\langle\rangle Na \backslash Sf)), !Nt(s(m)) \Rightarrow Sf}\ !L$$

$$\dfrac{[\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \Box \forall a \forall f((\langle\rangle Na \backslash Sf) \backslash (\langle\rangle Na \backslash Sf)), !Nt(s(m)) \Rightarrow Sf}{[\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \Box \forall a \forall f((\langle\rangle Na \backslash Sf) \backslash (\langle\rangle Na \backslash Sf)) \Rightarrow Sf/!Nt(s(m))}\ /R$$

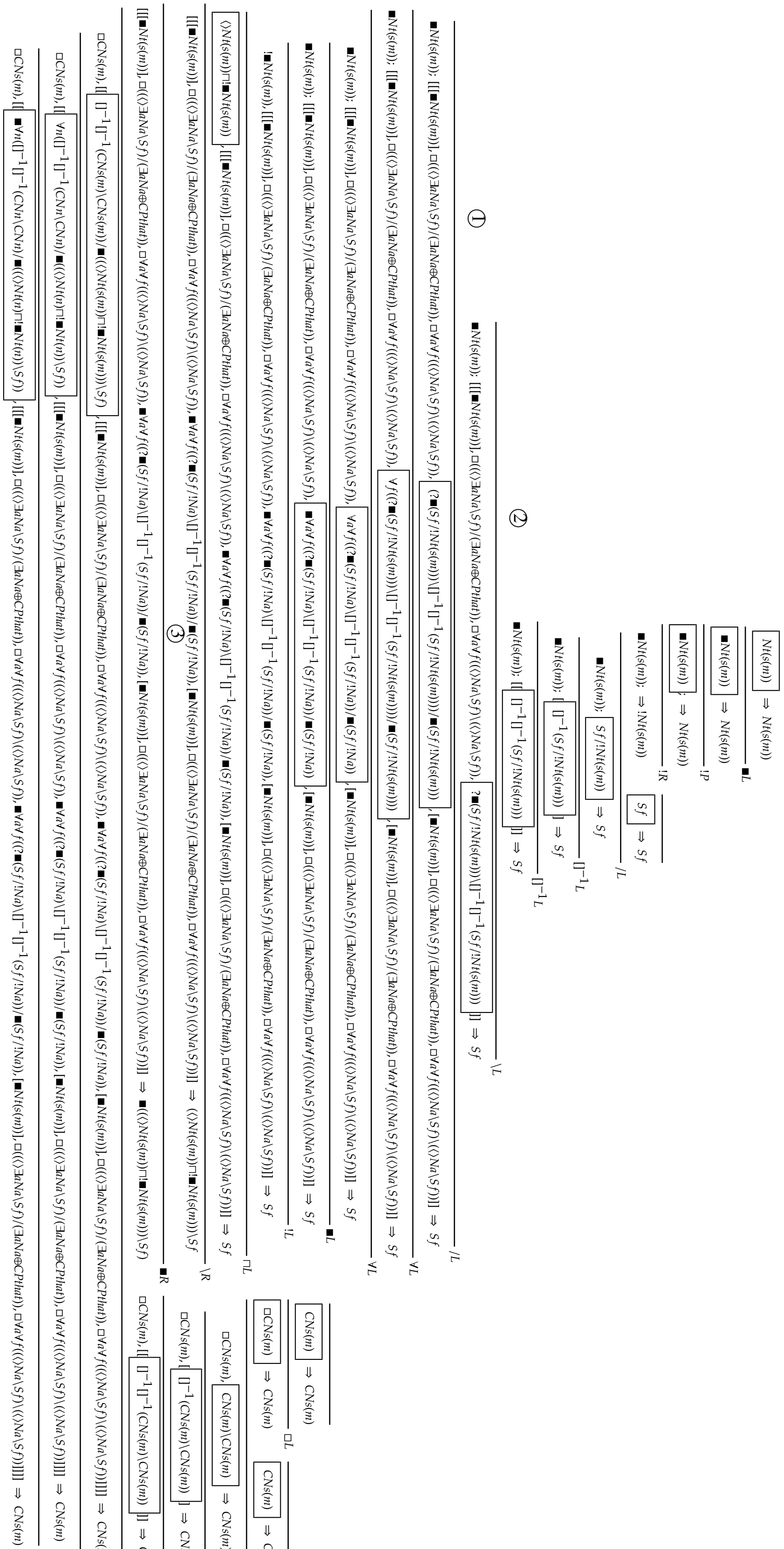
$$\dfrac{[\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \Box \forall a \forall f((\langle\rangle Na \backslash Sf) \backslash (\langle\rangle Na \backslash Sf)) \Rightarrow Sf/!Nt(s(m))}{[\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \Box \forall a \forall f((\langle\rangle Na \backslash Sf) \backslash (\langle\rangle Na \backslash Sf)) \Rightarrow \blacksquare(Sf/!Nt(s(m)))}\ \blacksquare R$$

①

$$\vdots$$

$$\dfrac{\vdots}{[\blacksquare Nt(s(m))], \Box((\langle\rangle \exists aNa \backslash Sf)/(\exists aNa \oplus CPthat)), \Box \forall a \forall f((\langle\rangle Na \backslash Sf) \backslash (\langle\rangle Na \backslash Sf)) \Rightarrow \boxed{?\blacksquare(Sf/!Nt(s(m)))}}\ ?R$$

②



This delivers semantics:

(268) $\lambda C[(\check{}\mathit{man}\ C) \wedge [(\check{}\mathit{yesterday}\ (\mathit{Past}\ ((\check{}\mathit{seee}\ C)\ j))) \wedge (\check{}\mathit{today}\ (\mathit{Past}\ ((\check{}\mathit{seee}\ C)\ b)))]]$

Note that (with bracket modalities) the CSC is respected in TLG as in phrase structure grammar and categorial grammar formalisms assuming like type coordination schemata, because there is no coordinator type instance conjoining sentences only one of which contains a gap.

14.2 Iterated coordination

We consider examples of iterated coordination.

14.2.1 Iterated coordination of addicity zero

Minimally we have the example:

(269) $[[[\mathbf{john}]+\mathbf{walks}+[\mathbf{mary}]+\mathbf{talks}+\mathbf{and}+[\mathbf{bill}]+\mathbf{sings}]] : Sf$

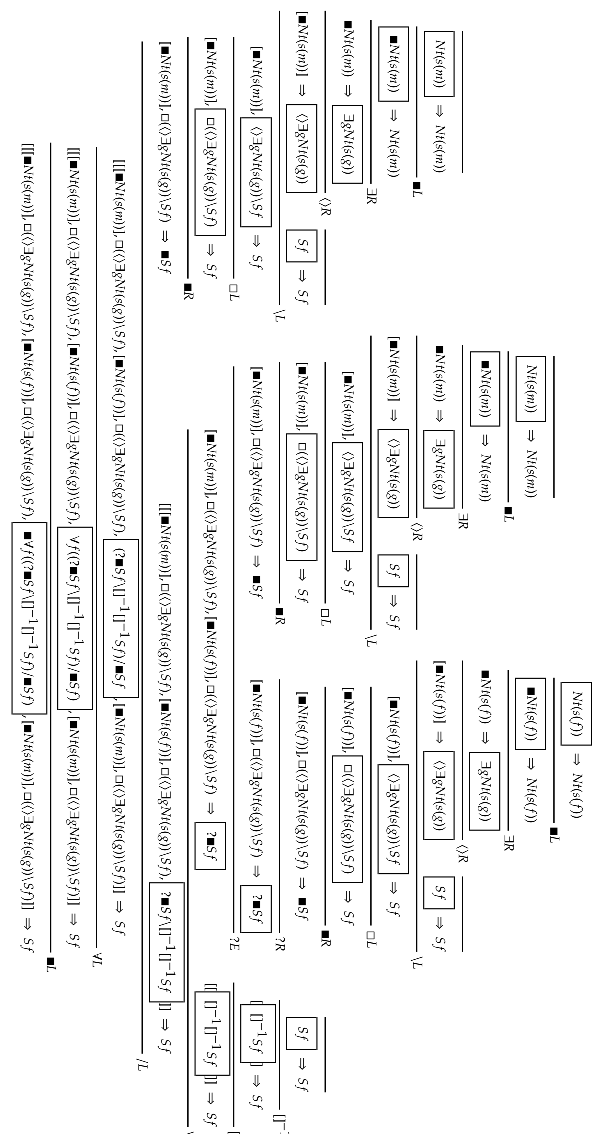
Appropriate lexical lookup yields:

(270)
$$\begin{array}{l}[[[\blacksquare Nt(s(m)) : j], \square(\langle\rangle\exists g Nt(s(g))\backslash Sf) : \hat{}\lambda A(\mathit{Pres}\ (\check{}\mathit{walk}\ A)),\\ [\blacksquare Nt(s(f)) : m], \square(\langle\rangle\exists g Nt(s(g))\backslash Sf) : \hat{}\lambda B(\mathit{Pres}\ (\check{}\mathit{talk}\ B)),\\ \blacksquare\forall f((?\blacksquare Sf\backslash[]^{-1}[]^{-1}Sf)/\blacksquare Sf) : (\mathbf{\Phi^{n+}}\ \mathit{0}\ \mathit{and}), [\blacksquare Nt(s(m)) : b],\\ \square(\langle\rangle\exists g Nt(s(g))\backslash Sf) : \hat{}\lambda C(\mathit{Pres}\ (\check{}\mathit{sing}\ C))]] \ \Rightarrow\ Sf\end{array}$$

The coordination combinator ($\mathbf{\Phi^{n+}}$ 0 *and*) is such that:

(271)
$$\begin{array}{l}(((\mathbf{\Phi^{n+}}\ 0\ \mathit{and})\ x)\ [y, z]) =\\ [y \wedge (((\mathbf{\Phi^{n+}}\ 0\ \mathit{and})\ x)\ [z])] =\\ [y \wedge [z \wedge x]]\end{array}$$

There is the derivation:



This delivers semantics:

(272) $[(Pres\ (\check{}walk\ j)) \wedge [(Pres\ (\check{}talk\ m)) \wedge (Pres\ (\check{}sing\ b))]]$

14.2.2 Iterated coordination of addicity one

There is the example of verb phrase iterated coordination:

(273) **[john]+[[walks+talks+and+sings]]** : Sf

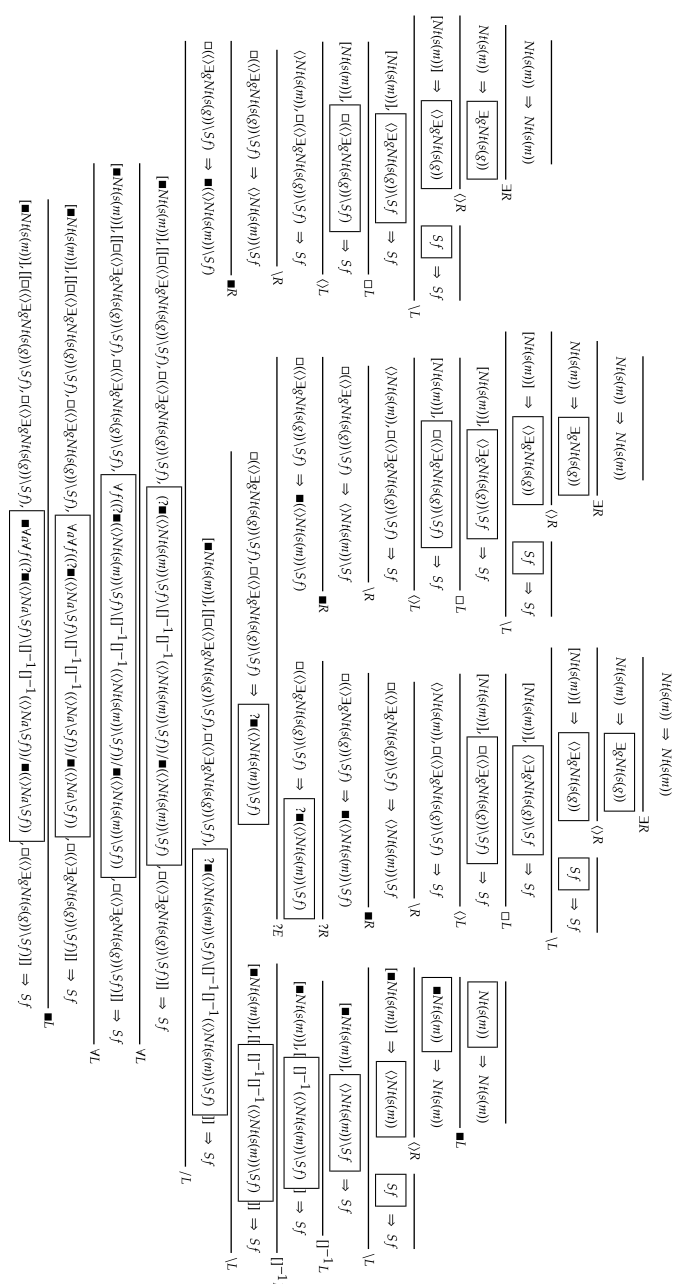
Appropriate lexical insertion yields:

(274) $[\blacksquare Nt(s(m)) : j], [[\square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda A(Pres\ (\check{}walk\ A)),$
$\square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda B(Pres\ (\check{}talk\ B)),$
$\blacksquare\forall a\forall f((?\blacksquare(\langle\rangle Na\backslash Sf)\backslash[]^{-1}[]^{-1}(\langle\rangle Na\backslash Sf))/\blacksquare(\langle\rangle Na\backslash Sf)) : (\mathbf{\Phi^{n+}}\ (s\ 0)\ and),$
$\square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda C(Pres\ (\check{}sing\ C))]] \Rightarrow Sf$

The coordinator lexical semantics ($\mathbf{\Phi^{n+}}$ $(s\ 0)$ *and*) is such that:

(275) $((((\mathbf{\Phi^{n+}}\ (s\ 0)\ \mathit{and})\ x)\ [y,z])\ w) =$
$(((\mathbf{\Phi^{n+}}\ 0\ \mathit{and})\ (x\ w))\ (\alpha^{+}\ [y,z]\ w)) =$
$(((\mathbf{\Phi^{n+}}\ 0\ \mathit{and})\ (x\ w))\ [(y\ w)|(\alpha^{+}\ [z]\ w)]) =$
$(((\mathbf{\Phi^{n+}}\ 0\ \mathit{and})\ (x\ w))\ [(y\ w),(z\ w)]) =$
$[(y\ w) \wedge (((\mathbf{\Phi^{n+}}\ 0\ \mathit{and})\ (x\ w))\ [(z\ w)])] =$
$[(y\ w) \wedge [(z\ w) \wedge (x\ w)]]$

There is the derivation:



This delivers semantics:

(276) $[(\mathit{Pres}\ (\check{}\mathit{walk}\ j)) \wedge [(\mathit{Pres}\ (\check{}\mathit{talk}\ j)) \wedge (\mathit{Pres}\ (\check{}\mathit{sing}\ j))]]$

14.2.3 Iterated coordination of addicity two

There is the example of transitive verb phrase iterated coordination:

(277) **[john]+[[praises+likes+and+will+love]]+london** : Sf

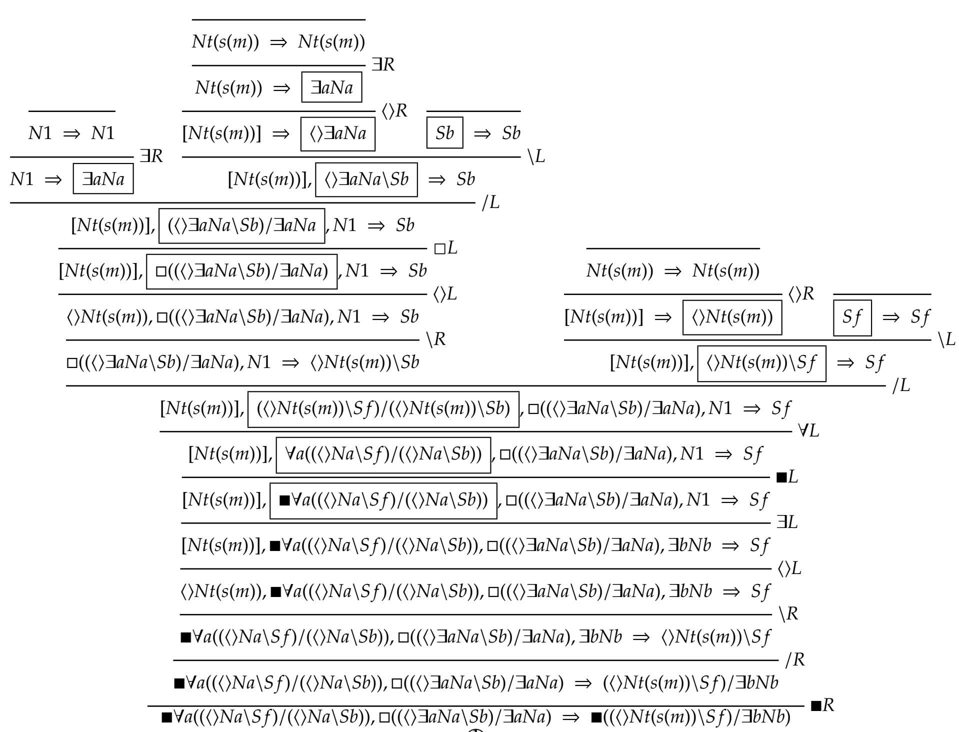
Lexical insertion yields:

(278) $[\blacksquare Nt(s(m)) : j], [[\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}praise\ A)\ B)),$
$\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda C\lambda D(Pres\ ((\check{}like\ C)\ D)),$
$\blacksquare\forall f\forall a((?\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash[\,]^{-1}[\,]^{-1}((\langle\rangle Na\backslash Sf)/\exists bNb))/\blacksquare((\langle\rangle Na\backslash Sf)/\exists bNb)) :$
$(\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and), \blacksquare\forall a((\langle\rangle Na\backslash Sf)/(\langle\rangle Na\backslash Sb)) : \lambda E\lambda F(Fut\ (E\ F)),$
$\Box((\langle\rangle\exists aNa\backslash Sb)/\exists aNa) : \hat{}\lambda G\lambda H((\check{}love\ G)\ H)]], \blacksquare Nt(s(n)) : l \ \Rightarrow\ Sf$

The coordination combinator semantics is such that:

(279) $(((((\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and)\ x)\ [y,z])\ w)\ u) =$
$((((\mathbf{\Phi^{n+}}\ (s\ 0)\ and)\ (x\ w))\ (\alpha^{+}\ [y,z]\ w))\ u) =$
$((((\mathbf{\Phi^{n+}}\ (s\ 0)\ and)\ (x\ w))\ [(y\ w),(z\ w)])\ u) =$
$(((\mathbf{\Phi^{n+}}\ 0\ and)\ ((x\ w)\ u))\ (\alpha^{+}\ [(y\ w),(z\ w)]\ u)) =$
$(((\mathbf{\Phi^{n+}}\ 0\ and)\ ((x\ w)\ u))\ [((y\ w)\ u),((z\ w)\ u)]) =$
$[((y\ w)\ u) \wedge [((z\ w)\ u) \wedge ((x\ w)\ u)]]$

There is the derivation:



①

$$\dfrac{}{N1 \Rightarrow N1} \qquad \dfrac{N1 \Rightarrow N1}{N1 \Rightarrow \boxed{\exists aNa}}\ \exists R$$

$$\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))} \qquad \dfrac{Nt(s(m)) \Rightarrow Nt(s(m))}{Nt(s(m)) \Rightarrow \boxed{\exists gNt(s(g))}}\ \exists R \qquad \dfrac{Nt(s(m)) \Rightarrow \boxed{\exists gNt(s(g))}}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g))}}\ \langle\rangle R$$

$$\dfrac{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g))} \qquad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{[Nt(s(m))], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} \Rightarrow Sf}\ \backslash L$$

$$\dfrac{N1 \Rightarrow \boxed{\exists aNa} \qquad [Nt(s(m))], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} \Rightarrow Sf}{[Nt(s(m))], \boxed{(\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa}, N1 \Rightarrow Sf}\ /L$$

$$\dfrac{[Nt(s(m))], \boxed{(\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa}, N1 \Rightarrow Sf}{[Nt(s(m))], \boxed{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)}, N1 \Rightarrow Sf}\ \Box L$$

$$\dfrac{[Nt(s(m))], \boxed{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)}, N1 \Rightarrow Sf}{[Nt(s(m))], \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf}\ \exists L$$

$$\dfrac{[Nt(s(m))], \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf}{\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf}\ \langle\rangle L$$

$$\dfrac{\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf}{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}\ \backslash R$$

$$\dfrac{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb}\ /R$$

$$\dfrac{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb}{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow \blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)}\ \blacksquare R$$

$$\dfrac{}{N2 \Rightarrow N2} \qquad \dfrac{N2 \Rightarrow N2}{N2 \Rightarrow \boxed{\exists aNa}}\ \exists R$$

$$\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))} \qquad \dfrac{Nt(s(m)) \Rightarrow Nt(s(m))}{Nt(s(m)) \Rightarrow \boxed{\exists gNt(s(g))}}\ \exists R \qquad \dfrac{Nt(s(m)) \Rightarrow \boxed{\exists gNt(s(g))}}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g))}}\ \langle\rangle R$$

$$\dfrac{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g))} \qquad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{[Nt(s(m))], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} \Rightarrow Sf}\ \backslash L$$

$$\dfrac{N2 \Rightarrow \boxed{\exists aNa} \qquad [Nt(s(m))], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} \Rightarrow Sf}{[Nt(s(m))], \boxed{(\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa}, N2 \Rightarrow Sf}\ /L$$

$$\dfrac{[Nt(s(m))], \boxed{(\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa}, N2 \Rightarrow Sf}{[Nt(s(m))], \boxed{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)}, N2 \Rightarrow Sf}\ \Box L$$

$$\dfrac{[Nt(s(m))], \boxed{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa)}, N2 \Rightarrow Sf}{[Nt(s(m))], \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf}\ \exists L$$

$$\dfrac{[Nt(s(m))], \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf}{\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf}\ \langle\rangle L$$

$$\dfrac{\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf}{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}\ \backslash R$$

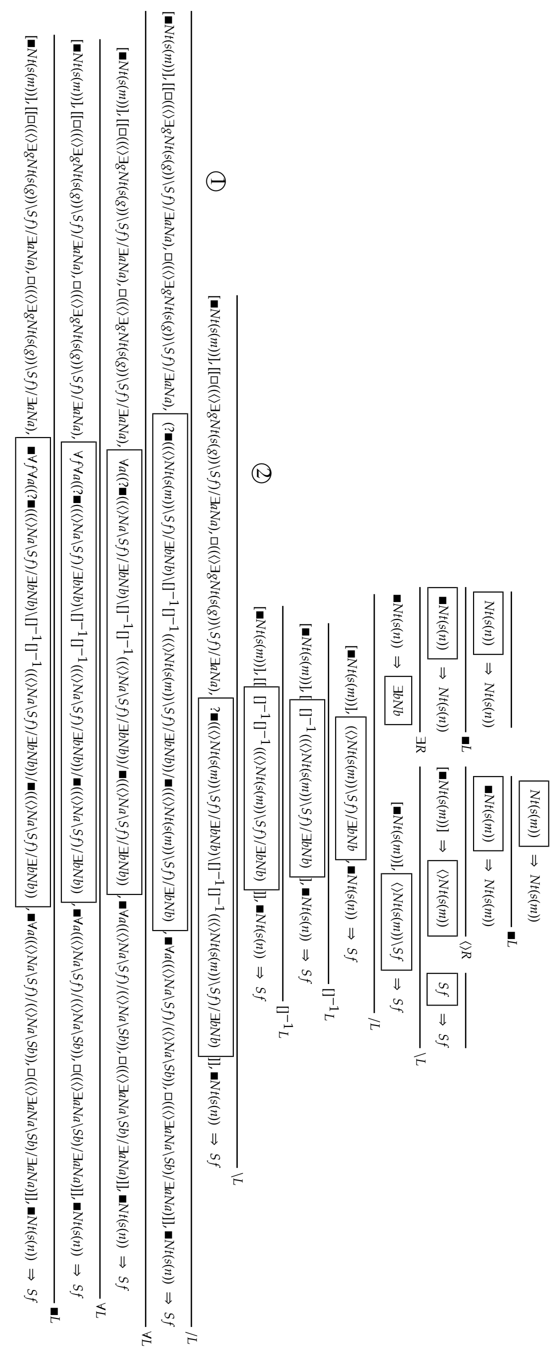
$$\dfrac{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \exists bNb \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb}\ /R$$

$$\dfrac{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb}{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow \blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)}\ \blacksquare R$$

$$\dfrac{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow \blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)}{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow \boxed{?\blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)}}\ ?R$$

$$\dfrac{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow \blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb) \qquad \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow \boxed{?\blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)}}{\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa), \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) \Rightarrow \boxed{?\blacksquare((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)}}\ ?E$$

②



All this assigns the correct semantics:

(280) $[(Pres\ ((\check{}praise\ l)\ j)) \wedge [(Pres\ ((\check{}like\ l)\ j)) \wedge (Fut\ ((\check{}love\ l)\ j))]]$

14.3 Coordination of unlike types

In the following we have coordinate unlike types with nominal and adjectival complementation of *is*.

(281) **[bond]**+**is**+**[[007+and+teetotal]]** : Sf

Together with a suitable coordinator type, a polymorphic assignment to the copula of the form $(N\backslash S)/(N\oplus(CN/CN))$ predicts such coordination under a *like* type scheme (Morrill 1990[57]; Johnson and Bayer 1995[31]; Bayer 1996[10]). Lexical lookup of types yields the following, where the coordinator is of the form $(X\backslash[\,]^{-1}[\,]^{-1}X)/X$ where $X = (N\backslash S)/(N\oplus((CN/CN)\sqcup(CN\backslash CN)))$.

(282) $[\blacksquare Nt(s(m)) : b], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I))) :$
$\lambda A\lambda B(Pres\ (A\text{->}C.[B = C]; D.((D\ \lambda E[E = B])\ B))), [[\blacksquare\forall gNt(s(g)) : 007,$
$\blacksquare\forall f\forall a((\blacksquare((((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))))\backslash(\langle\rangle Na\backslash Sf))\backslash$
$[]^{-1}[]^{-1}((((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))))\backslash(\langle\rangle Na\backslash Sf)))/$
$\blacksquare((((\langle\rangle Na\backslash Sf)/(\exists bNb\oplus\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))))\backslash(\langle\rangle Na\backslash Sf))) :$
$\lambda F\lambda G\lambda H\lambda I[((G\ H)\ I)\wedge((F\ H)\ I)], \Box\forall n(CNn/CNn) : {}^\wedge\lambda J\lambda K[(J\ K)\wedge({}^\vee teetotal\ K)]]]$
$\Rightarrow\ Sf$

The derivation is as shown below.

$$\frac{CNA\Rightarrow CNA \qquad \boxed{CNA}\Rightarrow CNA}{\boxed{CNA/CNA}, CNA\Rightarrow CNA}\ /L$$
$$\frac{\boxed{CNA/CNA}, CNA\Rightarrow CNA}{\boxed{\forall n(CNn/CNn)}, CNA\Rightarrow CNA}\ \forall L$$
$$\frac{\boxed{\forall n(CNn/CNn)}, CNA\Rightarrow CNA}{\boxed{\Box\forall n(CNn/CNn)}, CNA\Rightarrow CNA}\ \Box L$$
$$\frac{\boxed{\Box\forall n(CNn/CNn)}, CNA\Rightarrow CNA}{\Box\forall n(CNn/CNn)\Rightarrow\boxed{CNA/CNA}}\ \sqcup R$$
$$\frac{\Box\forall n(CNn/CNn)\Rightarrow\boxed{CNA/CNA}}{\Box\forall n(CNn/CNn)\Rightarrow\boxed{\exists g(CNg/CNg)}}\ \exists R$$
$$\frac{\Box\forall n(CNn/CNn)\Rightarrow\boxed{\exists g(CNg/CNg)}}{\Box\forall n(CNn/CNn)\Rightarrow\boxed{\exists bNb\oplus\exists g(CNg/CNg)}}\ \oplus R$$
$$\frac{Nt(s(m))\Rightarrow Nt(s(m))}{[Nt(s(m))]\Rightarrow\boxed{\langle\rangle Nt(s(m))}}\ \langle\rangle R$$
$$\frac{[Nt(s(m))]\Rightarrow\boxed{\langle\rangle Nt(s(m))} \qquad \boxed{Sf}\Rightarrow Sf}{[Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf}\Rightarrow Sf}\ \backslash L$$
$$\frac{\Box\forall n(CNn/CNn)\Rightarrow\boxed{\exists bNb\oplus\exists g(CNg/CNg)} \qquad [Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf}\Rightarrow Sf}{[Nt(s(m))], \boxed{(\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg))}, \Box\forall n(CNn/CNn)\Rightarrow Sf}\ /L$$
$$\frac{[Nt(s(m))], \boxed{(\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg))}, \Box\forall n(CNn/CNn)\Rightarrow Sf}{\langle\rangle Nt(s(m)), (\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)), \Box\forall n(CNn/CNn)\Rightarrow Sf}\ \langle\rangle L$$
$$\frac{\langle\rangle Nt(s(m)), (\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)), \Box\forall n(CNn/CNn)\Rightarrow Sf}{(\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)), \Box\forall n(CNn/CNn)\Rightarrow\langle\rangle Nt(s(m))\backslash Sf}\ \backslash R$$
$$\frac{(\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)), \Box\forall n(CNn/CNn)\Rightarrow\langle\rangle Nt(s(m))\backslash Sf}{\Box\forall n(CNn/CNn)\Rightarrow((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf)}\ \backslash R$$
$$\frac{\Box\forall n(CNn/CNn)\Rightarrow((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf)}{\Box\forall n(CNn/CNn)\Rightarrow\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf))}\ \blacksquare R$$

①

$$\frac{\boxed{Nt(s(A))}\Rightarrow Nt(s(A))}{\boxed{\forall gNt(s(g))}\Rightarrow Nt(s(A))}\ \forall L$$
$$\frac{\boxed{\forall gNt(s(g))}\Rightarrow Nt(s(A))}{\boxed{\blacksquare\forall gNt(s(g))}\Rightarrow Nt(s(A))}\ \blacksquare L$$
$$\frac{\boxed{\blacksquare\forall gNt(s(g))}\Rightarrow Nt(s(A))}{\blacksquare\forall gNt(s(g))\Rightarrow\boxed{\exists bNb}}\ \exists R$$
$$\frac{\blacksquare\forall gNt(s(g))\Rightarrow\boxed{\exists bNb}}{\blacksquare\forall gNt(s(g))\Rightarrow\boxed{\exists bNb\oplus\exists g(CNg/CNg)}}\ \oplus R$$
$$\frac{Nt(s(m))\Rightarrow Nt(s(m))}{[Nt(s(m))]\Rightarrow\boxed{\langle\rangle Nt(s(m))}}\ \langle\rangle R$$
$$\frac{[Nt(s(m))]\Rightarrow\boxed{\langle\rangle Nt(s(m))} \qquad \boxed{Sf}\Rightarrow Sf}{[Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf}\Rightarrow Sf}\ \backslash L$$
$$\frac{\blacksquare\forall gNt(s(g))\Rightarrow\boxed{\exists bNb\oplus\exists g(CNg/CNg)} \qquad [Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf}\Rightarrow Sf}{[Nt(s(m))], \boxed{(\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg))}, \blacksquare\forall gNt(s(g))\Rightarrow Sf}\ /L$$
$$\frac{[Nt(s(m))], \boxed{(\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg))}, \blacksquare\forall gNt(s(g))\Rightarrow Sf}{\langle\rangle Nt(s(m)), (\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)), \blacksquare\forall gNt(s(g))\Rightarrow Sf}\ \langle\rangle L$$
$$\frac{\langle\rangle Nt(s(m)), (\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)), \blacksquare\forall gNt(s(g))\Rightarrow Sf}{(\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)), \blacksquare\forall gNt(s(g))\Rightarrow\langle\rangle Nt(s(m))\backslash Sf}\ \backslash R$$
$$\frac{(\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)), \blacksquare\forall gNt(s(g))\Rightarrow\langle\rangle Nt(s(m))\backslash Sf}{\blacksquare\forall gNt(s(g))\Rightarrow((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf)}\ \backslash R$$
$$\frac{\blacksquare\forall gNt(s(g))\Rightarrow((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf)}{\blacksquare\forall gNt(s(g))\Rightarrow\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/(\exists bNb\oplus\exists g(CNg/CNg)))\backslash(\langle\rangle Nt(s(m))\backslash Sf))}\ \blacksquare R$$

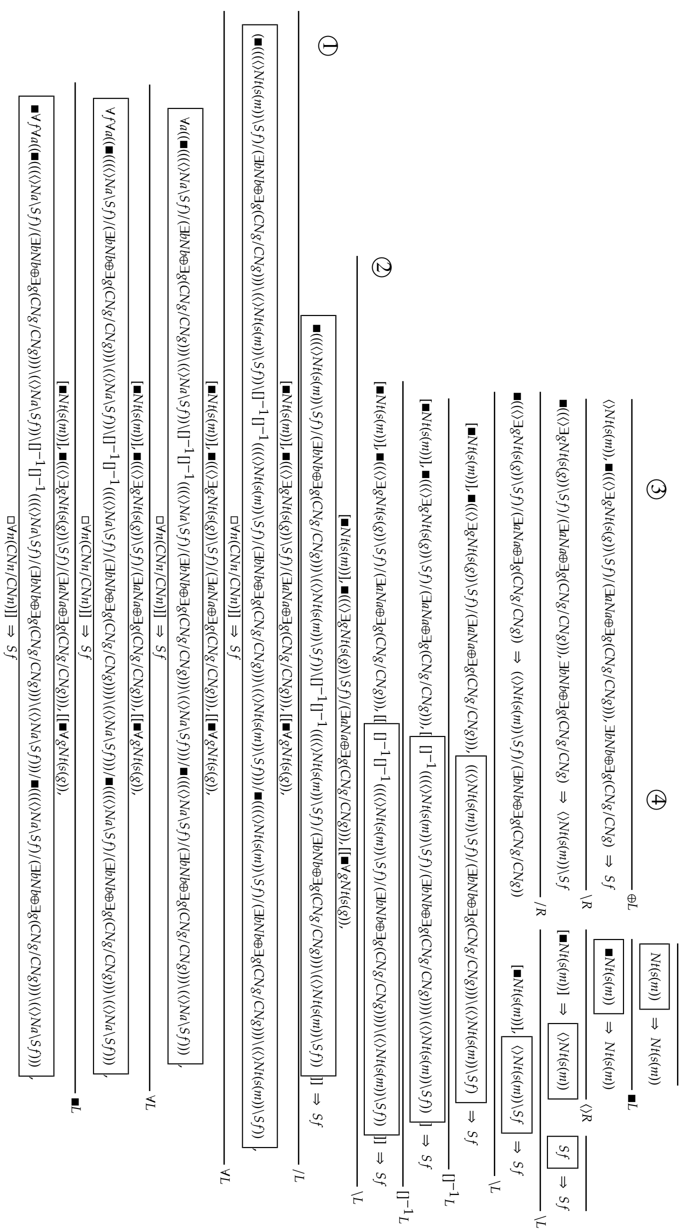
②

$$
\dfrac{
\dfrac{\dfrac{\dfrac{}{N1 \Rightarrow N1}}{N1 \Rightarrow \boxed{\exists aNa}}\,\exists R}{N1 \Rightarrow \boxed{\exists aNa\oplus\exists g(CNg/CNg)}}\,\oplus R
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}}{Nt(s(m)) \Rightarrow \boxed{\exists gNt(s(g))}}\,\exists R}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g))}}\,\langle\rangle R \qquad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{[Nt(s(m))], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} \Rightarrow Sf}\,\backslash L
}{
\dfrac{[Nt(s(m))], \boxed{(\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg)}, N1 \Rightarrow Sf}{
\dfrac{[Nt(s(m))], \boxed{\blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg)))}, N1 \Rightarrow Sf}{
\dfrac{[Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), \exists bNb \Rightarrow Sf}{\langle\rangle Nt(s(m)), \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), \exists bNb \Rightarrow Sf}\,\langle\rangle L
}\,\exists L
}\,\blacksquare L
}\,/L
$$

③

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\vdots}{CN5581/CN5581 \Rightarrow \boxed{\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))}}\,\exists R}{CN5581/CN5581 \Rightarrow \boxed{\exists g((CNg/CNg)\sqcup(CNg\backslash CNg)) - I}}\,-R}{CN5581/CN5581 \Rightarrow \boxed{\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I)}}\,\oplus R
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}}{Nt(s(m)) \Rightarrow \boxed{\exists gNt(s(g))}}\,\exists R}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists gNt(s(g))}}\,\langle\rangle R \qquad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{[Nt(s(m))], \boxed{\langle\rangle\exists gNt(s(g))\backslash Sf} \Rightarrow Sf}\,\backslash L
}{
\dfrac{[Nt(s(m))], \boxed{(\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus(\exists g((CNg/CNg)\sqcup(CNg\backslash CNg))-I))}, CN5581/CN5581 \Rightarrow Sf}{
\dfrac{[Nt(s(m))], \boxed{\blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g((CNg/CNg)))}, CN5581/CN5581 \Rightarrow Sf}{
\dfrac{[Nt(s(m))], \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), \exists g(CNg/CNg) \Rightarrow Sf}{\langle\rangle Nt(s(m)), \blacksquare((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa\oplus\exists g(CNg/CNg))), \exists g(CNg/CNg) \Rightarrow Sf}\,\langle\rangle L
}\,\exists L
}\,\blacksquare L
}\,/L
$$

④



This yields semantics:

(283) $[(Pres\ [b = 007]) \wedge (Pres\ ({}^{\vee}teetotal\ b))]$.

The same account can be given for other verbs, for example our polymorphic type for **saw** will allow under a suitable coordinator type ‘unlike’ coordination such as **John saw the facts and that Mary had been right**.

Chapter 15

The examples of Morrill, Valentín & Fadda (2011)

In this chapter we analyse the displacement examples of the article Morrill, Valentín and Fadda (2011[79]) presenting the displacement calculus.

15.1 English examples

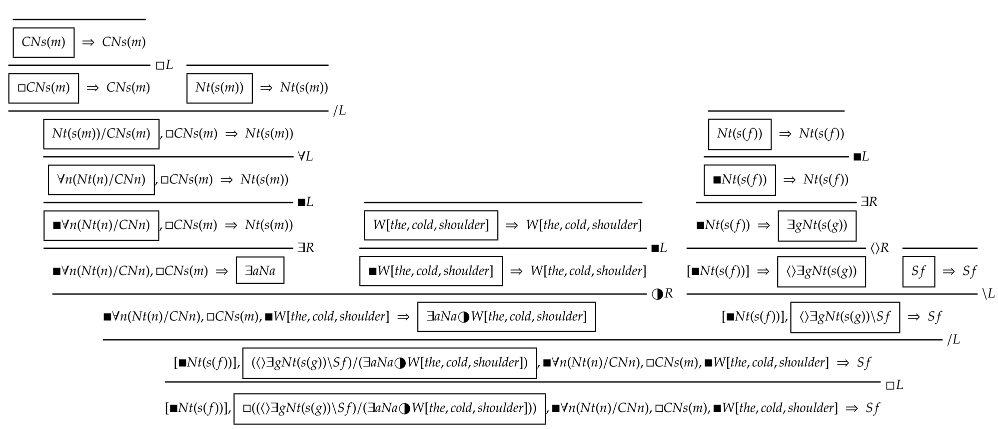
The first example, however, is modified in view of Morrill and Valentín (2014[72]). It is a discontinuous idiom. (We include the indexation of CatLog, which contains the numeration of the source, within the example displays.)

(284) (tdc(43)) **[mary]+gave+the+man+the+cold+shoulder** : Sf

Lexical lookup yields:

(285) $[\blacksquare Nt(s(f)) : m], \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/(\exists aNa \circleddash W[the, cold, shoulder])) :$
$\hat{}\lambda A\lambda B(Past\ ((\check{}shun\ A)\ B)), \blacksquare\forall n(Nt(n)/CNn) : \iota, \square CNs(m) : man, \blacksquare W[the, cold, shoulder] : 0$
$\Rightarrow\ Sf$

There is the derivation:



This delivers semantics:

(286) $(Past\ ((\check{}shun\ (\iota\ \check{}man))\ m))$

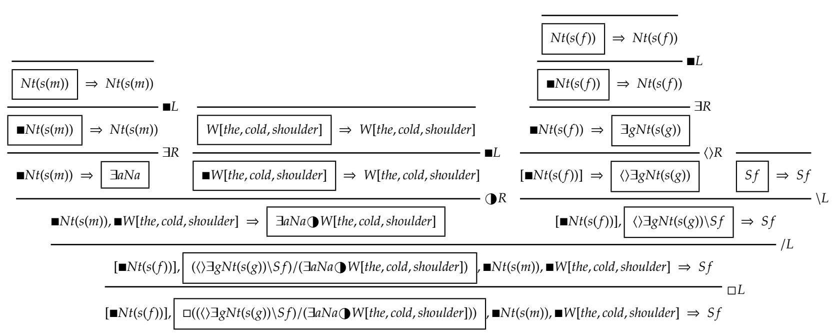
Similarly:

(287) (tdc(4343)) [**mary**]+**gave**+**john**+**the**+**cold**+**shoulder** : Sf

Lexical lookup yields:

(288) $[\blacksquare Nt(s(f)) : m], \Box(((\langle\rangle \exists g Nt(s(g)) \backslash Sf)/(\exists a Na \circ\!\!\!\blacktriangleright W[the, cold, shoulder])) :$
$\hat{}\lambda A \lambda B(Past\ ((\check{}shun\ A)\ B)), \blacksquare Nt(s(m)) : j, \blacksquare W[the, cold, shoulder] : 0 \ \Rightarrow\ Sf$

There is the derivation:



This delivers semantics:

(289) $(Past\ ((\check{}shun\ j)\ m))$

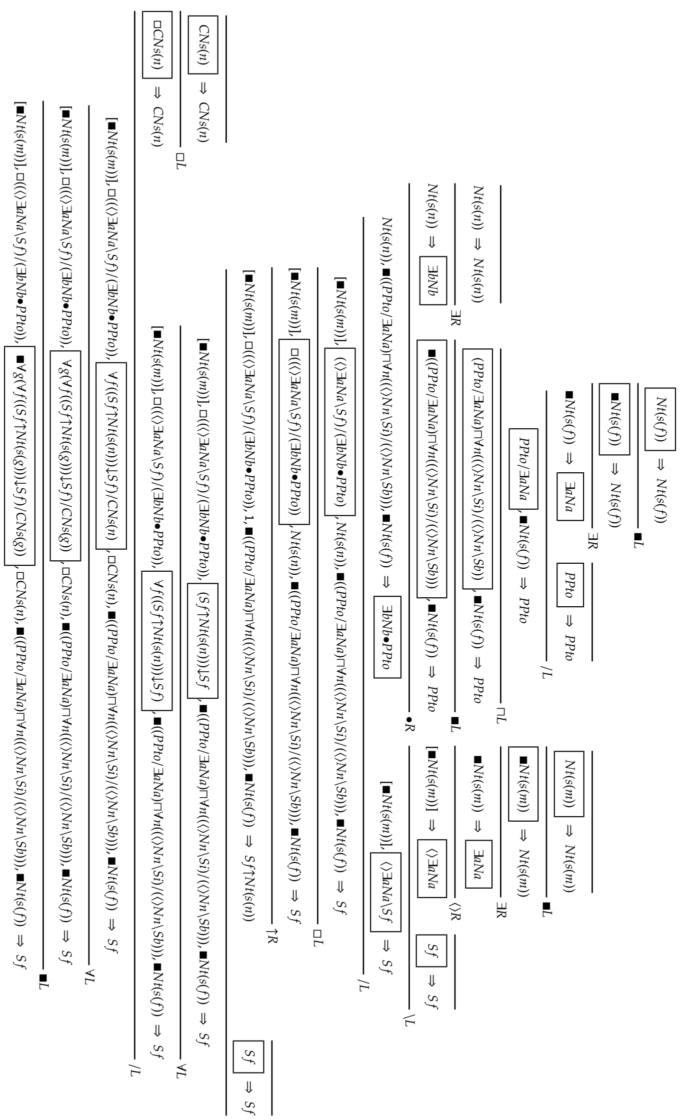
The next example has medial quantification:

(290) (tdc(47)) [**john**]+**gave**+**every**+**book**+**to**+**mary** : Sf

Lexical lookup yields:

(291) $[\blacksquare Nt(s(m)) : j], \Box(((\langle\rangle \exists a Na \backslash Sf)/(\exists b Nb \bullet PPto)) : \hat{}\lambda A \lambda B(Past\ (((\check{}give\ \pi_2 A)\ \pi_1 A)\ B)),$
$\blacksquare \forall g(\forall f((Sf \uparrow Nt(s(g))) \downarrow Sf)/CNs(g)) : \lambda C \lambda D \forall E[(C\ E) \rightarrow (D\ E)], \Box CNs(n) : book,$
$\blacksquare((PPto/\exists a Na) \sqcap \forall n((\langle\rangle Nn \backslash Si)/(\langle\rangle Nn \backslash Sb))) : \lambda F F, \blacksquare Nt(s(f)) : m \ \Rightarrow\ Sf$

There is the derivation:



This delivers semantics:

(292) $\forall C[(\check{\ }book\ C) \rightarrow (Past\ (((\check{\ }give\ m)\ C)\ j))]$

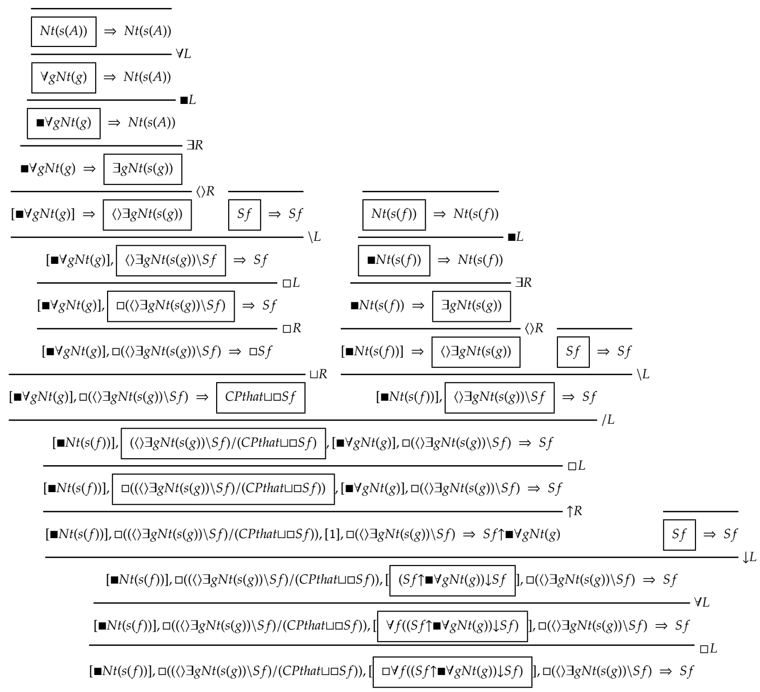
The following example has subordinate clause existential quantification, exhibiting de re/de dicto ambiguity:

(293) (tdc(50)) **[mary]+thinks+[someone]+left** : Sf

Lexical lookup yields:

(294) $[\blacksquare Nt(s(f)) : m], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/(CPthat\sqcup\Box Sf)) : \hat{\ }\lambda A\lambda B(Pres\ ((\check{\ }think\ A)\ B)),$
$[\Box\forall f((Sf{\uparrow}\blacksquare\forall gNt(g)){\downarrow}Sf) : \hat{\ }\lambda C\exists D[(\check{\ }person\ D) \wedge (C\ D)]], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) :$
$\hat{\ }\lambda E(Pres\ (\check{\ }leave\ E)) \Rightarrow Sf$

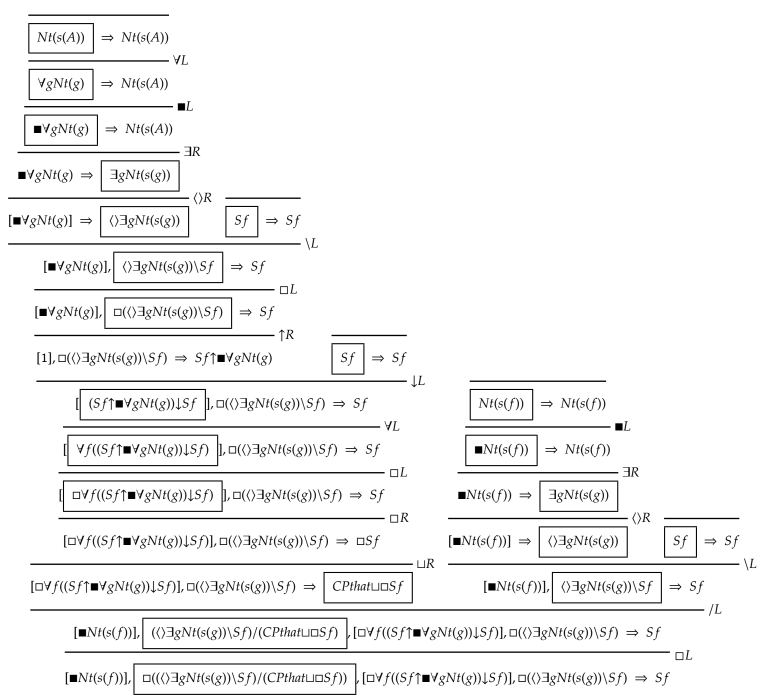
There is the de re derivation:



This delivers semantics:

(295) $\exists B[(\check{}person\ B) \wedge (Pres\ ((\check{}think\ \hat{}(Pres\ (\check{}leave\ B)))\ m))]$

And the de dicto derivation:



This delivers semantics:

(296) $(Pres\ ((\check{}think\ \hat{}\exists D[(\check{}person\ D) \wedge (Pres\ (\check{}leave\ D))]) \ m))$

The next example exhibits classical quantifier scope ambiguity:

(297) (tdc(53)) **[everyone]+loves+someone** : Sf

There is the subject wide scope reading (cf. everyone loves their (respective) mother) and the object wide scope reading (cf. everyone loves (one and) the (same) queen). Lexical lookup yields:

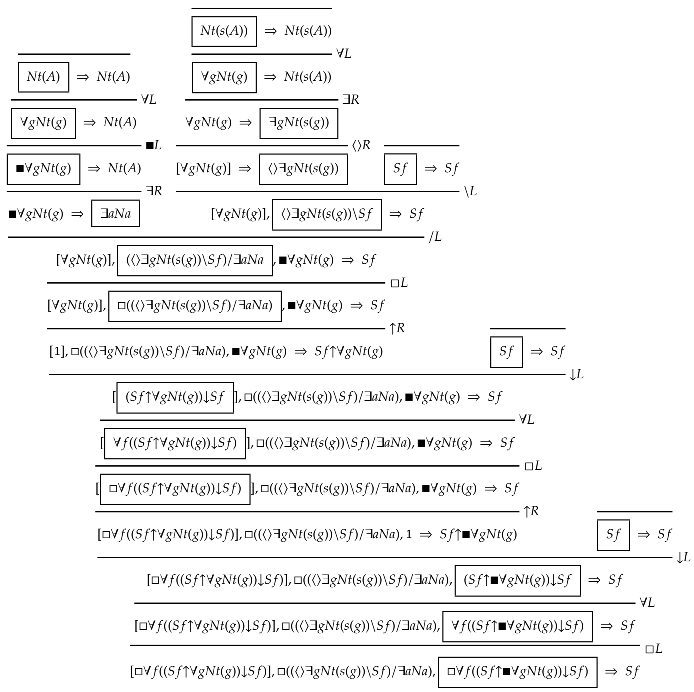
(298) $[\Box\forall f((Sf\uparrow\forall gNt(g))\downarrow Sf) : \hat{}\lambda A\forall B[(\check{}person\ B) \rightarrow (A\ B)]], \Box(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) :$
$\hat{}\lambda C\lambda D(Pres\ ((\check{}love\ C)\ D)), \Box\forall f((Sf\uparrow\blacksquare\forall gNt(g))\downarrow Sf) : \hat{}\lambda E\exists F[(\check{}person\ F) \wedge (E\ F)] \ \Rightarrow\ Sf$

There is the subject wide scope derivation as follows in which the subject quantifier is processed closest to the root:

This delivers semantics:

(299) $\forall B[(\check{}person\ B) \rightarrow \exists E[(\check{}person\ E) \wedge (Pres\ ((\check{}love\ E)\ B))]]$

And there is the object wide scope derivation as follows in which the object quantifier is processed closest to the root:



This delivers the semantics:

(300) $\exists B[(\check{}person\ B) \wedge \forall E[(\check{}person\ E) \rightarrow (Pres\ ((\check{}love\ B)\ E))]]$

The next example is of medial relativisation:

(301) (tdc(54)) **dog**+[[**that**+[**mary**]+**saw**+**today**]] : $CNs(n)$

But it is not analysed as in Morrill, Valentín and Fadda (2011[79]) but as in Morrill (2011[81]), Chapter 5. Note the double brackets for the strong island relative clause. The lexical lookup yields:

(302) $\square CNs(n) : dog, [[\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap!\blacksquare Nt(n))\backslash Sf)) :$
$\lambda A\lambda B\lambda C[(B\ C) \wedge (A\ C)], [\blacksquare Nt(s(f)) : m], \square((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)) :$
$\hat{}\lambda D\lambda E(Past\ ((D\text{-}{>}F.(\check{}seee\ F); G.(\check{}seet\ G))\ E)), \square\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) :$
$\hat{}\lambda H\lambda I(\check{}today\ (H\ I))]] \Rightarrow CNs(n)$

There is the derivation:

$$Nt(s(n)) \Rightarrow Nt(s(n))$$
$$\blacksquare Nt(s(n)) \Rightarrow Nt(s(n)) \quad \blacksquare L$$
$$\blacksquare Nt(s(n)); \Rightarrow Nt(s(n)) \quad !P$$
$$\blacksquare Nt(s(n)); \Rightarrow \exists aNa \quad \exists R$$
$$\blacksquare Nt(s(n)); \Rightarrow \exists aNa \oplus CPthat \quad \oplus R$$

$$Nt(s(f)) \Rightarrow Nt(s(f))$$
$$Nt(s(f)) \Rightarrow \exists aNa \quad \exists R$$
$$[Nt(s(f))] \Rightarrow \langle\rangle\exists aNa \quad \langle\rangle R$$
$$Sf \Rightarrow Sf$$
$$[Nt(s(f))], \langle\rangle\exists aNa\backslash Sf \Rightarrow Sf \quad \backslash L$$

$$\blacksquare Nt(s(n)); [Nt(s(f))], (\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat) \Rightarrow Sf \quad /L$$
$$\blacksquare Nt(s(n)); [Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)) \Rightarrow Sf \quad \Box L$$
$$\blacksquare Nt(s(n)); \langle\rangle Nt(s(f)), \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)) \Rightarrow Sf \quad \langle\rangle L$$
$$\blacksquare Nt(s(n)); \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)) \Rightarrow \langle\rangle Nt(s(f))\backslash Sf \quad \backslash R$$

$$Nt(s(f)) \Rightarrow Nt(s(f))$$
$$\blacksquare Nt(s(f)) \Rightarrow Nt(s(f)) \quad \blacksquare L$$
$$[\blacksquare Nt(s(f))] \Rightarrow \langle\rangle Nt(s(f)) \quad \langle\rangle R$$
$$Sf \Rightarrow Sf$$
$$[\blacksquare Nt(s(f))], \langle\rangle Nt(s(f))\backslash Sf \Rightarrow Sf \quad \backslash L$$

$$\blacksquare Nt(s(n)); [\blacksquare Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), (\langle\rangle Nt(s(f))\backslash Sf)\backslash(\langle\rangle Nt(s(f))\backslash Sf) \Rightarrow Sf \quad \backslash L$$
$$\blacksquare Nt(s(n)); [\blacksquare Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \forall f((\langle\rangle Nt(s(f))\backslash Sf)\backslash(\langle\rangle Nt(s(f))\backslash Sf)) \Rightarrow Sf \quad \forall L$$
$$\blacksquare Nt(s(n)); [\blacksquare Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) \Rightarrow Sf \quad \forall L$$
$$\blacksquare Nt(s(n)); [\blacksquare Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) \Rightarrow Sf \quad \Box L$$
$$!\blacksquare Nt(s(n)), [\blacksquare Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) \Rightarrow Sf \quad !L$$
$$\langle\rangle Nt(s(n))\sqcap!\blacksquare Nt(s(n)), [\blacksquare Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) \Rightarrow Sf \quad \sqcap L$$
$$[\blacksquare Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) \Rightarrow (\langle\rangle Nt(s(n))\sqcap!\blacksquare Nt(s(n)))\backslash Sf \quad \backslash R$$
$$[\blacksquare Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) \Rightarrow \blacksquare((\langle\rangle Nt(s(n))\sqcap!\blacksquare Nt(s(n)))\backslash Sf) \quad \blacksquare R$$

$$CNs(n) \Rightarrow CNs(n)$$
$$\Box CNs(n) \Rightarrow CNs(n) \quad \Box L$$
$$CNs(n) \Rightarrow CNs(n)$$
$$\Box CNs(n), CNs(n)\backslash CNs(n) \Rightarrow CNs(n) \quad \backslash L$$
$$\Box CNs(n), [\,[]^{-1}(CNs(n)\backslash CNs(n))\,] \Rightarrow CNs(n) \quad []^{-1}L$$
$$\Box CNs(n), [[\,[]^{-1}[]^{-1}(CNs(n)\backslash CNs(n))\,]] \Rightarrow CNs(n) \quad []^{-1}L$$

$$\Box CNs(n), [[\,[]^{-1}[]^{-1}(CNs(n)\backslash CNs(n))/\blacksquare((\langle\rangle Nt(s(n))\sqcap!\blacksquare Nt(s(n)))\backslash Sf), [\blacksquare Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))]] \Rightarrow CNs(n) \quad /L$$
$$\Box CNs(n), [[\,\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap!\blacksquare Nt(n))\backslash Sf)), [\blacksquare Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))]] \Rightarrow CNs(n) \quad \forall L$$
$$\Box CNs(n), [[\,\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap!\blacksquare Nt(n))\backslash Sf)), [\blacksquare Nt(s(f))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\oplus CPthat)), \Box\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))]] \Rightarrow CNs(n) \quad \blacksquare L$$

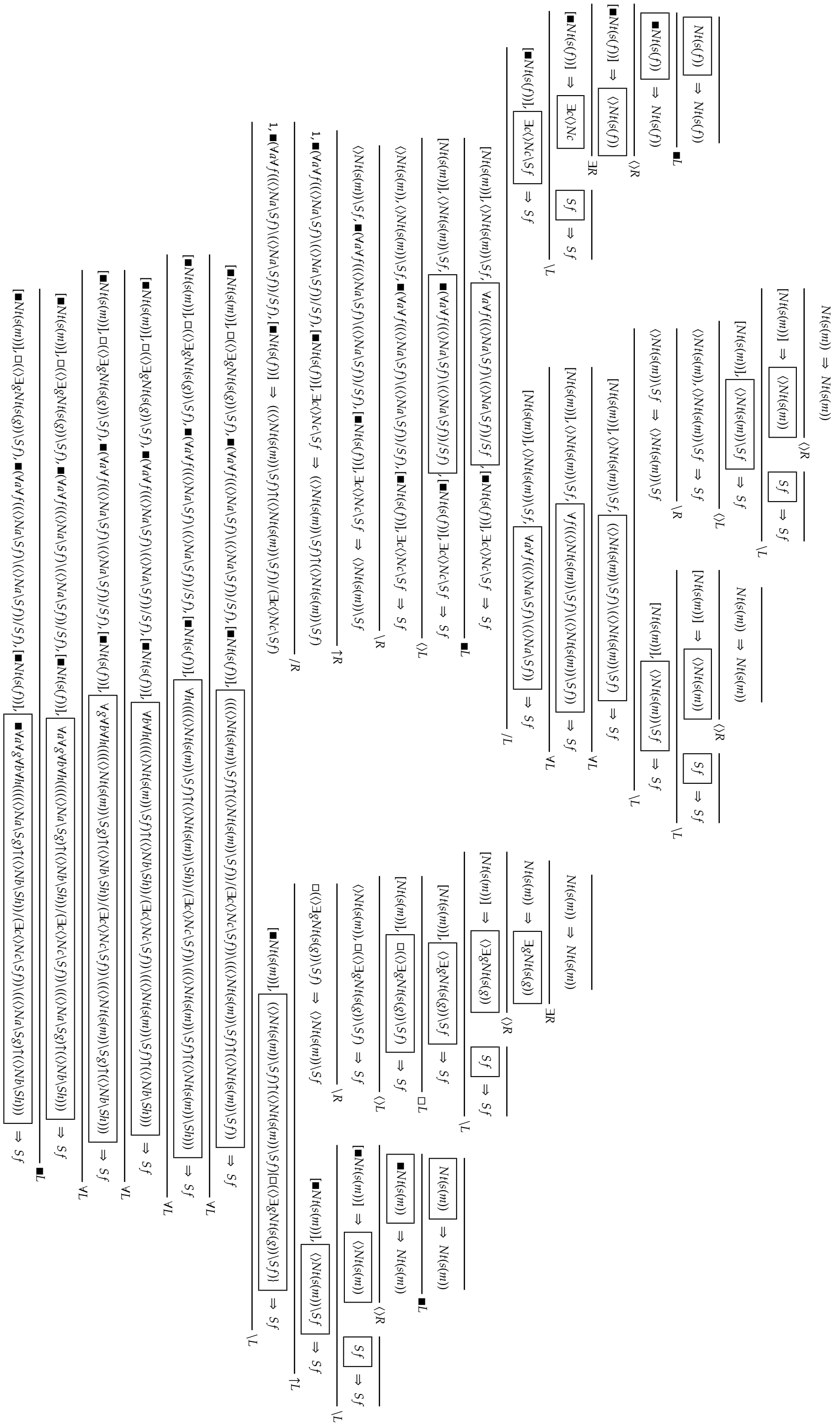
This delivers semantics:

(303) $\lambda C[(\check{}dog\ C) \wedge (\check{}today\ (Past\ ((\check{}seee\ C)\ m)))]$

The next example is of VP ellipsis:

(304) (tdc(58a)) [**john**]+**slept**+**before**+[**mary**]+**did** : Sf

(305) $[\blacksquare Nt(s(m)) : j], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda A(Past\ (\check{}sleep\ A)), \blacksquare(\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))/Sf) : \lambda B\lambda C\lambda D((before\ B)\ (C\ D)), [\blacksquare Nt(s(f)) : m], \blacksquare\forall a\forall g\forall b\forall h((((\langle\rangle Na\backslash Sg)\uparrow(\langle\rangle Nb\backslash Sh))/(\exists c\langle\rangle Nc\backslash Sf))\backslash(((\langle\rangle Na\backslash Sg)\uparrow(\langle\rangle Nb\backslash Sh))) : \lambda E\lambda F((E\ F)\ F) \Rightarrow Sf$



This delivers the semantics:

(306) ((*before* (*Past* (˘*sleep m*))) (*Past* (˘*sleep j*)))

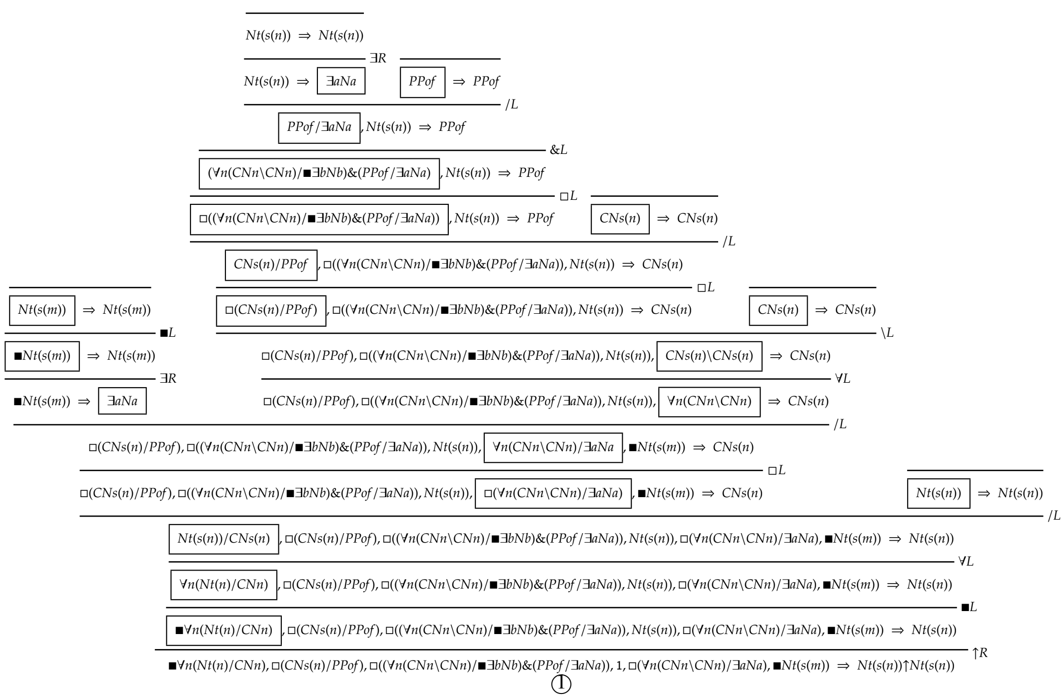
In tdc(64) there is medial pied-piping:

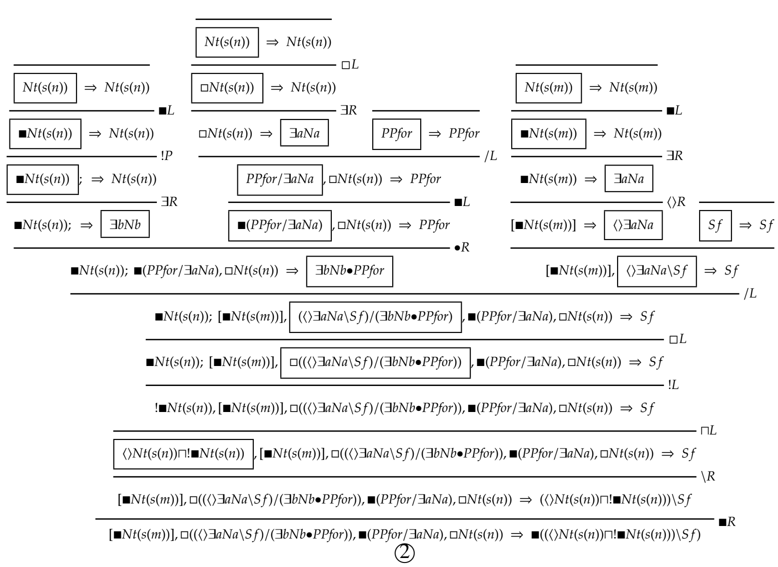
(307) (tdc(64)) **mountain+[[the+painting+of+which+by+cezanne+[john]+sold+for+tenmilliondollars]]** : $CNs(n)$

Lexical lookup yields:

(308) $\Box CNs(n) : mountain, [[\blacksquare\forall n(Nt(n)/CNn) : \iota, \Box(CNs(n)/PPof) : \hat{\ }\lambda A((\check{\ }of\ A)\ \check{\ }painting),$
$\Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)) : \hat{\ }(\check{\ }of, \lambda BB), \blacksquare\forall n\forall m((Nt(n)\uparrow Nt(m))\downarrow$
$([]^{-1}[]^{-1}(CNm\backslash CNm)/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf))) : \lambda C\lambda D\lambda E\lambda F[(E\ F) \wedge (D\ (C\ F))],$
$\Box(\forall n(CNn\backslash CNn)/\exists aNa) : \hat{\ }\lambda G\lambda H((\check{\ }by\ G)\ H), \blacksquare Nt(s(m)) : c, [\blacksquare Nt(s(m)) : j],$
$\Box((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPfor)) : \hat{\ }\lambda I\lambda J(Past\ (((\check{\ }sell\ \pi_2 I)\ \pi_1 I)\ J)),$
$\blacksquare(PPfor/\exists aNa) : \lambda KK, \Box Nt(s(n)) : tenmilliondollars]] \Rightarrow CNs(n)$

There is the derivation:





$$\boxed{CNs(n)} \Rightarrow CNs(n)$$

$\square L$

$$\boxed{\square CNs(n)} \Rightarrow CNs(n) \qquad \boxed{CNs(n)} \Rightarrow CNs(n)$$

$\backslash L$

$$\square CNs(n), \boxed{CNs(n)\backslash CNs(n)} \Rightarrow CNs(n)$$

$[\,]^{-1}L$

$$\square CNs(n), [\boxed{[\,]^{-1}(CNs(n)\backslash CNs(n))}] \Rightarrow CNs(n)$$

$[\,]^{-1}L$

②

$$\square CNs(n), [[\boxed{[\,]^{-1}[\,]^{-1}(CNs(n)\backslash CNs(n))}]] \Rightarrow CNs(n)$$

$/L$

$$\square CNs(n), [[\boxed{[\,]^{-1}[\,]^{-1}(CNs(n)\backslash CNs(n))/\blacksquare((\langle\rangle Nt(s(n))\sqcap !\blacksquare Nt(s(n)))\backslash Sf)}, [\blacksquare Nt(s(m))], \square((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPfor)), \blacksquare(PPfor/\exists aNa), \square Nt(s(n))]] \Rightarrow CNs(n)$$

①

$\downarrow L$

$$\square CNs(n), [[\blacksquare\forall n(Nt(n)/CNn), \square(CNs(n)/PPof), \square((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \boxed{(Nt(s(n))\uparrow Nt(s(n)))\downarrow([\,]^{-1}[\,]^{-1}(CNs(n)\backslash CNs(n))/\blacksquare((\langle\rangle Nt(s(n))\sqcap !\blacksquare Nt(s(n)))\backslash Sf))}, \square(\forall n(CNn\backslash CNn)/\exists aNa), \blacksquare Nt(s(m)), [\blacksquare Nt(s(m))], \square((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPfor)), \blacksquare(PPfor/\exists aNa), \square Nt(s(n))]] \Rightarrow CNs(n)$$

$\forall L$

$$\square CNs(n), [[\blacksquare\forall n(Nt(n)/CNn), \square(CNs(n)/PPof), \square((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \boxed{\forall m((Nt(s(n))\uparrow Nt(m))\downarrow([\,]^{-1}[\,]^{-1}(CNm\backslash CNm)/\blacksquare((\langle\rangle Nt(s(n))\sqcap !\blacksquare Nt(s(n)))\backslash Sf)))}, \square(\forall n(CNn\backslash CNn)/\exists aNa), \blacksquare Nt(s(m)), [\blacksquare Nt(s(m))], \square((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPfor)), \blacksquare(PPfor/\exists aNa), \square Nt(s(n))]] \Rightarrow CNs(n)$$

$\forall L$

$$\square CNs(n), [[\blacksquare\forall n(Nt(n)/CNn), \square(CNs(n)/PPof), \square((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \boxed{\forall n\forall m((Nt(n)\uparrow Nt(m))\downarrow([\,]^{-1}[\,]^{-1}(CNm\backslash CNm)/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf)))}, \square(\forall n(CNn\backslash CNn)/\exists aNa), \blacksquare Nt(s(m)), [\blacksquare Nt(s(m))], \square((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPfor)), \blacksquare(PPfor/\exists aNa), \square Nt(s(n))]] \Rightarrow CNs(n)$$

$\blacksquare L$

$$\square CNs(n), [[\blacksquare\forall n(Nt(n)/CNn), \square(CNs(n)/PPof), \square((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)), \boxed{\blacksquare\forall n\forall m((Nt(n)\uparrow Nt(m))\downarrow([\,]^{-1}[\,]^{-1}(CNm\backslash CNm)/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf)))}, \square(\forall n(CNn\backslash CNn)/\exists aNa), \blacksquare Nt(s(m)), [\blacksquare Nt(s(m))], \square((\langle\rangle\exists aNa\backslash Sf)/(\exists bNb\bullet PPfor)), \blacksquare(PPfor/\exists aNa), \square Nt(s(n))]] \Rightarrow CNs(n)$$

This delivers semantics:

(309) $\lambda D[(\check{}mountain\ D) \wedge (Past\ (((\check{}sell\ \check{}tenmilliondollars)\ (\iota\ ((\check{}by\ c)\ ((\check{}of\ D)\ \check{}painting))))\ j))]$

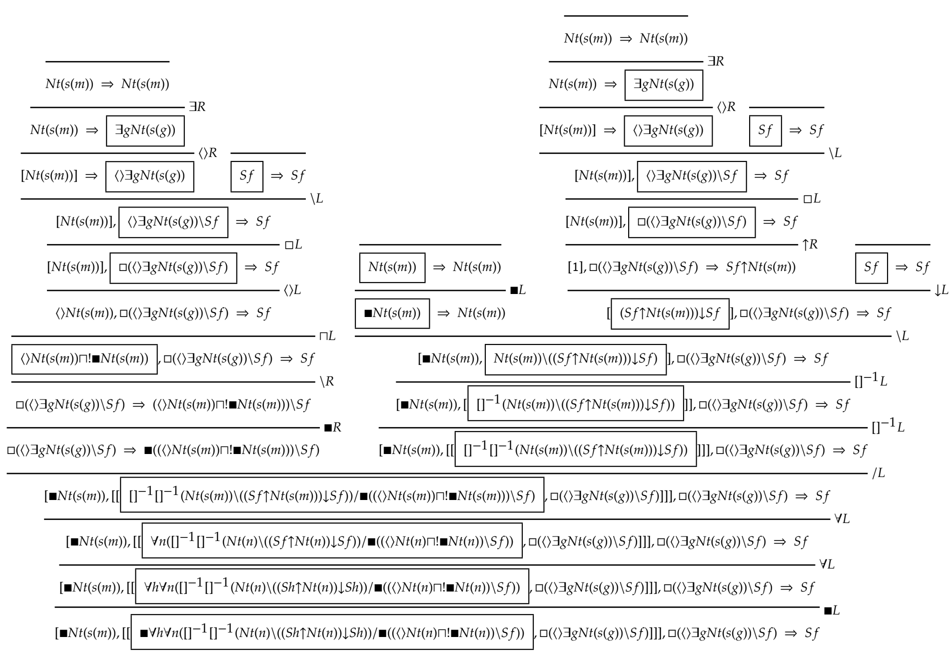
In appositive relativisation a relative clause, marked off by a prosodic phrase, modifies a full noun phrase:

(310) (tdc(67)) **[john+[[who+jogs]]]+sneezed** : Sf

The lexical lookup yields:

(311) $[\blacksquare Nt(s(m)) : j, [[\blacksquare\forall h\forall n([]^{-1}[]^{-1}(Nt(n)\backslash((Sh\uparrow Nt(n))\downarrow Sh))/\blacksquare((\langle\rangle Nt(n)\sqcap!\blacksquare Nt(n))\backslash Sf)) :$
$\lambda A\lambda B\lambda C[(A\ B) \wedge (C\ B)], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda D(Pres\ (\check{}jog\ D))]]], \Box(\langle\rangle\exists gNt(s(g))\backslash Sf) :$
$\hat{}\lambda E(Past\ (\check{}sneeze\ E))\ \Rightarrow\ Sf$

There is the derivation:



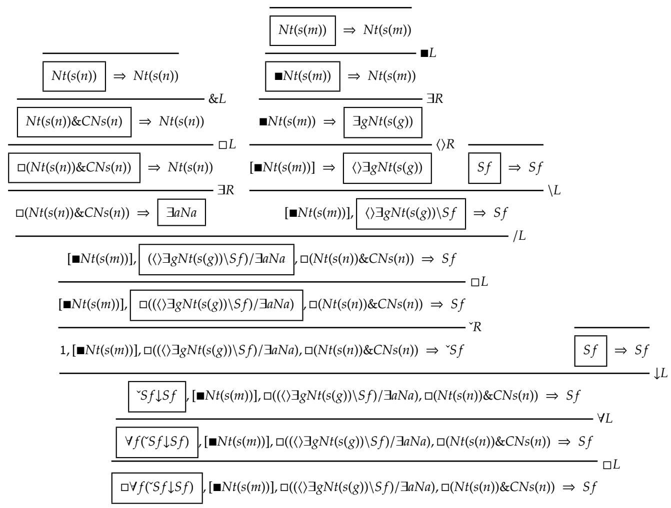
This delivers semantics:

(312) $[(Pres\ (\check{}jog\ j)) \wedge (Past\ (\check{}sneeze\ j))]$

There follow four placements of a parenthetical adverbial within a sentence all derived from the same types and all assigning the same semantics. First:

(313) (tdc(70a)) **fortunately+[john]+has+perseverance** : Sf

(314) $\Box\forall f(\check{}Sf\downarrow Sf) : fortunately, [\blacksquare Nt(s(m)) : j], \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) :$
$\hat{}\lambda A\lambda B(Pres\ ((\check{}have\ A)\ B)), \Box(Nt(s(n))\&CNs(n)) : \hat{}((gen\ \check{}perseverance), \check{}perseverance)$
$\Rightarrow\ Sf$

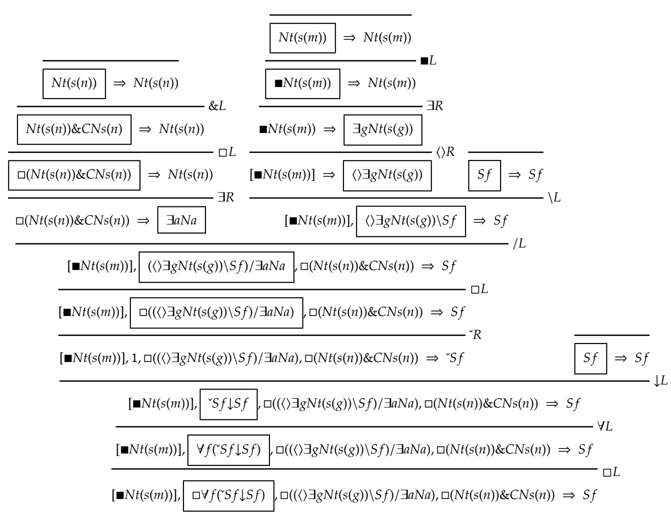


(315) $(\check{}fortunately\ (Pres\ ((\check{}have\ (gen\ \check{}perseverance))\ j)))$

Second:

(316) (tdc(70b)) **[john]+fortunately+has+perseverance** : Sf

(317) $[\blacksquare Nt(s(m)) : j], \Box\forall f(\check{}Sf{\downarrow}Sf) : fortunately, \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) :$
$\hat{}\lambda A\lambda B(Pres\ ((\check{}have\ A)\ B)), \Box(Nt(s(n))\&CNs(n)) : \hat{}((gen\ \check{}perseverance), \check{}perseverance)$
$\Rightarrow\ Sf$

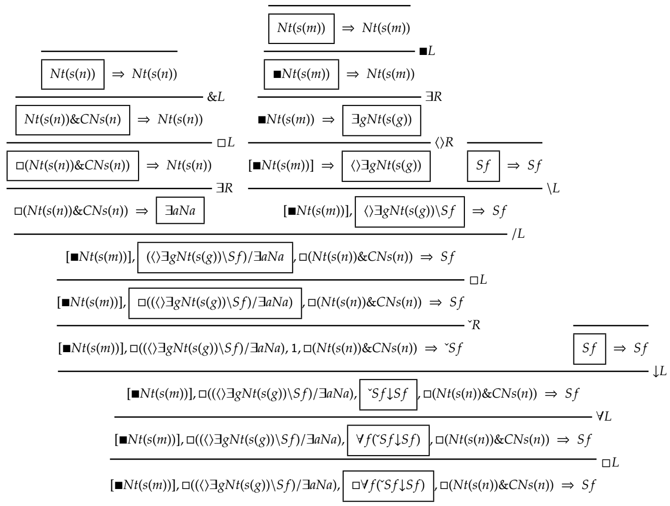


(318) $(\check{}fortunately\ (Pres\ ((\check{}have\ (gen\ \check{}perseverance))\ j)))$

Third:

(319) (tdc(70c)) **[john]+has+fortunately+perseverance** : Sf

(320) $[\blacksquare Nt(s(m)) : j], \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}have\ A)\ B)), \Box\forall f(\check{}Sf{\downarrow}Sf) :$
$fortunately, \Box(Nt(s(n))\&CNs(n)) : \hat{}((gen\ \check{}perseverance), \check{}perseverance)\ \Rightarrow\ Sf$

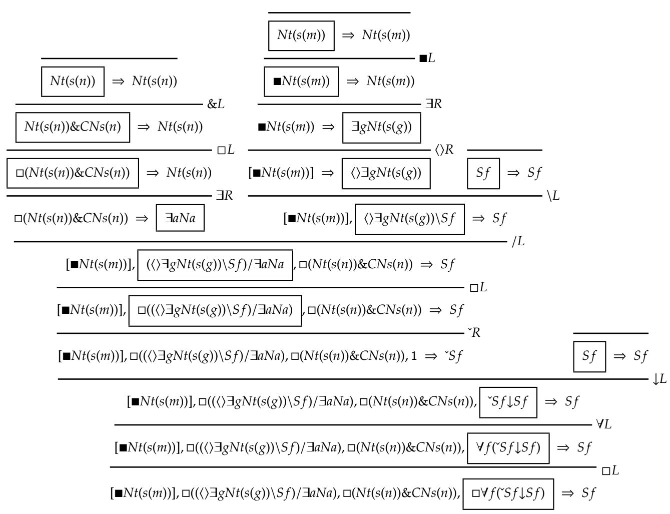


(321) $(\check{}fortunately\ (Pres\ ((\check{}have\ (gen\ \check{}perseverance))\ j)))$

And fourth:

(322) (tdc(70d)) **[john]+has+perseverance+fortunately** : Sf

(323) $[\blacksquare Nt(s(m)) : j], \square(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}have\ A)\ B)),$
$\square(Nt(s(n))\&CNs(n)) : \hat{}((gen\ \check{}perseverance), \check{}perseverance), \square\forall f(\check{}Sf{\downarrow}Sf) : fortunately$
$\Rightarrow\ Sf$



(324) $(\check{}fortunately\ (Pres\ ((\check{}have\ (gen\ \check{}perseverance))\ j)))$

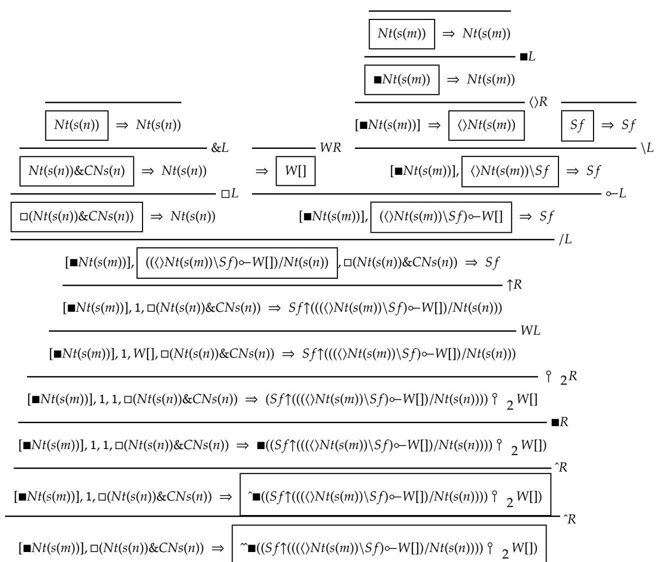
The next example is gapping:

(325) (tdc(73)) **[[[john]+studies+logic+and+[charles]+phonetics]]** : Sf

(Note the double brackets for the coordinate structure strong island.) The treatment of the example, however, is modified according to Morrill and Valentín (2014[72]) in view of the observations of Kubota and Levine (2012[38]). Lexical lookup yields:

(326) $[[\blacksquare Nt(s(m)) : j], \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Pres\ ((\check{}study\ A)\ B)),$
$\square(Nt(s(n))\&CNs(n)) : \hat{}((gen\ \check{}logic), \check{}logic), \blacksquare\forall w\forall a\forall b\forall f((\blacksquare((Sf\uparrow((((\langle\rangle Na\backslash Sf)\circ\!-Ww)/Nb))⫯_2 Ww)\backslash[]^{-1}[]^{-1}((Sf\uparrow((((\langle\rangle Na\backslash Sf)\circ\!-Ww)/Nb))⫯_2 Ww))/\hat{}\check{}\blacksquare((Sf\uparrow((((\langle\rangle Na\backslash Sf)\circ\!-Ww)/Nb))⫯_2 Ww)) : \lambda C\lambda D\lambda E[(D\ E) \wedge (C\ E)], [\blacksquare Nt(s(m)) : c], \square(Nt(s(n))\&CNs(n)) :$
$\hat{}((gen\ \check{}phonetics), \check{}phonetics)]] \Rightarrow Sf$

There is the derivation:



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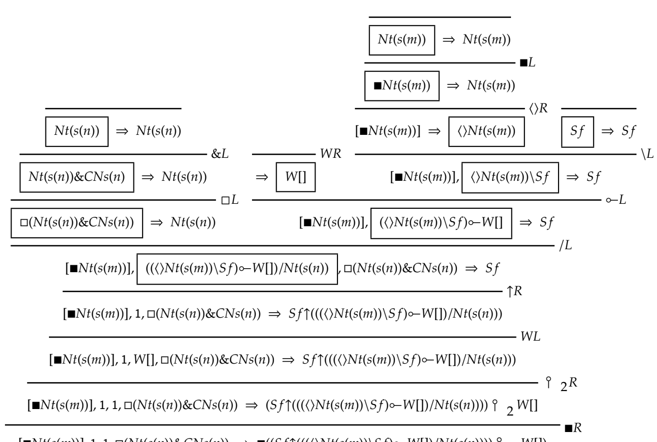


②

This delivers semantics:

(327) $[(Pres\ ((\check{\ }study\ (gen\ \check{\ }logic))\ j)) \wedge (Pres\ ((\check{\ }study\ (gen\ \check{\ }phonetics))\ c))]$

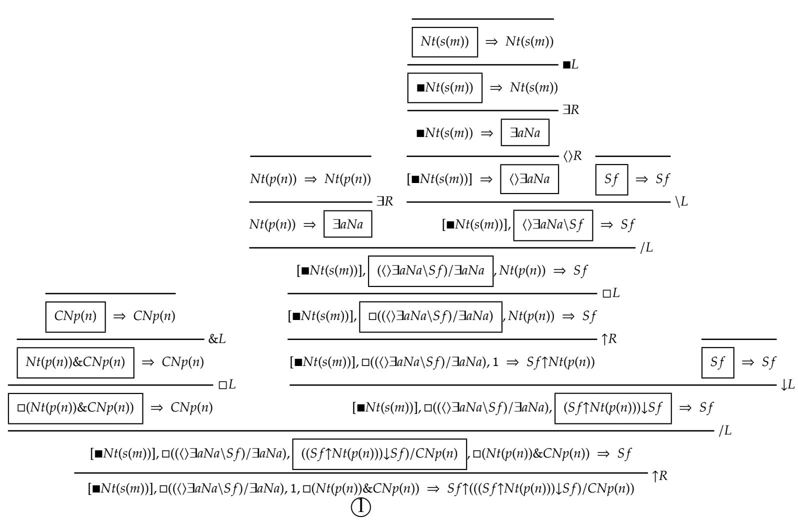
Example tdc(75) contains comparative subdeletion:

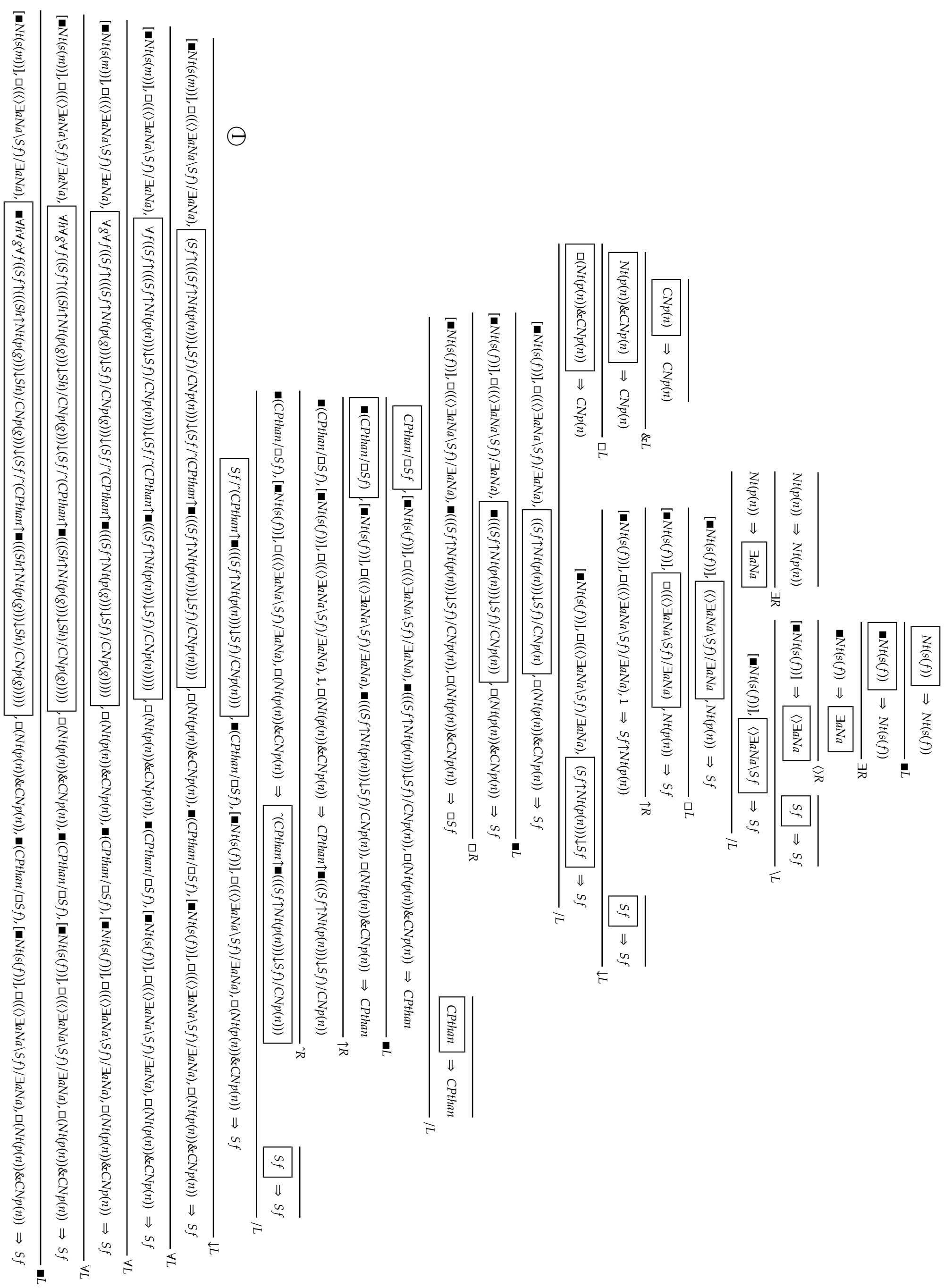
(328) (tdc(75)) **[john]+ate+more+donuts+than+[mary]+bought+bagels** : Sf

Lexical lookup yields:

(329) $[\blacksquare Nt(s(m)) : j], \square(((\langle\rangle\exists aNa\backslash Sf)/\exists aNa) : \hat{\ }\lambda A\lambda B(Past\ ((\check{\ }eat\ A)\ B)),$
$\blacksquare\forall h\forall g\forall f((Sf\uparrow(((Sh\uparrow Nt(p(g)))\downarrow Sh)/CNp(g)))\downarrow(Sf/\hat{\ }(CPthan\uparrow$
$\blacksquare(((Sh\uparrow Nt(p(g)))\downarrow Sh)/CNp(g))))) : \lambda C\lambda D[|\lambda E(C\ \lambda F\lambda G[(F\ E) \wedge (G\ E)])| >$
$|\lambda H\check{\ }(D\ \lambda I\lambda J[(I\ H) \wedge (J\ H)])|], \square(Nt(p(n))\&CNp(n)) : \hat{\ }((gen\ \check{\ }donuts), \check{\ }donuts),$
$\blacksquare(CPthan/\square Sf) : \lambda KK, [\blacksquare Nt(s(f)) : m], \square(((\langle\rangle\exists aNa\backslash Sf)/\exists aNa) :$
$\hat{\ }\lambda L\lambda M(Past\ ((\check{\ }buy\ L)\ M)), \square(Nt(p(n))\&CNp(n)) : \hat{\ }((gen\ \check{\ }bagels), \check{\ }bagels)\ \Rightarrow\ Sf$

There is the derivation:





This delivers semantics:

(330) $[|\lambda C[(\check{}donuts\ C) \wedge (Past\ ((\check{}eat\ C)\ j))]| > |\lambda F[(\check{}bagels\ F) \wedge (Past\ ((\check{}buy\ F)\ m))]|]$

Finally, there is the medial reflexivisation:

(331) (tdc(86a)) **[john]+bought+himself+coffee** : Sf

The lexical lookup yields:

(332) $[\blacksquare Nt(s(m)) : j], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\bullet\exists aNa)) : \hat{}\lambda A\lambda B(Past\ (((\check{}buy\ \pi_1 A)\ \pi_2 A)\ B)),$
$\blacksquare\forall f(((\langle\rangle Nt(s(m))\backslash Sf)\uparrow Nt(s(m)))\downarrow(\langle\rangle Nt(s(m))\backslash Sf)) : \lambda C\lambda D((C\ D)\ D),$
$\Box(Nt(s(n))\&CNs(n)) : \hat{}((gen\ \check{}coffee), \check{}coffee) \Rightarrow Sf$

There is the derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}}{Nt(s(m)) \Rightarrow \boxed{\exists aNa}}\exists R \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(n))} \Rightarrow Nt(s(n))}}{\boxed{Nt(s(n))\&CNs(n)} \Rightarrow Nt(s(n))}\&L}{\boxed{\Box(Nt(s(n))\&CNs(n))} \Rightarrow Nt(s(n))}\Box L}{\Box(Nt(s(n))\&CNs(n)) \Rightarrow \boxed{\exists aNa}}\exists R}{Nt(s(m)), \Box(Nt(s(n))\&CNs(n)) \Rightarrow \boxed{\exists aNa\bullet\exists aNa}}\bullet R \quad \dfrac{\dfrac{\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}}{Nt(s(m)) \Rightarrow \boxed{\exists aNa}}\exists R}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle\exists aNa}}\langle\rangle R \quad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{[Nt(s(m))], \boxed{\langle\rangle\exists aNa\backslash Sf} \Rightarrow Sf}\backslash L}{[Nt(s(m))], \boxed{(\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\bullet\exists aNa)}, Nt(s(m)), \Box(Nt(s(n))\&CNs(n)) \Rightarrow Sf}/L}{[Nt(s(m))], \boxed{\Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\bullet\exists aNa))}, Nt(s(m)), \Box(Nt(s(n))\&CNs(n)) \Rightarrow Sf}\Box L}{\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\bullet\exists aNa)), Nt(s(m)), \Box(Nt(s(n))\&CNs(n)) \Rightarrow Sf}\langle\rangle L}{\Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\bullet\exists aNa)), Nt(s(m)), \Box(Nt(s(n))\&CNs(n)) \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}\backslash R}{\Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\bullet\exists aNa)), 1, \Box(Nt(s(n))\&CNs(n)) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)\uparrow Nt(s(m))}\uparrow R \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\boxed{Nt(s(m))} \Rightarrow Nt(s(m))}}{\boxed{\blacksquare Nt(s(m))} \Rightarrow Nt(s(m))}\blacksquare L}{[\blacksquare Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))}}\langle\rangle R \quad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{[\blacksquare Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf}\backslash L}{[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\bullet\exists aNa)), \boxed{((\langle\rangle Nt(s(m))\backslash Sf)\uparrow Nt(s(m)))\downarrow(\langle\rangle Nt(s(m))\backslash Sf)}, \Box(Nt(s(n))\&CNs(n)) \Rightarrow Sf}\downarrow L}{[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\bullet\exists aNa)), \boxed{\forall f(((\langle\rangle Nt(s(m))\backslash Sf)\uparrow Nt(s(m)))\downarrow(\langle\rangle Nt(s(m))\backslash Sf))}, \Box(Nt(s(n))\&CNs(n)) \Rightarrow Sf}\forall L}{[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/(\exists aNa\bullet\exists aNa)), \boxed{\blacksquare\forall f(((\langle\rangle Nt(s(m))\backslash Sf)\uparrow Nt(s(m)))\downarrow(\langle\rangle Nt(s(m))\backslash Sf))}, \Box(Nt(s(n))\&CNs(n)) \Rightarrow Sf}\blacksquare L$$

This delivers semantics:

(333) $(Past\ (((\check{}buy\ j)\ (gen\ \check{}coffee))\ j))$

15.2 Dutch verb raising and cross-serial dependency

kan : $(NA\backslash Si)\downarrow(NA\backslash Sf) : \lambda B\lambda C((isable\ (B\ C))\ C)$
las : $NA\backslash(Nt(s(B))\backslash Sf) : read$
wil : $(NA\backslash Si)\downarrow(NA\backslash Sf) : \lambda B\lambda C((want\ (B\ C))\ C)$
kan : $Q/\hat{}(Sf\uparrow((NA\backslash Si)\downarrow(NA\backslash Sf))) : \lambda B(B\ \lambda C\lambda D((isable\ (C\ D))\ D))$
las : $Q/\hat{}(Sf\uparrow(NA\backslash(Nt(s(B))\backslash Sf))) : \lambda C(C\ read)$
wil : $Q/\hat{}(Sf\uparrow((NA\backslash Si)\downarrow(NA\backslash Sf))) : \lambda B(B\ \lambda C\lambda D((want\ (C\ D))\ D))$
alles : $(SA\uparrow Nt(s(n)))\downarrow SA : \lambda B\forall C[(thing\ C) \rightarrow (B\ C)]$
boeken : $Np(n) : books$
cecilia : $Nt(s(f)) : c$
de : $Nt(s(A))/CNA : the$
helpen : $\triangleright^{-1}((NA\backslash Si)\downarrow(NB\backslash(NA\backslash Si))) : \lambda C\lambda D((help\ (C\ D))\ D)$
henk : $Nt(s(m)) : h$
jan : $Nt(s(m)) : j$
kunnen : $\triangleright^{-1}((NA\backslash Si)\downarrow(NA\backslash Si)) : \lambda B\lambda C((isable\ (B\ C))\ C)$
lezen : $\triangleright^{-1}(NA\backslash(NB\backslash Si)) : read$
nijlpaarden : $CNp(n) : hippos$
voeren : $\triangleright^{-1}(NA\backslash(NB\backslash Si)) : feed$
zag : $(Nt(s(A))\backslash Si)\downarrow(NB\backslash(Nt(s(A))\backslash Sf)) : \lambda C\lambda D((saw\ (C\ D))\ D)$

(d(1)) **jan+boeken+las** : Sf

$Nt(s(m)) : j, Np(n) : books, NA\backslash(Nt(s(B))\backslash Sf) : read \Rightarrow Sf$

$$\dfrac{\overline{Np(n) \Rightarrow Np(n)} \quad \dfrac{\overline{Nt(s(m)) \Rightarrow Nt(s(m))} \quad \overline{\boxed{Sf} \Rightarrow Sf}}{Nt(s(m)), \boxed{Nt(s(m))\backslash Sf} \Rightarrow Sf}\backslash L}{Nt(s(m)), Np(n), \boxed{Np(n)\backslash(Nt(s(m))\backslash Sf)} \Rightarrow Sf}\backslash L$$

((*read books*) *j*)

$Nt(s(m)) : j, Np(n) : books, Q/\hat{}(Sf\uparrow(NA\backslash(Nt(s(B))\backslash Sf))) : \lambda C(C\ read) \Rightarrow Sf$

(d(2)) **jan+boeken+kan+lezen** : Sf

$Nt(s(m)) : j, Np(n) : books, (NA\backslash Si)\downarrow(NA\backslash Sf) : \lambda B\lambda C((isable\ (B\ C))\ C), \triangleright^{-1}(ND\backslash(NE\backslash Si)) : read \Rightarrow Sf$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{Np(n)\Rightarrow Np(n)} \quad \dfrac{\overline{Nt(s(m))\Rightarrow Nt(s(m))} \quad \overline{\boxed{Si\{1\}}\Rightarrow Si}}{Nt(s(m)),\boxed{Nt(s(m))\backslash Si\{1\}}\Rightarrow Si}\backslash L}{Nt(s(m)),Np(n),\boxed{Np(n)\backslash(Nt(s(m))\backslash Si)\{1\}}\Rightarrow Si}\backslash L}{Nt(s(m)),Np(n),1,\boxed{\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))}\Rightarrow Si}\triangleright^{-1}L}{Np(n),1,\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))\Rightarrow Nt(s(m))\backslash Si}\backslash R \quad \dfrac{\overline{Nt(s(m))\Rightarrow Nt(s(m))}\quad \overline{\boxed{Sf}\Rightarrow Sf}}{Nt(s(m)),\boxed{Nt(s(m))\backslash Sf}\Rightarrow Sf}\backslash L}{Nt(s(m)),Np(n),\boxed{(Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash Sf)},\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))\Rightarrow Sf}\downarrow L$$

((*isable* ((*read books*) *j*)) *j*)

$Nt(s(m)) : j, Np(n) : books, Q/\hat{}(Sf\uparrow((NA\backslash Si)\downarrow(NA\backslash Sf))) : \lambda B(B\ \lambda C\lambda D((isable\ (C\ D))\ D)), \triangleright^{-1}(NE\backslash(NF\backslash Si)) : read \Rightarrow Sf$

(d(3)) **jan+boeken+wil+kunnen+lezen** : Sf

$Nt(s(m)) : j, Np(n) : books, (NA\backslash Si)\downarrow(NA\backslash Sf) : \lambda B\lambda C((want\ (B\ C))\ C), \triangleright^{-1}((ND\backslash Si)\downarrow(ND\backslash Si)) : \lambda E\lambda F((isable\ (E\ F))\ F), \triangleright^{-1}(NG\backslash(NH\backslash Si)) : read \Rightarrow Sf$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{Np(n)\Rightarrow Np(n)} \quad \dfrac{\overline{Nt(s(m))\Rightarrow Nt(s(m))} \quad \overline{\boxed{Si\{1\}}\Rightarrow Si}}{Nt(s(m)),\boxed{Nt(s(m))\backslash Si\{1\}}\Rightarrow Si}\backslash L}{Nt(s(m)),Np(n),\boxed{Np(n)\backslash(Nt(s(m))\backslash Si)\{1\}}\Rightarrow Si}\backslash L}{Nt(s(m)),Np(n),1,\boxed{\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))}\Rightarrow Si}\triangleright^{-1}L}{Np(n),1,\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))\Rightarrow Nt(s(m))\backslash Si}\backslash R \quad \dfrac{\overline{Nt(s(m))\Rightarrow Nt(s(m))}\quad \overline{\boxed{Si\{1\}}\Rightarrow Si}}{Nt(s(m)),\boxed{Nt(s(m))\backslash Si\{1\}}\Rightarrow Si}\backslash L}{Nt(s(m)),Np(n),\boxed{(Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash Si)\{1\}},\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))\Rightarrow Si}\downarrow L}{Nt(s(m)),Np(n),1,\boxed{\triangleright^{-1}((Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash Si))},\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))\Rightarrow Si}\triangleright^{-1}L}{Np(n),1,\triangleright^{-1}((Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash Si)),\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))\Rightarrow Nt(s(m))\backslash Si}\backslash R \quad \dfrac{\overline{Nt(s(m))\Rightarrow Nt(s(m))}\quad \overline{\boxed{Sf}\Rightarrow Sf}}{Nt(s(m)),\boxed{Nt(s(m))\backslash Sf}\Rightarrow Sf}\backslash L}{Nt(s(m)),Np(n),\boxed{(Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash Sf)},\triangleright^{-1}((Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash Si)),\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))\Rightarrow Sf}\downarrow L$$

((*want* ((*isable* ((*read books*) *j*)) *j*)) *j*)

$Nt(s(m)) : j, Np(n) : books, Q/\hat{}(Sf\uparrow((NA\backslash Si)\downarrow(NA\backslash Sf))) : \lambda B(B\ \lambda C\lambda D((want\ (C\ D))\ D)), \triangleright^{-1}((NE\backslash Si)\downarrow(NE\backslash Si)) : \lambda F\lambda G((isable\ (F\ G))\ G), \triangleright^{-1}(NH\backslash(NI\backslash Si)) : read \Rightarrow Sf$

(d(4)) **jan+alles+las** : Sf

$Nt(s(m)) : j, (SA\uparrow Nt(s(n)))\downarrow SA : \lambda B\forall C[(thing\ C) \to (B\ C)], ND\backslash(Nt(s(E))\backslash Sf) : read \Rightarrow Sf$

$$\dfrac{\dfrac{\dfrac{\overline{Nt(s(n)) \Rightarrow Nt(s(n))} \quad \dfrac{\overline{Nt(s(m)) \Rightarrow Nt(s(m))} \quad \overline{\boxed{Sf} \Rightarrow Sf}}{Nt(s(m)), \boxed{Nt(s(m))\backslash Sf} \Rightarrow Sf}\backslash L}{Nt(s(m)), Nt(s(n)), \boxed{Nt(s(n))\backslash(Nt(s(m))\backslash Sf)} \Rightarrow Sf}\backslash L}{Nt(s(m)), 1, Nt(s(n))\backslash(Nt(s(m))\backslash Sf) \Rightarrow Sf\uparrow Nt(s(n))}\uparrow R \quad \overline{\boxed{Sf} \Rightarrow Sf}}{Nt(s(m)), \boxed{(Sf\uparrow Nt(s(n)))\downarrow Sf}, Nt(s(n))\backslash(Nt(s(m))\backslash Sf) \Rightarrow Sf}\downarrow L$$

$\forall B[(thing\ B) \rightarrow ((read\ B)\ j)]$

$Nt(s(m)) : j, (SA\uparrow Nt(s(n)))\downarrow SA : \lambda B\forall C[(thing\ C) \rightarrow (B\ C)], Q/\hat{}(Sf\uparrow(ND\backslash(Nt(s(E))\backslash Sf))) : \lambda F(F\ read) \Rightarrow Sf$

(d(5)) **jan+alles+kan+lezen** : Sf

$Nt(s(m)) : j, (SA\uparrow Nt(s(n)))\downarrow SA : \lambda B\forall C[(thing\ C) \rightarrow (B\ C)], (ND\backslash Si)\downarrow(ND\backslash Sf) : \lambda E\lambda F((isable\ (E\ F))\ F), \triangleright^{-1}(NG\backslash(NH\backslash Si)) : read \Rightarrow Sf$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{Nt(s(n)) \Rightarrow Nt(s(n))} \quad \dfrac{\overline{Nt(s(m)) \Rightarrow Nt(s(m))} \quad \overline{\boxed{Si\{1\}} \Rightarrow Si}}{Nt(s(m)), \boxed{Nt(s(m))\backslash Si\{1\}} \Rightarrow Si}\backslash L}{Nt(s(m)), Nt(s(n)), \boxed{Nt(s(n))\backslash(Nt(s(m))\backslash Si)\{1\}} \Rightarrow Si}\backslash L}{Nt(s(m)), Nt(s(n)), 1, \boxed{\triangleright^{-1}(Nt(s(n))\backslash(Nt(s(m))\backslash Si))} \Rightarrow Si}\triangleright^{-1}L}{Nt(s(n)), 1, \triangleright^{-1}(Nt(s(n))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Nt(s(m))\backslash Si}\backslash R \quad \dfrac{\overline{Nt(s(m)) \Rightarrow Nt(s(m))} \quad \overline{\boxed{Sf} \Rightarrow Sf}}{Nt(s(m)), \boxed{Nt(s(m))\backslash Sf} \Rightarrow Sf}\backslash L}{Nt(s(m)), Nt(s(n)), \boxed{(Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash Sf)}, \triangleright^{-1}(Nt(s(n))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Sf}\downarrow L}{Nt(s(m)), 1, (Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash Sf), \triangleright^{-1}(Nt(s(n))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Sf\uparrow Nt(s(n))}\uparrow R \quad \overline{\boxed{Sf} \Rightarrow Sf}}{Nt(s(m)), \boxed{(Sf\uparrow Nt(s(n)))\downarrow Sf}, (Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash Sf), \triangleright^{-1}(Nt(s(n))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Sf}\downarrow L$$

$\forall B[(thing\ B) \rightarrow ((isable\ ((read\ B)\ j))\ j)]$

$Nt(s(m)) : j, (SA\uparrow Nt(s(n)))\downarrow SA : \lambda B\forall C[(thing\ C) \rightarrow (B\ C)], Q/\hat{}(Sf\uparrow((ND\backslash Si)\downarrow(ND\backslash Sf))) : \lambda E(E\ \lambda F\lambda G((isable\ (F\ G))\ G)), \triangleright^{-1}(NH\backslash(NI\backslash Si)) : read \Rightarrow Sf$

(d(6)) **jan+cecilia+henk+de+nijlpaarden+zag+helpen+voeren** : Sf

$Nt(s(m)) : j, Nt(s(f)) : c, Nt(s(m)) : h, Nt(s(A))/CNA : the, CNp(n) : hippos, (Nt(s(B))\backslash Si)\downarrow(NC\backslash(Nt(s(B))\backslash Sf)) : \lambda D\lambda E((saw\ (D\ E))\ E), \triangleright^{-1}((NF\backslash Si)\downarrow(NG\backslash(NF\backslash Si))) : \lambda H\lambda I((help\ (H\ I))\ I), \triangleright^{-1}(NJ\backslash(NK\backslash Si)) : feed \Rightarrow Sf$

$$\dfrac{\dfrac{}{CNp(n) \Rightarrow CNp(n)} \quad \dfrac{}{\boxed{Nt(s(p(n)))} \Rightarrow Nt(s(p(n)))}}{\boxed{Nt(s(p(n)))/CNp(n)}, CNp(n) \Rightarrow Nt(s(p(n)))} /L$$

$$\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))} \quad \dfrac{}{\boxed{Si\{1\}} \Rightarrow Si}}{Nt(s(m)), \boxed{Nt(s(m))\backslash Si\{1\}} \Rightarrow Si} \backslash L$$

$$\dfrac{\boxed{Nt(s(p(n)))/CNp(n)}, CNp(n) \Rightarrow Nt(s(p(n))) \quad Nt(s(m)), \boxed{Nt(s(m))\backslash Si\{1\}} \Rightarrow Si}{Nt(s(m)), Nt(s(p(n)))/CNp(n), CNp(n), \boxed{Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si)\{1\}} \Rightarrow Si} \backslash L$$

$$\dfrac{Nt(s(m)), Nt(s(p(n)))/CNp(n), CNp(n), \boxed{Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si)\{1\}} \Rightarrow Si}{Nt(s(m)), Nt(s(p(n)))/CNp(n), CNp(n), 1, \boxed{\triangleright^{-1}(Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si))} \Rightarrow Si} \triangleright^{-1}L$$

$$\dfrac{Nt(s(m)), Nt(s(p(n)))/CNp(n), CNp(n), 1, \boxed{\triangleright^{-1}(Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si))} \Rightarrow Si}{Nt(s(p(n)))/CNp(n), CNp(n), 1, \triangleright^{-1}(Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Nt(s(m))\backslash Si} \backslash R$$

$$\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))} \quad \dfrac{}{\boxed{Si\{1\}} \Rightarrow Si}}{Nt(s(m)), \boxed{Nt(s(m))\backslash Si\{1\}} \Rightarrow Si} \backslash L$$

$$\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))} \quad Nt(s(m)), \boxed{Nt(s(m))\backslash Si\{1\}} \Rightarrow Si}{Nt(s(m)), Nt(s(m)), \boxed{Nt(s(m))\backslash(Nt(s(m))\backslash Si)\{1\}} \Rightarrow Si} \backslash L$$

$$\dfrac{Nt(s(p(n)))/CNp(n), CNp(n), 1, \triangleright^{-1}(Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Nt(s(m))\backslash Si \quad Nt(s(m)), Nt(s(m)), \boxed{Nt(s(m))\backslash(Nt(s(m))\backslash Si)\{1\}} \Rightarrow Si}{Nt(s(m)), Nt(s(m)), Nt(s(p(n)))/CNp(n), CNp(n), \boxed{(Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash(Nt(s(m))\backslash Si))\{1\}}, \triangleright^{-1}(Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Si} \downarrow L$$

$$\dfrac{Nt(s(m)), Nt(s(m)), Nt(s(p(n)))/CNp(n), CNp(n), \boxed{(Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash(Nt(s(m))\backslash Si))\{1\}}, \triangleright^{-1}(Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Si}{Nt(s(m)), Nt(s(m)), Nt(s(p(n)))/CNp(n), CNp(n), 1, \boxed{\triangleright^{-1}((Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash(Nt(s(m))\backslash Si)))}, \triangleright^{-1}(Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Si} \triangleright^{-1}L$$

$$\dfrac{Nt(s(m)), Nt(s(m)), Nt(s(p(n)))/CNp(n), CNp(n), 1, \boxed{\triangleright^{-1}((Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash(Nt(s(m))\backslash Si)))}, \triangleright^{-1}(Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Si}{Nt(s(m)), Nt(s(p(n)))/CNp(n), CNp(n), 1, \triangleright^{-1}((Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash(Nt(s(m))\backslash Si))), \triangleright^{-1}(Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Nt(s(m))\backslash Si} \backslash R$$

$$\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))} \quad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{Nt(s(m)), \boxed{Nt(s(m))\backslash Sf} \Rightarrow Sf} \backslash L$$

$$\dfrac{\dfrac{}{Nt(s(f)) \Rightarrow Nt(s(f))} \quad Nt(s(m)), \boxed{Nt(s(m))\backslash Sf} \Rightarrow Sf}{Nt(s(m)), Nt(s(f)), \boxed{Nt(s(f))\backslash(Nt(s(m))\backslash Sf)} \Rightarrow Sf} \backslash L$$

$$\dfrac{Nt(s(m)), Nt(s(p(n)))/CNp(n), CNp(n), 1, \triangleright^{-1}((Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash(Nt(s(m))\backslash Si))), \triangleright^{-1}(Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Nt(s(m))\backslash Si \quad Nt(s(m)), Nt(s(f)), \boxed{Nt(s(f))\backslash(Nt(s(m))\backslash Sf)} \Rightarrow Sf}{Nt(s(m)), Nt(s(f)), Nt(s(m)), Nt(s(p(n)))/CNp(n), CNp(n), \boxed{(Nt(s(m))\backslash Si)\downarrow(Nt(s(f))\backslash(Nt(s(m))\backslash Sf))}, \triangleright^{-1}((Nt(s(m))\backslash Si)\downarrow(Nt(s(m))\backslash(Nt(s(m))\backslash Si))), \triangleright^{-1}(Nt(s(p(n)))\backslash(Nt(s(m))\backslash Si)) \Rightarrow Sf} \downarrow L$$

$(((saw\ (((help\ ((feed\ (the\ hippos))\ h))\ h)\ c))\ c)\ j)$

(d(7)) **wil+jan+boeken+lezen** : Q

$(NA\backslash Si){\downarrow}(NA\backslash Sf) : \lambda B\lambda C((want\ (B\ C))\ C), Nt(s(m)) : j, Np(n) : books, \triangleright^{-1}(ND\backslash(NE\backslash Si)) : read \Rightarrow Q$

$Q/\hat{}(Sf{\uparrow}((NA\backslash Si){\downarrow}(NA\backslash Sf))) : \lambda B(B\ \lambda C\lambda D((want\ (C\ D))\ D)), Nt(s(m)) : j, Np(n) : books, \triangleright^{-1}(NE\backslash(NF\backslash Si)) : read \Rightarrow Q$

$$\overline{Nt(s(m)) \Rightarrow Nt(s(m))} \qquad \overline{\boxed{Si\{1\}} \Rightarrow Si}$$
$$Nt(s(m)), \boxed{Nt(s(m))\backslash Si\{1\}} \Rightarrow Si \quad \backslash L$$
$$\overline{Np(n) \Rightarrow Np(n)}$$
$$Nt(s(m)), Np(n), \boxed{Np(n)\backslash(Nt(s(m))\backslash Si)\{1\}} \Rightarrow Si \quad \backslash L$$
$$Nt(s(m)), Np(n), 1, \boxed{\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))} \Rightarrow Si \quad \triangleright^{-1}L$$
$$Np(n), 1, \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Nt(s(m))\backslash Si \quad \backslash R$$
$$\overline{Nt(s(m)) \Rightarrow Nt(s(m))} \qquad \overline{\boxed{Sf} \Rightarrow Sf}$$
$$Nt(s(m)), \boxed{Nt(s(m))\backslash Sf} \Rightarrow Sf \quad \backslash L$$
$$Nt(s(m)), Np(n), \boxed{(Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf)}, \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Sf \quad {\downarrow}L$$
$$Nt(s(m)), Np(n), 1, \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf)) \quad {\uparrow}R$$
$$Nt(s(m)), Np(n), \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow \boxed{\hat{}(Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf)))} \quad \hat{}R$$
$$\overline{\boxed{Q} \Rightarrow Q}$$
$$\boxed{Q/\hat{}(Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf)))}, Nt(s(m)), Np(n), \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Q \quad /L$$

$((want\ ((read\ books)\ j))\ j)$

(d(8)) **jan+wil+boeken+lezen** : $Nt(s(m)){\bullet}\hat{}(Q{\uparrow}Nt(s(m)))$

$Nt(s(m)) : j, (NA\backslash Si){\downarrow}(NA\backslash Sf) : \lambda B\lambda C((want\ (B\ C))\ C), Np(n) : books, \triangleright^{-1}(ND\backslash(NE\backslash Si)) : read \Rightarrow Nt(s(m)){\bullet}\hat{}(Q{\uparrow}Nt(s(m)))$

$Nt(s(m)) : j, Q/\hat{}(Sf{\uparrow}((NA\backslash Si){\downarrow}(NA\backslash Sf))) : \lambda B(B\ \lambda C\lambda D((want\ (C\ D))\ D)), Np(n) : books, \triangleright^{-1}(NE\backslash(NF\backslash Si)) : read \Rightarrow Nt(s(m)){\bullet}\hat{}(Q{\uparrow}Nt(s(m)))$

$$\overline{Np(n) \Rightarrow Np(n)} \qquad \overline{\boxed{Si\{1\}} \Rightarrow Si}$$
$$Np(n), \boxed{Np(n)\backslash Si\{1\}} \Rightarrow Si \quad \backslash L$$
$$\overline{Nt(s(m)) \Rightarrow Nt(s(m))}$$
$$Np(n), Nt(s(m)), \boxed{Nt(s(m))\backslash(Np(n)\backslash Si)\{1\}} \Rightarrow Si \quad \backslash L$$
$$Np(n), Nt(s(m)), 1, \boxed{\triangleright^{-1}(Nt(s(m))\backslash(Np(n)\backslash Si))} \Rightarrow Si \quad \triangleright^{-1}L$$
$$Nt(s(m)), 1, \triangleright^{-1}(Nt(s(m))\backslash(Np(n)\backslash Si)) \Rightarrow Np(n)\backslash Si \quad \backslash R$$
$$\overline{Np(n) \Rightarrow Np(n)} \qquad \overline{\boxed{Sf} \Rightarrow Sf}$$
$$Np(n), \boxed{Np(n)\backslash Sf} \Rightarrow Sf \quad \backslash L$$
$$Np(n), Nt(s(m)), \boxed{(Np(n)\backslash Si){\downarrow}(Np(n)\backslash Sf)}, \triangleright^{-1}(Nt(s(m))\backslash(Np(n)\backslash Si)) \Rightarrow Sf \quad {\downarrow}L$$
$$Np(n), Nt(s(m)), 1, \triangleright^{-1}(Nt(s(m))\backslash(Np(n)\backslash Si)) \Rightarrow Sf{\uparrow}((Np(n)\backslash Si){\downarrow}(Np(n)\backslash Sf)) \quad {\uparrow}R$$
$$Np(n), Nt(s(m)), \triangleright^{-1}(Nt(s(m))\backslash(Np(n)\backslash Si)) \Rightarrow \boxed{\hat{}(Sf{\uparrow}((Np(n)\backslash Si){\downarrow}(Np(n)\backslash Sf)))} \quad \hat{}R$$
$$\overline{\boxed{Q} \Rightarrow Q}$$
$$\boxed{Q/\hat{}(Sf{\uparrow}((Np(n)\backslash Si){\downarrow}(Np(n)\backslash Sf)))}, Np(n), Nt(s(m)), \triangleright^{-1}(Nt(s(m))\backslash(Np(n)\backslash Si)) \Rightarrow Q \quad /L$$
$$Q/\hat{}(Sf{\uparrow}((Np(n)\backslash Si){\downarrow}(Np(n)\backslash Sf))), Np(n), 1, \triangleright^{-1}(Nt(s(m))\backslash(Np(n)\backslash Si)) \Rightarrow Q{\uparrow}Nt(s(m)) \quad {\uparrow}R$$
$$\overline{Nt(s(m)) \Rightarrow Nt(s(m))}$$
$$Q/\hat{}(Sf{\uparrow}((Np(n)\backslash Si){\downarrow}(Np(n)\backslash Sf))), Np(n), \triangleright^{-1}(Nt(s(m))\backslash(Np(n)\backslash Si)) \Rightarrow \boxed{\hat{}(Q{\uparrow}Nt(s(m)))} \quad \hat{}R$$
$$Nt(s(m)), Q/\hat{}(Sf{\uparrow}((Np(n)\backslash Si){\downarrow}(Np(n)\backslash Sf))), Np(n), \triangleright^{-1}(Nt(s(m))\backslash(Np(n)\backslash Si)) \Rightarrow \boxed{Nt(s(m)){\bullet}\hat{}(Q{\uparrow}Nt(s(m)))} \quad {\bullet}R$$

$(j, \lambda A((want\ ((read\ A)\ books))\ books))$

$$\dfrac{Nt(s(m)) \Rightarrow Nt(s(m)) \qquad \boxed{Si\{1\}} \Rightarrow Si}{Nt(s(m)), \boxed{Nt(s(m))\backslash Si\{1\}} \Rightarrow Si}\,\backslash L$$

$$\dfrac{Np(n) \Rightarrow Np(n) \qquad Nt(s(m)), \boxed{Nt(s(m))\backslash Si\{1\}} \Rightarrow Si}{Nt(s(m)), Np(n), \boxed{Np(n)\backslash(Nt(s(m))\backslash Si)\{1\}} \Rightarrow Si}\,\backslash L$$

$$\dfrac{Nt(s(m)), Np(n), \boxed{Np(n)\backslash(Nt(s(m))\backslash Si)\{1\}} \Rightarrow Si}{Nt(s(m)), Np(n), 1, \boxed{\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))} \Rightarrow Si}\,\triangleright^{-1}L$$

$$\dfrac{Nt(s(m)), Np(n), 1, \boxed{\triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si))} \Rightarrow Si}{Np(n), 1, \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Nt(s(m))\backslash Si}\,\backslash R$$

$$\dfrac{Nt(s(m)) \Rightarrow Nt(s(m)) \qquad \boxed{Sf} \Rightarrow Sf}{Nt(s(m)), \boxed{Nt(s(m))\backslash Sf} \Rightarrow Sf}\,\backslash L$$

$$\dfrac{Np(n), 1, \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Nt(s(m))\backslash Si \qquad Nt(s(m)), \boxed{Nt(s(m))\backslash Sf} \Rightarrow Sf}{Nt(s(m)), Np(n), \boxed{(Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf)}, \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Sf}\,{\downarrow}L$$

$$\dfrac{Nt(s(m)), Np(n), \boxed{(Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf)}, \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Sf}{Nt(s(m)), Np(n), 1, \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf))}\,{\uparrow}R$$

$$\dfrac{Nt(s(m)), Np(n), 1, \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf))}{Nt(s(m)), Np(n), \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow \boxed{\hat{}(Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf)))}}\,\hat{}R$$

$$\dfrac{Nt(s(m)), Np(n), \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow \boxed{\hat{}(Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf)))} \qquad \boxed{Q} \Rightarrow Q}{\boxed{Q/\hat{}(Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf)))}, Nt(s(m)), Np(n), \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Q}\,/L$$

$$\dfrac{\boxed{Q/\hat{}(Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf)))}, Nt(s(m)), Np(n), \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Q}{Q/\hat{}(Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf))), 1, Np(n), \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Q{\uparrow}Nt(s(m))}\,{\uparrow}R$$

$$\dfrac{Q/\hat{}(Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf))), 1, Np(n), \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow Q{\uparrow}Nt(s(m))}{Q/\hat{}(Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf))), Np(n), \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow \boxed{\hat{}(Q{\uparrow}Nt(s(m)))}}\,\hat{}R$$

$$\dfrac{Nt(s(m)) \Rightarrow Nt(s(m)) \qquad Q/\hat{}(Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf))), Np(n), \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow \boxed{\hat{}(Q{\uparrow}Nt(s(m)))}}{Nt(s(m)), Q/\hat{}(Sf{\uparrow}((Nt(s(m))\backslash Si){\downarrow}(Nt(s(m))\backslash Sf))), Np(n), \triangleright^{-1}(Np(n)\backslash(Nt(s(m))\backslash Si)) \Rightarrow \boxed{Nt(s(m))\bullet\hat{}(Q{\uparrow}Nt(s(m)))}}\,\bullet R$$

$(j, \lambda A((\mathit{want}\ ((\mathit{read}\ \mathit{books})\ A))\ A))$

Chapter 16

Relativization

In this chapter we present an account of relativisation.

16.1 Routes we do not take

Szaboloczi (1983[88]) and Steedman (1987[86]) aim to account for parasitic gaps in combinatory categorial grammar (CCG) by means of the combinator **S** such that $\mathbf{S}\ x\ y\ z\ =\ (x\ z)\ (y\ z)$, for example positing a combinatory schema:

(334) $y\colon Y/Z, x\colon (Y\backslash X)/Z \Rightarrow \mathbf{S}\ x\ y\colon X/Z$

Such a schema makes no sense from the point of view of the logicisation of grammar pursued here. The rule is not Lambek-valid and any semantics validating it would also validate schemata which overgenerate massively. So much the worse, the proponents of CCG would say, for grammar as logic: grammar is a formal system but not a logic, and one should not care about things like soundness and completeness.

CCG and type logical grammar agree on the task of defining syntax and semantics of the (object) natural language. What is curious about CCG is that at the same time it declines to consider syntax (proof theory) and semantics (model theory) of the (meta-)linguistic formalism. A CCG account of parasitic gaps, which employs just the directional slashes and a minimum of combinatory schemata, must capture the effects of structural inhibition (islands) and structural facilitation (parasiticy) by good fortune in the interaction of the combinatory schemata chosen and the categorial types occurring in grammar. Here control of structural inhibition by bracket modalities and control of structural facilitation by subexponentials are separated in an analysis recognising the distinct algebraic roles of variation from an associative and from a linear regime. This type logical approach lets us state our analysis with clarity in the knowledge that whatever the empirical adequacy, the metatheoretical facts are what they are. In CCG the metatheory is not logically investigated.

It is interesting to ask why we treat medial extraction here with ! rather than with $\uparrow$ as illustrated in Section 8.4 of Chapter 8 (cf. also Moortgat 1988[52]; Muskens 2003[82]; Mihaliček and Pollard 2012[49]; Barker and Shan 2015[8]; and Kubota and Levine 2015[40]). The answer is that, on the one hand, $\uparrow$ as defined does not respect island constraints and, on the other hand, $\uparrow$ does not extend to parasitic gaps: it is unclear how a single local inference rule can account for unbounded recursive nesting of parasitic gaps in subislands. Our treatment in terms of ! both respects islands, and extends to (unbounded numbers of) parasitic gaps through iteration of contraction.[1]

An option available in both CCG and type logical grammar is to attempt to analyse the nonlinearity of parasitic extraction not syntactically but lexically. Thus for example Jansche and Vasishth (2002[30]) propose induction of parasitic gaps in adverbial clauses by a lexicalised gap-duplicating effect in the adverbial head. All contexts allowing parasitic gaps would require a corresponding gap-duplicating lexical ambiguity. The appeal to lexical ambiguity in lexical grammar formalisms is as frequent as it is untenable.

[1] We note that the discontinuity operators serve to account for the pied-piping aspect of relativisation (e.g. Morrill, Valentín and Fadda 2011[79]), see example (307) in Chapter 15.

Every ambiguity of every item doubles the lexical insertion search space. And in the case in hand there is to our knowledge no independent evidence, such as difference in meaning, for positing lexical ambiguity underlying parasitic extraction. We continue on the assumption that it is indeed a syntactic phenomenon.

16.2 Relativisation

Our account of relativisation rests on the lexical projection of islands by argument bracketing ($\langle\rangle$) and value antibracketing ($[\,]^{-1}$), and a single relative pronoun type of overall shape $R/((\langle\rangle N\sqcap !N)\backslash S)$ for both subject and object relativisation. In analysis of the body of relative clauses the higher order succedent argument essentially of form $\langle\rangle N\sqcap !N$ is lowered into the antecedent according to the deduction theorem; in subject relativisation $\langle\rangle N$ is selected by conjunction left, and satisfies the (bracketed) subject valency.

In object relativisation $!N$ is selected. When the $!L$ rule is applied to $!N$, the hypothetical subtype N moves into the stoup, from where it can move by $!P$ to any (nonisland) position in its zone, realising nonparasitic extraction.

However, in addition it can be copied by $!C$ to the stoup of a newly created weak island domain, realising parasitic extraction. The N in the outer stoop can be copied by $!C$ repeatedly, capturing that there may be parasitic gaps in any number of local weak islands; at the end of this process it moves by $!P$ to a host position in its zone. The N in an inner stoup can also be copied by $!C$ to the stoup of any number of newly created weak subislands, and so on recursively, capturing that parasitic gaps can also be hosts to further parasitic gaps; finally the stoup contents are copied by $!P$ to an extraction site in their zone.

In this section we analyse examples illustrating the account of relativisation. (cf. Morrill 2011, Chapter 5). The first example is a minimal subject relativisation; note that the relative clause is doubly bracketed, corresponding to the fact that relative clauses are strong islands:[2]

(335) **man**+[[**that**+**walks**]] : $CNs(m)$

Lexical lookup yields the following, where there is semantically inactive additive conjunction of the hypothetical subtypes $\langle\rangle N$ for subject relativisation and $!\blacksquare N$ for object relativisation; the (semantically inactive) modality on the object gap subtype is to permit object relativisation from embedded modal/intensional domains:[3]

(336) $\square CNs(m) : man, [[\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf)) :$
$\lambda A\lambda B\lambda C[(B\ C) \wedge (A\ C)], \square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda D(Pres\ (\check{}walk\ D))]] \Rightarrow CNs(m)$

There is the derivation in Figure 16.1, which starts with the relative clause doubly bracketed (this will always be the case for relativisation). After elimination of the outer (semantically inactive) modality of the relative pronoun, universal left instantiates it to agree with masculine singular. Then $/L$ partitions in such a way as to select the intransitive verb body of the relative clause as argument of the relative pronoun. In the righthand, value, subtree two antibracket eliminations cancel the double brackets before the head common noun is modified. In the lefthand, argument, subtree (inactive) box right is enabled since the antecedent is modalised, and under right then lowers the additively conjoined hypothetical subtype into the antecedent. Observe how in the lefthand subtree $\sqcap$L selects the subject relativisation hypothetical subtype $\langle\rangle Nt(s(m))$; the remaining subderivation is the usual intransitive sentence analysis. This delivers the required semantics:

(337) $\lambda C[(\check{}man\ C) \wedge (Pres\ (\check{}walk\ C))]$

The next sentence contains a minimal example of object relativisation:

(338) [**the**+**man**+[[**that**+[**mary**]+**loves**]]+**walks** : Sf

Lexical lookup yields:

[2] As we will see relative clauses themselves, being doubly bracketed, will not allow parasitic gaps.

[3] The body of the relative clause is marked as a (semantically inactive) modal domain in order to make it a scope island. Thus where, say, **everyone** has a type $\square((S\uparrow N)\downarrow S)$ the unmodalised hypothetical subtype N cannot be bound outside the modal domain of the body of a relative clause in which **everyone** occurs.

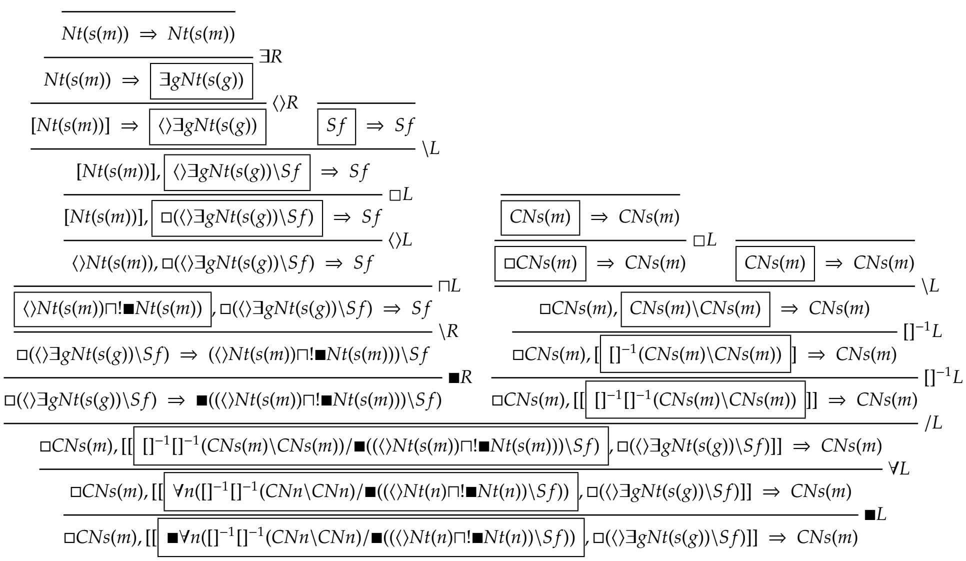


Figure 16.1: Derivation for **man that walks**

(339) $[\blacksquare\forall n(Nt(n)/CNn) : \iota, \square CNs(m) : man, [[\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/$
$\blacksquare(((\langle\rangle Nt(n)\sqcap!\blacksquare Nt(n))\backslash Sf)) : \lambda A\lambda B\lambda C[(B\ C) \wedge (A\ C)], [\blacksquare Nt(s(f)) : m],$
$\square(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda D\lambda E(Pres\ ((\check{}love\ D)\ E))]]], \square(\langle\rangle\exists gNt(s(g))\backslash Sf) :$
$\hat{}\lambda F(Pres\ (\check{}walk\ F)) \Rightarrow Sf$

There is the derivation given in Figure 16.2. The lowest four inferences prepare the subject of the intransitive matrix verb and the next three prepare the relative clause modification itself, argument to the subject definite article. The analysis of the complex common noun phrase starts in the minor premise of the lowest $/L$ with (semantically inactive) modality left, and for all left instantiating agreement to masculine singular. At the middle $/L$, the righthand subtree cancels the double brackets with the relative pronoun value antibrackets and the lefthand subtree selects the body of the relative clause as the semantically inactive modalised higher-order subject-and-object polymorphic relative pronoun argument type. After (semantically inactive) modality right, licensed since the antecedent types are modalised, the conjoined hypothetical subject is lowered by $\backslash R$ into the antecedent. Observe how $\sqcap L$ selects the object relativisation hypothetical subtype $!\blacksquare Nt(s(m))$ and how this subsequently percolates in the stoup, passing in particular into the minor premise branch of the upper $/L$ inference and hence satisfying the object valency of the transitive verb **love**; subject and intransitive verb phrase are analysed as usual. This delivers the required semantics:

(340) $(Pres\ (\check{}walk\ (\iota\ \lambda D[(\check{}man\ D) \wedge (Pres\ ((\check{}love\ D)\ m))])))$

An example with longer-distance object relativisation, in the context of an entire sentence, is:

(341) **[the**+**man**+**[[that**+**[john]**+**thinks**+**[mary]**+**loves]]]**+**walks** : Sf

Lexical lookup yields the following; note how the propositional attitude verb is polymorphic between a complementised and an uncomplementised sentential argument, expressed with a semantically inactive additive disjunction:

(342) $[\blacksquare\forall n(Nt(n)/CNn) : \iota, \square CNs(m) : man, [[\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/$
$\blacksquare(((\langle\rangle Nt(n)\sqcap!\blacksquare Nt(n))\backslash Sf)) : \lambda A\lambda B\lambda C[(B\ C) \wedge (A\ C)], [\blacksquare Nt(s(m)) : j],$
$\square(((\langle\rangle\exists gNt(s(g))\backslash Sf)/(CPthat\sqcup\square Sf)) : \hat{}\lambda D\lambda E(Pres\ ((\check{}think\ D)\ E)),$
$[\blacksquare Nt(s(f)) : m], \square(((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda F\lambda G(Pres\ ((\check{}love\ F)\ G))]]],$
$\square(\langle\rangle\exists gNt(s(g))\backslash Sf) : \hat{}\lambda H(Pres\ (\check{}walk\ H)) \Rightarrow Sf$

Figure 16.2: Derivation for **The man that Mary loves walks**

,

There is the derivation given in Figure 16.3. Inference up as far as ① brings us to analysis of the complex common noun phrase in the lefthand subtree. The following preparation of the relative pronoun and double bracket cancellation of its value are as usual. After modality right and under right on the relative pronoun higher-order argument, $\sqcap L$ selects the object relativisation hypothetical subtype and $!L$ moves this into the stoup. In the stoup it percolates to the subordinate clause, (observe how $\sqcup R$ selects the uncomplementised sentential argument type of the propositional attitude verb) and there $!P$ moves it into position to satisfy the embedded clause object valency. This delivers the correct semantics:

(343) $(Pres\ (\check{}walk\ (\iota\ \lambda D[(\check{}man\ D) \wedge (Pres\ ((\check{}think\ \hat{}(Pres\ ((\check{}love\ D)\ m)))\ j))])))$

There follows an example of medial object relativisation (the gap is in a non-peripheral position left of the adverb):

(344) **man**+[[**that**+[**mary**]+**likes**+**today**]] : $CNs(m)$

Appropriate lexical lookup yields:

(345) $\square CNs(m) : man, [[\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap!\blacksquare Nt(n))\backslash Sf)) :$
$\lambda A\lambda B\lambda C[(B\ C) \wedge (A\ C)], [\blacksquare Nt(s(f)) : m], \square((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) :$
$\hat{}\lambda D\lambda E(Pres\ ((\check{}like\ D)\ E)), \square\forall a\forall f((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf)) : \hat{}\lambda F\lambda G(\check{}today\ (F\ G))]] \Rightarrow CNs(m)$

There is the derivation in Figure 16.4. Analysis of the complex common noun phrase begins at the lefthand subtree ①. After modality right and conditionalisation of the conjoined hypothetical subtype, additive conjunction left applies to this latter to select the object relativisation subtype, which then moves into the stoup. After preparation of the adverb the stoup contents pass into its argument subbranch. Note how the object relativisation hypothetical gap subtype percolates in the stoup to satisfy the transitive verb object valency. The semantics delivered is:

(346) $\lambda C[(\check{}man\ C) \wedge (\check{}today\ (Pres\ ((\check{}like\ C)\ m)))]$

As we see in (347), **assure** has an obligatory extraction valency: its second object cannot be realized canonically, by lexical material, but must correspond to an extraction dependency (Kayne 1984[34]):

(347) a. ∗ John assures Mary Bill to be reliable.
b. the man that John assures Mary to be reliable

This can be captured by assigning the verb a type in which that extraction valency is marked by the universal subexponential modality: **assure**: $(((N\backslash S)/(N\backslash S))/!N)/N$. Then (347a) is correctly blocked because $N \Rightarrow !N$ is not derivable whereas (347b) is correctly derived because $!N \Rightarrow !N$ is derivable:

(348)
$$\dfrac{\dfrac{N \Rightarrow N}{!N \Rightarrow N}\,!L}{!N \Rightarrow !N}\,!R$$

As we remarked as the beginning of Section 12 subjects are weak islands (the Subject Condition of Chomsky 1973[13]); accordingly in our CatLog2 fragment there is no derivation of simple relativisation from a subject such as:

(349) **man**+[[**that**+[**the**+**friends**+**of**]+**walk**]] : $CNs(m)$

This is because **walk** projects brackets around its subject, but the permutation of the ! hypothetical gap subtype issued by the relative pronoun is limited to its zone and cannot penetrate a bracketed subzone. Roughly, the derivation blocks at ∗ in:

(350)
$$\dfrac{\dfrac{\dfrac{[N/CN, CN/PP, PP/N, N], N\backslash S \Rightarrow S}{N; [N/CN, CN/PP, PP/N], N\backslash S \Rightarrow S}\,{*}!P}{!N, [N/CN, CN/PP, PP/N], N\backslash S \Rightarrow S}\,!L}{[N/CN, CN/PP, PP/N], N\backslash S \Rightarrow !N\backslash S}\,\backslash R$$

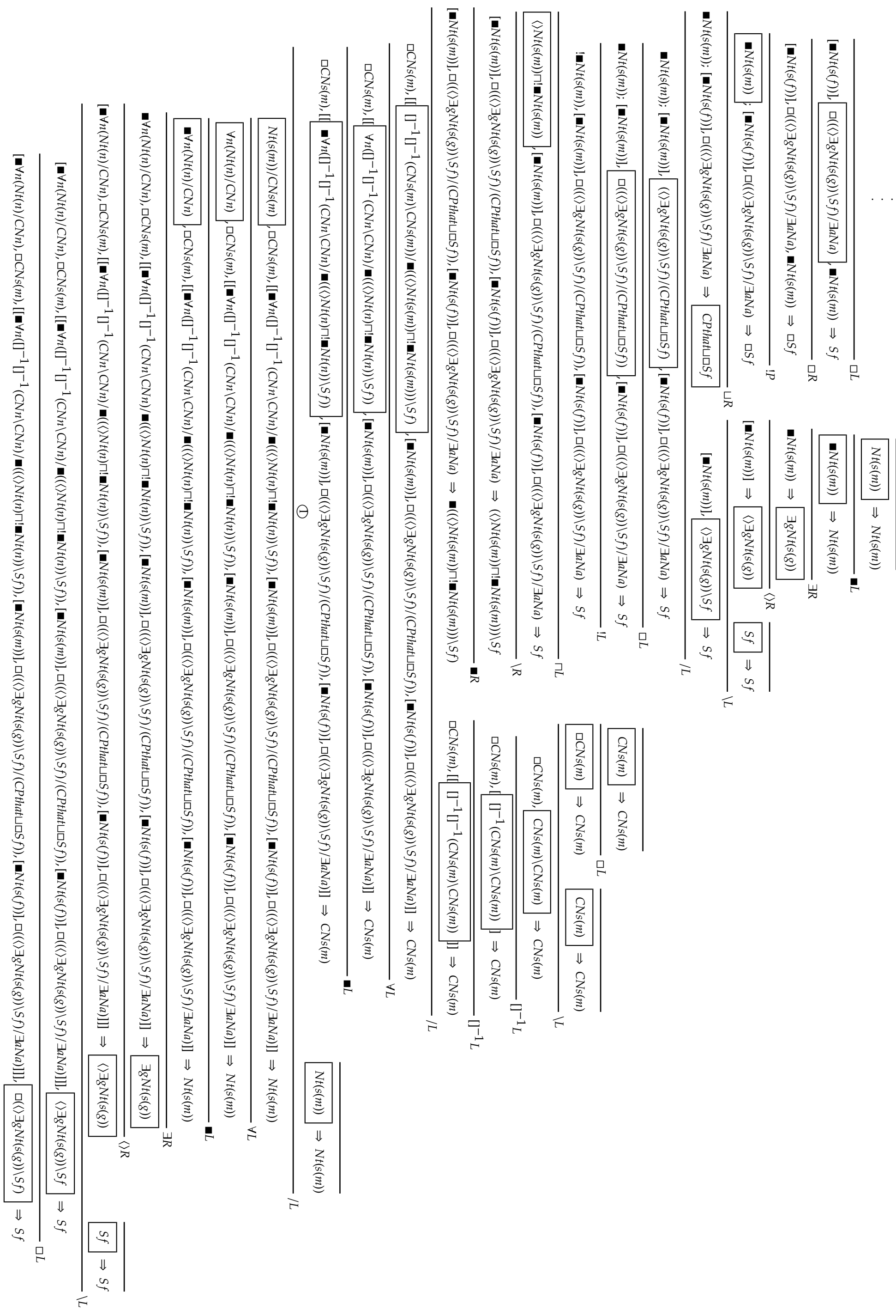


Figure 16.3: Derivation for **The man that John thinks Mary loves walks**

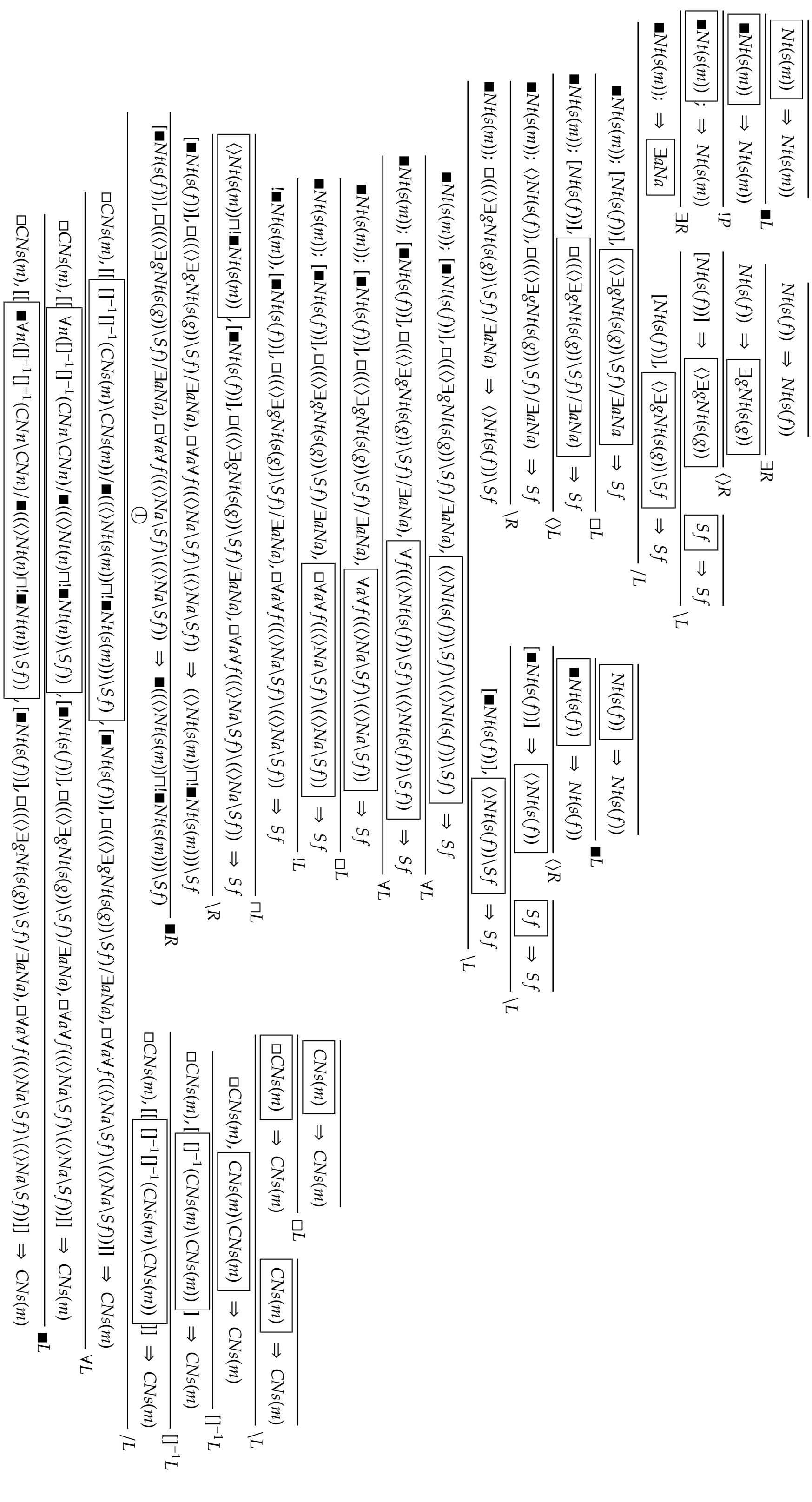


Figure 16.4: Derivation of medial relativisation: **man that Mary likes today**

However, a weak island 'parasitic' gap can be licensed by a host gap:

(351) **man**+[[**that**+**the**+**friends**+**of**+**admire**]] : $CNs(m)$

Lexical lookup yields:[4]

(352) $\Box CNs(m) : man, [[\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap !\blacksquare Nt(n))\backslash Sf)) :$
$\lambda A\lambda B\lambda C[(B\ C) \wedge (A\ C)], \blacksquare\forall n(Nt(n)/CNn) : \iota, \Box(CNp/PPof) : friends,$
$\Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)) : \hat{}(\check{}of, \lambda DD),$
$\Box(((\langle\rangle(\exists aNa - \exists gNt(s(g)))\backslash Sf)/\exists aNa) : \hat{}\lambda E\lambda F(Pres\ ((\check{}admire\ E)\ F))]] \Rightarrow CNs(m)$

There is the derivation given in Figure 16.5, where the use of contraction $!C$, involving brackets and stoups, corresponds to generating the parasitic gap. The object relativisation hypothetical subtype moves into the stoup at depth seven in the lefthand subtree (before this the analysis is standard). Contraction then applies copying the gap type into the stoup of a newly created bracketed domain around the subordinate subject. Applications of $!P$ then move the stoup contents into the object position of **admire** (host) and **of** (parasitic).

[4] We gloss over the use of 'difference' here to mark non-third person singular; its use depends on *absence* of derivability (negation as failure) which of course cannot easily be displayed.

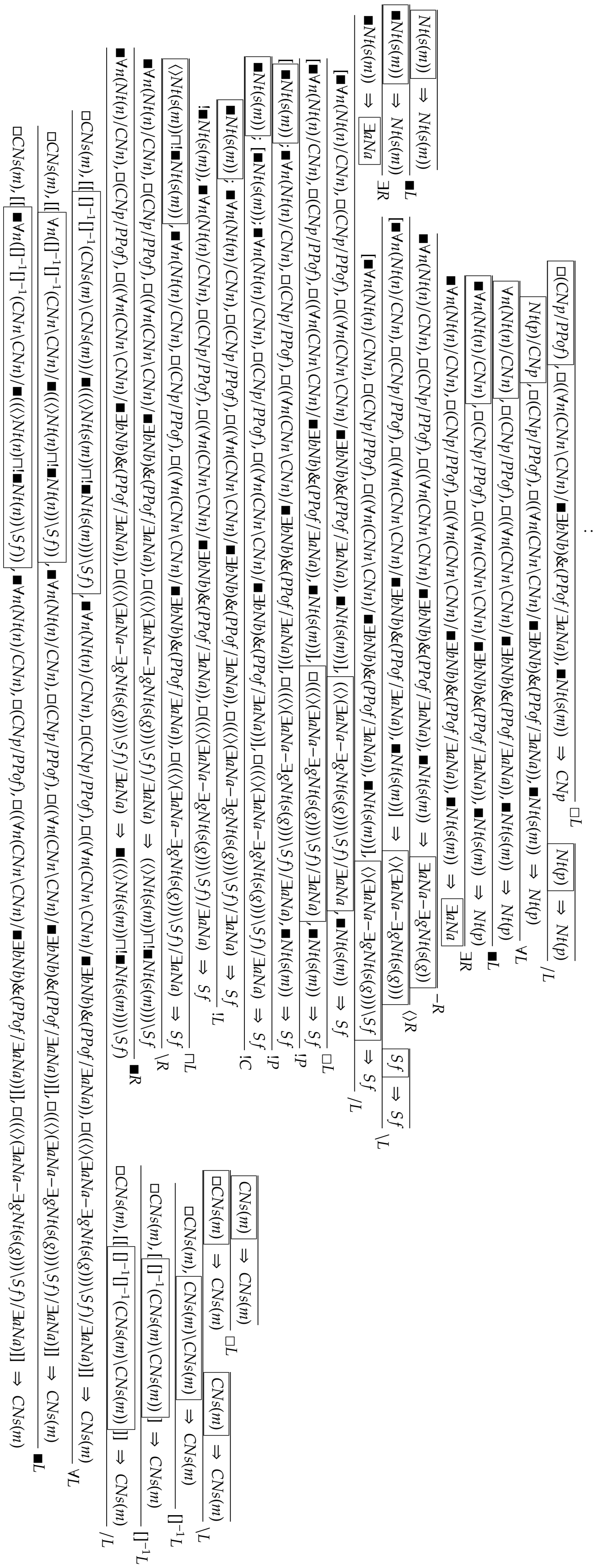


Figure 16.5: Derivation of **man that the friends of admire**

This delivers the following semantics in which the gap variable is multiply bound:

(353) $\lambda C[(\check{}man\ C) \wedge (Pres\ ((\check{}admire\ C)\ (\iota\ (\check{}friends\ C))))]$

Parasitic extraction from strong islands such as coordinate structures is not acceptable:

(354)∗that$_i$ Mary showed [[John and the friends of t_i]] to t_i

This is successfully blocked because strong islands are doubly bracketed. Although contraction could apply twice to introduce two bracketings, a copy of the hypothetical gap subtype would remain trapped in the stoup at the intermediate level of bracketing, blocking overall derivation. Likewise, as we remarked in footnote 2, parasitic extraction is not possible from relative clauses themselves, for the same reason: a superfluous gap subtype would remain in between the double brackets required for the strong island.

A parasitic gap can also appear in an adverbial weak island:

(355) **paper+[[that+[john]+filed+without+reading]]** : $CNs(n)$

Lexical lookup for this example yields:

(356) $\Box CNs(n) : paper, [[\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap!\blacksquare Nt(n))\backslash Sf)) :$
$\lambda A\lambda B\lambda C[(B\ C) \wedge (A\ C)], [\blacksquare Nt(s(m)) : j], \Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) :$
$\hat{}\lambda D\lambda E(Past\ ((\check{}file\ D)\ E)), \blacksquare\forall a\forall f([]^{-1}((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))/(\langle\rangle Na\backslash Spsp)) :$
$\lambda F\lambda G\lambda H[(G\ H) \wedge \neg(F\ H)], \Box((\langle\rangle\exists aNa\backslash Spsp)/\exists aNa) : \hat{}\lambda I\lambda J((\check{}read\ I)\ J)]] \Rightarrow CNs(n)$

There is the derivation given in Figure 16.6. This time at depth eight contraction copies the host stoup gap into the stoup of a newly created bracketed domain around the subordinate adverbial phrase. This delivers semantics:

(357) $\lambda C[(\check{}paper\ C) \wedge [(Past\ ((\check{}file\ C)\ j)) \wedge \neg((\check{}read\ C)\ j)]]$

In our final relativisation example the host gap licences two parasitic gaps, in the subject noun phrase and in an adverbial phrase:

(358) **paper+[[that+the+editor+of+filed+without+reading]]** : $CNs(n)$

Lexical lookup yields:

(359) $\Box CNs(n) : paper, [[\blacksquare\forall n([]^{-1}[]^{-1}(CNn\backslash CNn)/\blacksquare((\langle\rangle Nt(n)\sqcap!\blacksquare Nt(n))\backslash Sf)) :$
$\lambda A\lambda B\lambda C[(B\ C) \wedge (A\ C)], \blacksquare\forall n(Nt(n)/CNn) : \iota, \Box(\forall gCNs(g)/PPof) : editor,$
$\Box((\forall n(CNn\backslash CNn)/\blacksquare\exists bNb)\&(PPof/\exists aNa)) : \hat{}(\check{}of, \lambda DD),$
$\Box((\langle\rangle\exists gNt(s(g))\backslash Sf)/\exists aNa) : \hat{}\lambda E\lambda F(Past\ ((\check{}file\ E)\ F)),$
$\blacksquare\forall a\forall f([]^{-1}((\langle\rangle Na\backslash Sf)\backslash(\langle\rangle Na\backslash Sf))/(\langle\rangle Na\backslash Spsp)) : \lambda G\lambda H\lambda I[(H\ I) \wedge \neg(G\ I)],$
$\Box((\langle\rangle\exists aNa\backslash Spsp)/\exists aNa) : \hat{}\lambda J\lambda K((\check{}read\ J)\ K)]] \Rightarrow CNs(n)$

There is the derivation fragmented into Figures 16.7 and 16.8. There are two applications of contraction, at depth nine and ten, projecting brackets around the subordinate subject and adverbial phrase and giving rise to two parasitic gaps. This delivers the correct semantics:

(360) $\lambda C[(\check{}paper\ C) \wedge [(Past\ ((\check{}file\ C)\ (\iota\ (\check{}editor\ C)))) \wedge \neg((\check{}read\ C)\ (\iota\ (\check{}editor\ C)))]]$

16.3 Apparent exceptions

In this section we address three kinds of apparent exceptions to the account of relativisation given here.

First, there are examples in which there appears to be a parasitic gap which is not in an island. The following is example (8a) from Postal (1993[84]):

(361) man who$_i$ Mary convinced t_i that John wanted to visit t_i

And an anonymous L&P referee pointed out:

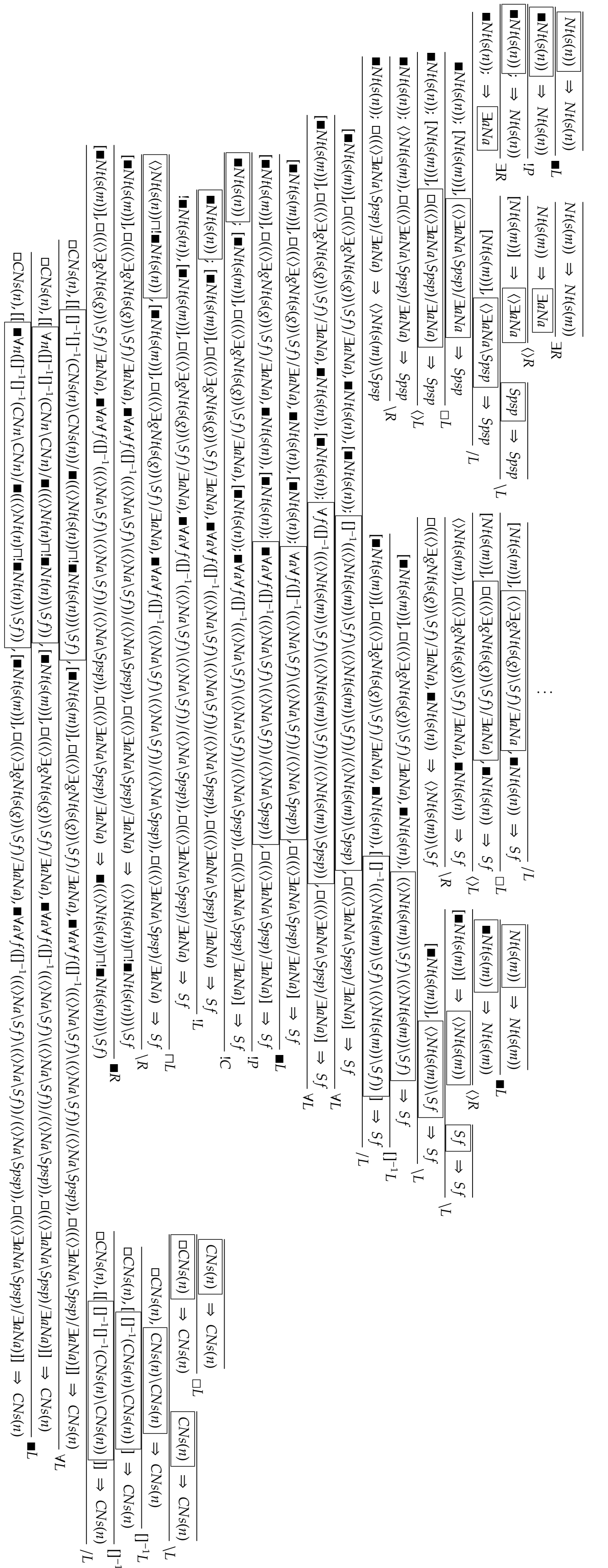


Figure 16.6: Derivation for **paper that John filed without reading**

,

Figure 16.7: Auxiliary derivations for **paper that the editor of filed without reading**

,

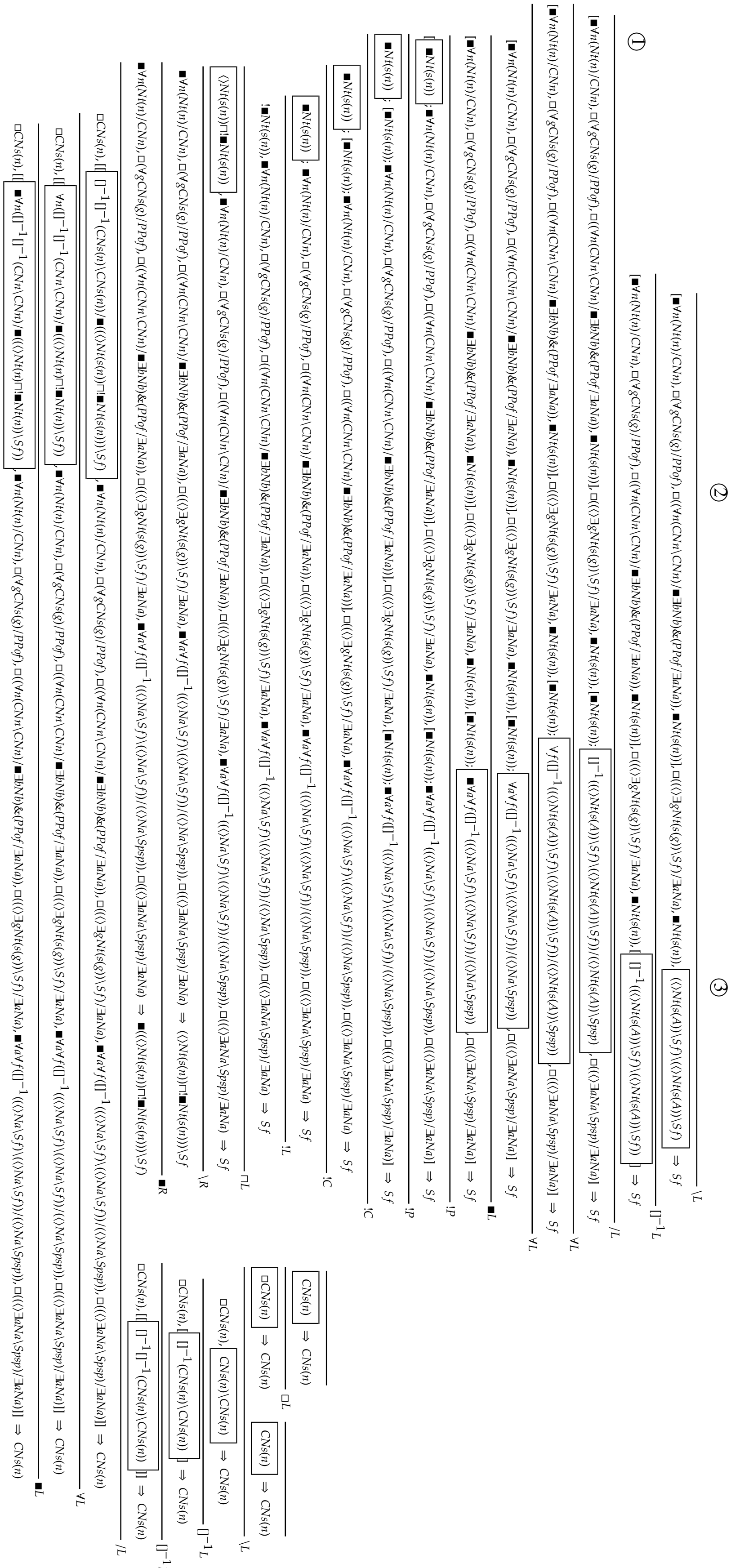


Figure 16.8: Main derivation for **paper that the editor of filed without reading**

(362) people whom$_i$ you sent pictures of t_i to t_i

In respect of such examples we suggest that although there *seems* to be no island, there *could* be one.[5] This is effected as follows for (361). Instead of a type of the form $((N\backslash S)/CP)/N$ for **convince** we assume $((N\backslash S)/CP)/(N\sqcup\langle\rangle N)$ where the semantically inactive additive disjunction disjunct N will be selected ordinarily, and $\langle\rangle N$ when there is parasitic extraction, as in (361). Similarly for (362) we assume for **picture** type $CN/(PP\sqcup\langle\rangle PP)$ where the second disjunct projects the brackets of a weak island.[6] Thus in examples such as the following the semantically inactive additive disjunction inference for **convince** of type $((N\backslash S)/\ CP)/(N\sqcup\langle\rangle N)$ will select N:

(363) a. man who$_i$ Mary convinced t_i that John wanted to visit Suzy
b. man who$_i$ Mary convinced the friends of t_i that John wanted to visit Suzy

But for (361) the semantically inactive additive disjunction inference for **convince** of type $((N\backslash S)/CP)/(N\sqcup\langle\rangle N)$ will select $\langle\rangle N$. Similarly for the picture noun case (362). This account is not explanatory, but it captures the facts.

Second, under certain processing conditions island violations are grammatical (Lakoff 1986[43]; Deane 1991[17]; Kluender 1992[36], 1998[37]; Kehler 2002[35]; Hofmeister and Sag 2010[28]).[7] We cannot undertake to offer a full account here of which processing conditions these are nor how the processing and combinatoric modules interact to produce this effect, but we do suggest how the weak island violation is combinatorially possible without changing the grammar. This is to assume a variant $\%!P$ of $!P$ as follows applicable under the right processing conditions:

(364) $$\frac{\Xi(\zeta;\Gamma_1,[\{A\colon x\};\Gamma_2],\Gamma_3)\Rightarrow B\colon\psi}{\Xi(\zeta\uplus\{A\colon x\};\Gamma_1,\Gamma_2,\Gamma_3)\Rightarrow B\colon\psi}\ \%!P$$

This allows extraction from weak islands, thus for example:

(365) $$\cfrac{\cfrac{\cfrac{\ldots,[\ldots,A,\ldots],\ldots\ \Rightarrow B}{\ldots,[A;\ldots,\ldots],\ldots\ \Rightarrow B}\ !P}{A;\ldots,\ldots,\ldots,\ldots\ \Rightarrow B}\ \%!P}{!A,\ldots,\ldots,\ldots,\ldots\ \Rightarrow B}\ !L$$

Thirdly, Levine and Hukari (2006[48]) cite an apparent example of 'symbiotic' extraction without a host gap:

(366) people that$_i$ fans of t_i gather from every continent just to listen to t_i

It is interesting that our response to the second issue predicts a possibility such as (366). An analysis can begin with the hypothetical subtype in the stoup emitting a parasitic gap by $!C$ in, say, the subject in the usual way. Then $\%!P$ can move the hypothetical subtype in the outer stoup into the stoup of the adjunt island. The two gap types in the two island stoups are then moved into their positions by $!P$'s.

16.4 Conclusion

The categorial analysis of gapping as like-type coordination was established in Steedman (1990[87]) and Hendriks (1995[26]). In the framework of HTLG Kubota and Levine (2012[38]) go further in that they provide like-type coordination for discontinuous gapping. Our analysis is inspired by that of Kubota and Levine (2012[38]; 2013[39]). However, our analysis of gapping represents an improvement on the HTLG analysis because we do not require two types for simple and discontinuous gapping: a single type suffices.

Finally, we have noted that the HTLG account of gapping suffers from determiner-transitive verb order inconsistency overgeneration; the same problem would arise for HTLG in relation to discontinuous determiner gapping:

[5] Tom Roeper, p.c.

[6] The argument pattern $X\sqcup\langle\rangle X$ is a general mechanism for an argument optional island X. Likewise the dual value pattern $X\sqcap[\,]^{-1}X$ is a general mechanism for a value optional island X.

[7] Kubota and Levine (2015[40], Section 4.6.2) argue for a version of TLG that freely overgenerates island constraint 'violations' in the syntax.

(367) a. ∗Some boy wants Everton to win and Mary (wants) (some) London club (to win)
b. ∗Mary wants some London club to win and (some) boy (wants) Everton (to win)

In addition to capturing simplex gapping (and determiner gapping) as a special case of complex gapping (and determiner gapping), our DTLG account does not have the order inconsistency overgeneration problem of determiner gapping and discontinuous determiner gapping.

Part V

CONCLUSION

Chapter 17

Going further

We give a sketch of second-order and higher-order displacement calculus.

17.1 Gapping

In a series of papers Kubota and Levine give an account of gapping and determiner gapping in terms of hybrid type logical grammar, including anomalous scopal interactions with auxiliaries and negative quantifiers.[1] We make three observations: i) under the counterpart assumptions that Kubota and Levine make, the existent displacement type logical grammar account of gapping already accounts for the scopal interactions, ii) Kubota and Levine overgenerate determiner-verb order inconsistencies in determiner gapping conjuncts whereas the immediate adaptation of their proposal to displacement type logical grammar does not do so, and iii) Kubota and Levine do not capture simplex gapping as a special case of complex gapping, but require distinct lexical entries for the two cases; we show how a generalisation of displacement type logical grammar allows both simplex and discontinuous gapping under a single type assignment.

Consider the following four types of gapping: simple gapping, discontinuous gapping, determiner gapping, and discontinuous determiner gapping.

(368) a. Leslie met Sandy and Robin (met) Bill.
b. John wants Watford to win and Daniel (wants) Chelsea (to win).

(369) a. Some dogs like Whiskas and (some) cats (like) Alpo.
b. Every cook wants Barça to win and (every) waiter (wants) Madrid (to win).

Kubota and Levine (2012[38]; 2013[39]; 2015[41]; henceforth K&L) develop an account of gapping in hybrid type logical grammar (HTLG), an extension of Lambek calculus admitting functional expressions in the phonological component.[2] K&L (2015[41]) provide a review of the literature and argue in broad terms the advantages of an analysis of gapping as hypothetical reasoning, to which we have nothing to add; we in turn review their type logical proposal. The Lambek rules of HTLG are as follows:[3]

[1]The contents of this section are to appear in *Natural Language and Linguistic Theory*.

[2]An anonymous NLLT referee questioned whether gapping is after all a purely combinatoric phenomenon citing split antecedent gapping (i), and non-ATB gapping (ii):

i) Sue goes running 6 times a week, and Alex lifts weights 3 times a week, but neither every day.
ii) Either Pat came with Chris and Sandy came with Kim, or Pat with Kim and the others were alone.

We cannot enter fully into this question here except to note that such examples do not show that there is *no* combinatoric component to gapping, but rather that it is a more generalised phenomenon in the case of iterated coordination, which we do not address here.

[3]We make some notational adjustments in order to smoothen comparison of hybrid TLG and displacement TLG.

(370)
$$\frac{\begin{array}{c}\vdots\\ \alpha\colon C/B\colon \phi\end{array}\quad\begin{array}{c}\vdots\\ \beta\colon B\colon \psi\end{array}}{\alpha{+}\beta\colon C\colon (\phi\ \psi)}/E \qquad \frac{\begin{array}{c}\vdots\\ \alpha\colon A\colon \phi\end{array}\quad\begin{array}{c}\vdots\\ \beta\colon A\backslash C\colon \psi\end{array}}{\alpha{+}\beta\colon C\colon (\psi\ \phi)}\backslash E$$

$$\frac{\begin{array}{c}\overline{b\colon B\colon y}^{\,n}\\ \vdots\\ \alpha{+}b\colon C\colon \chi\end{array}}{\alpha\colon C/B\colon \lambda y\chi}/I^n \qquad \frac{\begin{array}{c}\overline{a\colon A\colon x}^{\,n}\\ \vdots\\ a{+}\beta\colon C\colon \chi\end{array}}{\beta\colon A\backslash C\colon \lambda x\chi}\backslash I^n$$

In these rules $\alpha\colon A\colon\phi$ signifies a sign with phonology/prosodics α, syntactic type A, and semantics ϕ. The elimination (E) rules combine signs by concatenation prosodically, *modus ponens* type-logically, and functional application semantically; the introduction (I) rules of hypothetical reasoning conditionalise assumptions: when passing from premise to conclusion the assumption coindexed with the rule name n becomes closed. Every rule instance has a fresh index. The introduction rules are semantically interpreted by functional abstraction. To these directional rules HTLG adds inference rules for a nondirectional type constructor | interpreted by phonological functional application and phonological functional abstraction:

(371)
$$\frac{\begin{array}{c}\vdots\\ \alpha\colon C|B\colon \phi\end{array}\quad\begin{array}{c}\vdots\\ \beta\colon B\colon \psi\end{array}}{\alpha(\beta)\colon C\colon (\phi\ \psi)}|E \qquad \frac{\begin{array}{c}\overline{\phi\colon B\colon y}^{\,n}\\ \vdots\\ \alpha\colon C\colon \chi\end{array}}{\lambda\phi.\alpha\colon C|B\colon \lambda y\chi}|I^n$$

These are the characteristic rules of HTLG (for more details we refer the reader to the papers of Kubota and Levine).

In relation to gapping, K&L (2015[41]) (53) present a type assignment which is in essence:[4]

(372) $\lambda\sigma_2\lambda\sigma_1\lambda\phi.\sigma_1(\phi)+\mathbf{and}+\sigma_2(0)\colon (X|X)|X\colon \lambda x\lambda y\lambda z[(y\ z)\wedge(x\ z)]$
where $X = S|(VP/N)$

And they derive from this simple gapping such as (368a); see *ibid* (52) and (55). In our notation, their (52) is:

(373)
$$\frac{\dfrac{\mathbf{robin}\colon N\colon r \quad \dfrac{\overline{\phi\colon VP/N\colon x}^{\,i}\quad \mathbf{bill}\colon N\colon b}{\phi+\mathbf{bill}\colon VP\colon (x\ b)}E/}{\mathbf{robin}+\phi+\mathbf{bill}\colon S\colon ((x\ b)\ r)}E\backslash}{\lambda\phi.\mathbf{robin}+\phi+\mathbf{bill}\colon S|(VP/N)\colon \lambda x((x\ b)\ r)}I|^i$$

Continuing the derivation as in their (55) yields:

(374) **leslie + met + sandy + and + robin + bill**: $S\colon ((met\ s)\ l)\wedge((met\ b)\ r)$

K&L also assume in their (56) a raised type for an auxiliary:

(375) $\lambda\sigma.\sigma(\mathbf{must})\colon S|(S|(VP/VP))\colon \lambda x(Nec\ (x\ \lambda yy))$

They show that such assignments to auxiliaries license the auxiliary wide-scope reading of, say, **John must eat steak and Mary pizza**.

Likewise, K&L (2015[41]) (66) present a type assignment for determiner gapping which is:

(376) $\lambda\rho_2\lambda\rho_1\lambda\phi\lambda\sigma.\rho_1(\phi)(\sigma)+\mathbf{and}+\rho_2(0)(\lambda\chi\lambda\psi.\psi(\chi))\colon (X|X)|X\colon$
$\lambda x\lambda y\lambda z\lambda w[((y\ z)\ w)\wedge((x\ z)\ w)]$
where $X = (S|((S|(S|N))|CN))|(VP/N)$

[4]Throughout, VP abbreviates $N\backslash S$. We limit attention to gapping of the transitive verb category; so far as we are aware gapping in other categories raises no new issues differentiating between hybrid TLG and displacement TLG.

And they assume, K&L (2015[41]) (65), a raised type for a negative determiner of the form:

(377) $\lambda\rho.\rho(\lambda\phi\lambda\sigma.\sigma(\mathbf{no}+\phi))\colon S|((S|(S|(S|N)))|CN)\colon \lambda x(\neg(x\ \lambda y\lambda z\exists w[(y\ w)\wedge(z\ w)]))$

These enable derivation of the split scope reading of:

(378) No dog eats whiskas or cat alpo.

17.2 The scopal anomalies are already available in displacement TLG

K&L (2015[41]) state that 'our own analysis resembles most closely Morrill et al.'s (2011) (which is a refinement of Hendriks 1995)'. The displacement type logical grammar (DTLG) gapping assignment is as follows, where $\odot$ is discontinuous product, which is semantically interpreted by ordered pairing, and '$\pi_1 x$' selects the first component of x:[5]

(379) **and**: $(X\backslash X)/(X\odot I)\colon \lambda x\lambda y\lambda z[(y\ z)\wedge(\pi_1 x\ z)]$
where $X = S\uparrow(VP/N)$

Here, the symbols such as I and $\uparrow$ are connectives of the displacement calculus, which we define later. The corresponding determiner gapping assignment would be:

(380) **and**: $(X\backslash X)/((X\odot I)\odot I)\colon \lambda x\lambda y\lambda z\lambda w[((y\ z)\ w)\wedge((\pi_1\pi_1 x\ z)\ w)]$
where $X = (S\uparrow(VP/N))\uparrow(((S\uparrow N)\downarrow S)/CN)$

The auxiliary and negative determiner assignments would be:

(381) **must**: $(S\uparrow(VP/VP))\downarrow S\colon \lambda x(Nec\ (x\ \lambda yy))$

(382) **no**: $(S\uparrow(((S\uparrow N)\downarrow S)/CN))\downarrow S\colon \lambda x(\neg(x\ \lambda y\lambda z\exists w[(y\ w)\wedge(z\ w)]))$

Thus the analyses of gapping scope anomalies are already available in DTLG. Indeed K&L (2015[41]) state, 'To be fair, the core of our empirical results, so far as we can tell, seems to straightforwardly carry over to Morrill et al.'s (2011) system.' So K&L's type logical contribution is the determiner gapping, and the observation that raised auxiliary and negative determiner assignments capture the scopal anomalies in HTLG, but also in DTLG. Thus, although K&L couch their solution in terms of HTLG, HTLG and DTLG are on a par in respect of gapping and the scopal anomalies. Since this point appears to have been granted we do not elaborate on it further. But we will see a respect in which DTLG improves on HTLG, and another respect in which DTLG can be made to improve further on HTLG.

17.3 The HTLG determiner gapping assignment overgenerates

Kubota (p.c.) points out that the HTLG analysis of determiner gapping incorrectly predicts examples such as the following, where the determiner and the transitive verb orders are not consistent in the conjuncts:

(383) a. ∗Some dogs like Whiskas and I (like) (some) cats.
b. ∗I like some cats and (some) dogs (like) Whiskas.

This order inconsistency overgeneration arises because the $|((S|(S|N))|CN)$ and $|(VP/N)$ arguments in the types for determiner gapping do not distinguish the linear order of the phonological variables they bind. Thus, for example, in relation to (383a), both $\lambda v\lambda q.q+\mathbf{dogs}+v+\mathbf{Whiskas}$ and $\lambda v\lambda q.\mathbf{I}+v+q+\mathbf{cats}$ have type $(S|((S|(S|N))|CN))|(VP/N)$. Note that the issue has nothing to do with the order in which the quantifier and transitive verb arguments are abstracted, but rather with the left to right position of the variables they bind in the body of the phonological term, to which the type constructor | is insensitive. DTLG does not overgenerate in this way because the positions of discontinuity are indexed for left to right order.

[5] The notation π_1 (and π_2) represents first (and second) projection of an ordered pair, so that e.g. $\pi_1(\phi,\psi)=\phi$.

17.3.1 HTLG requires distinct type assignments for simplex and discontinuous gapping

Our analysis is inspired by that of Kubota and Levine, which assigns coordinator conjunct types and prosody as follows:

conjunt type & coordinator phonology	gapping	determiner gapping
simplex	$S\|(VP/N)$ $\lambda\sigma_2\lambda\sigma_1\lambda\phi.\sigma_1(\phi)+\mathbf{and}+\sigma_2(0)$ K&L (2012[38]) (5) K&L (2013[39]) (13) K&L (2015[41]) (53)	$(S\|(VP/N))\|((S\|(S\|N))\|CN)$ $\lambda\rho_2\lambda\rho_1\lambda\phi\lambda\sigma.\rho_1(\phi)(\sigma)+\mathbf{and}+\rho_2(\lambda\chi\lambda\psi.\psi(\chi))(0)$ K&L (2013[39]) (24) K&L (2015[41]) (66) (83)
discont.	$S\|(VP\|N)$ $\lambda\rho_2\lambda\rho_1\lambda\phi.\rho_1(\phi)+\mathbf{and}+\rho_2(\lambda\pi.\pi)$ K&L (2012[38]) (20)	$(S\|(VP\|N))\|((S\|(S\|N))\|CN)$ $\lambda\rho_2\lambda\rho_1\lambda\phi\lambda\sigma.\rho_1(\phi)(\sigma)+\mathbf{and}+\rho_2(\lambda\chi\lambda\psi.\psi(\chi))(\lambda\pi.\pi)$ (by extrapolation)

However, our analysis of gapping represents an improvement on the HTLG analysis in that we require only a single type for simple and discontinuous gapping. HTLG requires two types because simplex gapping conditionalises VP/N of sort *string* whereas discontinuous gapping conditionalises $VP|N$ of sort *function*, and these require two distinct prosodic operations: application to the empty string 0 (of sort *string*) and to the identity function $\lambda\pi.\pi$ (of sort *function*) respectively.

17.3.2 Our account

We give an account of simple and discontinuous gapping as in (368) and simplex and discontinuous determiner gapping as in (369) which is optimal in that a single coordinator type generates discontinuous gapping with simple gapping as a special case and a single coordinator type generates discontinuous determiner gapping with simplex determiner gapping as a special case. The account is expressed in an extension of displacement calculus (**D**; Morrill et al. 2011[79]) which we call second-order displacement calculus $\mathbf{D}^2$.

Second-order displacement calculus

Displacement calculus is a logic of discontinuous strings. By discontinuous strings we mean strings punctuated by a distinguished vocabulary item '1' called the *separator*. In contrast to HTLG, the phonological terms of displacement calculus have no lambda abstraction and, instead of a set of variable placeholders, DTLG has a single placeholder, the separator. The sort of a discontinuous string is the number of separators it contains. We notate by $L_i, i \geq 0$, the set of all strings of sort i with respect to some alphabet. We consider the operations concatenation, intercalation, and adjunction on discontinuous strings. Concatenation is represented in (384).

(384) $\boxed{\alpha} + \boxed{\beta}$

$=$

$\boxed{\alpha \,|\, \beta}$

concatenation $+ : L_i, L_j \to L_{i+j}$

For example, the concatenation of **Leslie**+1+**Sandy** and **and**+**Robin**+**Bill** is:

(385) **Leslie**+1+**Sandy** + **and**+**Robin**+**Bill**

=

Leslie+1+**Sandy**+**and**+**Robin**+**Bill**

Intercalation is represented in (386):

(386)

α	1	γ	$\times_k$	β

=

α	β	γ

intercalation $\times_k : L_{i+1}, L_j \to L_{i+j}$

For example, the intercalation at the second separator of 1+**dogs**+1+**Whiskas +and+cats+Alpo** and **like** is:

(387) 1+**dogs**+1+**Whiskas**+**and**+**cats**+**Alpo** $\times_2$ **like**
=
1+**dogs**+**like**+**Whiskas**+**and**+**cats**+**Alpo**

Finally, adjunction is represented in (388):

(388)

α	1	γ	1	ϵ	$^{\wedge k,l}$	β	1	δ

=

α	β	γ	δ	ϵ

adjunction $^{\wedge k,l} : L_{i+2}, L_{j+1} \to L_{i+j}$

For example, the adjunction at the first separators of **John**+1+**Watford**+1+**and**+ **Daniel**+ **Chelsea** and **wants**+1+**to**+**win** is:

(389) **John**+1+**Watford**+1+**and**+**Daniel**+**Chelsea** $^{\wedge 1,1}$ **wants**+1+**to**+**win**
=
John+**wants**+**Watford**+**to**+**win**+**and**+**Daniel**+**Chelsea**

We will have three families of type-constructors defined in relation to the three prosodic operations of concatenation, intercalation, and adjunction. The syntactic types are sorted $\mathbf{Tp}_0, \mathbf{Tp}_1, \mathbf{Tp}_2, \ldots$ according to the number of points of discontinuity $0, 1, 2, \ldots$ their expressions contain. The sets $\mathbf{Tp}_i$ of types of sort i are defined by mutual recursion in terms of sets $\mathcal{P}_i$ of primitive types of sort i as follows:

$\mathbf{Tp}_i$	::=	$\mathcal{P}_i$				
$\mathbf{Tp}_i$	::=	$\mathbf{Tp}_{i+j}/\mathbf{Tp}_j$	$T(C/B)$	=	$T(B)\to T(C)$	over
$\mathbf{Tp}_j$	::=	$\mathbf{Tp}_i\backslash\mathbf{Tp}_{i+j}$	$T(A\backslash C)$	=	$T(A)\to T(C)$	under
$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_i\bullet\mathbf{Tp}_j$	$T(A\bullet B)$	=	$T(A)\&T(B)$	continuous product
$\mathbf{Tp}_0$	::=	I	$T(I)$	=	$\top$	cont. unit
$\mathbf{Tp}_{i+1}$	::=	$\mathbf{Tp}_{i+j}\uparrow_k\mathbf{Tp}_j, 1 \le k \le i{+}1$	$T(C\uparrow_k B)$	=	$T(B)\to T(C)$	extract
$\mathbf{Tp}_j$	::=	$\mathbf{Tp}_{i+1}\downarrow_k\mathbf{Tp}_{i+j}, 1 \le k \le i{+}1$	$T(A\downarrow_k C)$	=	$T(A)\to T(C)$	infix
$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i+1}\odot_k\mathbf{Tp}_j, 1 \le k \le i{+}1$	$T(A\odot_k B)$	=	$T(A)\&T(B)$	disc. product
$\mathbf{Tp}_1$	::=	J	$T(J)$	=	$\top$	disc. unit
$\mathbf{Tp}_{i+2}$	::=	$\mathbf{Tp}_{i+j}\Uparrow_{k,l}\mathbf{Tp}_{j+1}, 1 \le k \le i{+}1, 1 \le l \le j{+}1$	$T(C\Uparrow_{k,l} B)$	=	$T(B)\to T(C)$	2nd order extract
$\mathbf{Tp}_{j+1}$	::=	$\mathbf{Tp}_{i+2}\Downarrow_{k,l}\mathbf{Tp}_{i+j}, 1 \le k \le i{+}1, 1 \le l \le j{+}1$	$T(A\Downarrow_{k,l} C)$	=	$T(A)\to T(C)$	2nd order infix
$\mathbf{Tp}_{i+j}$	::=	$\mathbf{Tp}_{i+2}\circledcirc_{k,l}\mathbf{Tp}_{j+1}, 1 \le k \le i{+}1, 1 \le l \le j{+}1$	$T(A\circledcirc_{k,l} B)$	=	$T(A)\&T(B)$	2nd order disc. product
$\mathbf{Tp}_2$	::=	K	$T(K)$	=	$\top$	2nd order disc. unit

The second column of this table shows the standard categorial semantic type map for the connectives.[6] Each type of sort i is interpreted as a set of (discontinuous) strings of sort i. The prosodic interpretation is as follows:

[6] For example, if $T(N) = e$ where e is the semantic type for individuals, and $T(S) = t$ where t is the semantic type corresponding to truth-values, we have then that $T(N\backslash S) = e \to t$.

$$
\begin{array}{rcl}
[[C/B]] & = & \{s_1 \mid \forall s_2 \in [[B]], s_1{+}s_2 \in [[C]]\}\\
[[A\backslash C]] & = & \{s_2 \mid \forall s_1 \in [[A]], s_1{+}s_2 \in [[C]]\}\\
[[A{\bullet}B]] & = & \{s_1{+}s_2 \mid s_1 \in [[A]]\ \&\ s_2 \in [[B]]\}\\
[[I]] & = & \{0\}\\[1em]
[[C{\uparrow_k}B]] & = & \{s_1 \mid \forall s_2 \in [[B]], s_1{\times_k}s_2 \in [[C]]\}\\
[[A{\downarrow_k}C]] & = & \{s_2 \mid \forall s_1 \in [[A]], s_1{\times_k}s_2 \in [[C]]\}\\
[[A{\odot_k}B]] & = & \{s_1{\times_k}s_2 \mid s_1 \in [[A]]\ \&\ s_2 \in [[B]]\}\\
[[J]] & = & \{1\}\\[1em]
[[C{\Uparrow_{k,l}}B]] & = & \{s_1 \mid \forall s_2 \in [[B]], s_1{}^{\wedge_{k,l}}s_2 \in [[C]]\}\\
[[A{\Downarrow_{k,l}}C]] & = & \{s_2 \mid \forall s_1 \in [[A]], s_1{}^{\wedge_{k,l}}s_2 \in [[C]]\}\\
[[A{\circledcirc_{k,l}}B]] & = & \{s_1{}^{\wedge_{k,l}}s_2 \mid s_1 \in [[A]]\ \&\ s_2 \in [[B]]\}\\
[[K]] & = & \{1{+}1{+}1\}
\end{array}
$$

Although linguistically only some of the power, and hence only some of the rules, are necessary here, the framework of DTLG complies with the modern logical paradigm of logic as an interpreted formal language, and aspiration to *soundness* (that everything said is true) and *completeness* (that everything true is said). This, to us, is the rock on which type *logical* grammar is founded. Thus, we present here all the rules so that the reader has the complete picture of which the gapping analysis uses just a part.

The rules for second-order displacement calculus fall into three groups for the concatenative, intercalative, and adjunctive connective families. Each family contains four connectives: the two implicational residuals, the conjunctive product, and product unit. Each connective has two rules, namely a rule of elimination (E), eliminating the connective reading from premise to conclusion, and a rule of introduction (I) introducing the connective reading from premise to conclusion.[7] Although there are many rules, the reader should be aware of the high degree of symmetry between them, and that the rules simply formalise the necessary and sufficient conditions for membership of syntactic types: the rules are essentially the result of restating the interpretation clauses given above.

The labelled natural deduction for second-order displacement calculus is as follows, where we use three conventions. Firstly, where **a** is a prosodic constant of sort i, the *vector* $\overrightarrow{\mathbf{a}}$ is $\mathbf{a}_0{+}1{+}\mathbf{a}_1{+}1{+}\cdots{+}1{+}\mathbf{a}_i$; for example, if **a** is of sort 1, $\overrightarrow{\mathbf{a}} = \mathbf{a}_0{+}1{+}\mathbf{a}_1$. Secondly, where α is a discontinuous string of sort $i > 0$, $\alpha\,|_k\,\beta$, $1 \le k \le i$, is the result of replacing the kth separator in α by β (counting from the left); for example **John**+1+**Watford**+1+**and**+**Daniel**+**Chelsea** $|_2$ **to**+**win** = **John**+1+**Watford**+**to**+**win**+**and**+**Daniel**+**Chelsea**. Thirdly, $\alpha\,||_k\,\beta$ abbreviates $\alpha\,|_k\,1{+}\beta{+}1$; for example **John**+1+**and**+**Daniel**+**Chelsea** $||_1$ **Watford** = **John**+1+**Watford**+1+**and**+**Daniel**+**Chelsea**.

Continuous family:

- Elimination rules for implications

$$
\frac{\begin{array}{c}\vdots\\ \alpha\colon C/B\colon \phi\end{array}\quad \begin{array}{c}\vdots\\ \beta\colon B\colon \psi\end{array}}{\alpha{+}\beta\colon C\colon (\phi\ \psi)}\ /E
\qquad\qquad
\frac{\begin{array}{c}\vdots\\ \alpha\colon A\colon \phi\end{array}\quad \begin{array}{c}\vdots\\ \beta\colon A\backslash C\colon \psi\end{array}}{\alpha{+}\beta\colon C\colon (\psi\ \phi)}\ \backslash E
$$

- Elimination rule for product

$$
\frac{\begin{array}{c}\vdots\\ \gamma\colon A{\bullet}B\colon \chi\end{array}\quad \begin{array}{c}\overline{\overrightarrow{a}\colon A\colon x}^{\,n}\quad \overline{\overrightarrow{b}\colon B\colon y}^{\,n}\\ \vdots\\ \delta(\overrightarrow{a}+\overrightarrow{b})\colon D\colon \omega(x,y)\end{array}}{\delta(\gamma)\colon D\colon \omega(\pi_1\chi, \pi_2\chi)}\ {\bullet}E^n
$$

[7] Except we omit the elimination rules for product units which, as well as being unmotivated linguistically, are awkward to formulate in the format used here.

- Introduction rules for implications

$$\dfrac{\begin{array}{c}\overline{\overrightarrow{b}: B: y}^{\,n}\\ \vdots \\ \alpha{+}\overrightarrow{b}: C: \chi\end{array}}{\alpha: C/B: \lambda y\chi}\,/I^n \qquad \dfrac{\begin{array}{c}\overline{\overrightarrow{a}: A: x}^{\,n}\\ \vdots \\ \overrightarrow{a}{+}\beta: C: \chi\end{array}}{\beta: A\backslash C: \lambda x\chi}\,\backslash I^n$$

- Introduction rules for product and product unit

$$\dfrac{\begin{array}{c}\vdots\\ \alpha: A: \phi\end{array} \qquad \begin{array}{c}\vdots\\ \beta: B: \psi\end{array}}{\alpha{+}\beta: A\bullet B: (\phi, \psi)}\,\bullet I \qquad \dfrac{}{0: I: 0}\,II$$

Discontinuous family:

- Elimination rules for implications

$$\dfrac{\begin{array}{c}\vdots\\ \alpha: C{\uparrow_k}B: \phi\end{array} \quad \begin{array}{c}\vdots\\ \beta: B: \psi\end{array}}{\alpha\,|_k\,\beta: C: (\phi\ \psi)}\,\uparrow E \qquad \dfrac{\begin{array}{c}\vdots\\ \alpha: A: \phi\end{array} \quad \begin{array}{c}\vdots\\ \beta: A{\downarrow_k}C: \psi\end{array}}{\alpha\,|_k\,\beta: C: (\psi\ \phi)}\,\downarrow E$$

- Elimination rule for product

$$\dfrac{\begin{array}{c}\vdots\\ \gamma: A\odot B: \chi\end{array} \quad \begin{array}{c}\overline{\overrightarrow{a}: A: x}^{\,n} \quad \overline{\overrightarrow{b}: B: y}^{\,n}\\ \vdots \\ \delta(\overrightarrow{a}\,|_k\,\overrightarrow{b}): D: \omega(x, y)\end{array}}{\delta(\gamma): D: \omega(\pi_1\chi, \pi_2\chi)}\,\odot E^n$$

- Introduction rules for implications

$$\dfrac{\begin{array}{c}\overline{\overrightarrow{b}: B: y}^{\,n}\\ \vdots \\ \alpha\,|_k\,\overrightarrow{b}: C: \chi\end{array}}{\alpha: C{\uparrow_k}B: \lambda y\chi}\,\uparrow I^n \qquad \dfrac{\begin{array}{c}\overline{\overrightarrow{a}: A: x}^{\,n}\\ \vdots \\ \overrightarrow{a}\,|_k\,\beta: C: \chi\end{array}}{\beta: A{\downarrow_k}C: \lambda x\chi}\,\downarrow I^n$$

- Introduction rules for product and product unit

$$\dfrac{\begin{array}{c}\vdots\\ \alpha: A: \phi\end{array} \qquad \begin{array}{c}\vdots\\ \beta: B: \psi\end{array}}{\alpha\,|_k\,\beta: A{\odot_k}B: (\phi, \psi)}\,\bullet I \qquad \dfrac{}{1: J: 0}\,IJ$$

Second-order discontinuous family:

- Elimination rules for implications

$$\frac{\begin{array}{c}\vdots\\ \alpha \|_k \gamma: C\Uparrow_{k,l}B: \phi\end{array} \quad \begin{array}{c}\vdots\\ \beta: B: \psi\end{array}}{\alpha |_k (\beta |_l \gamma): C: (\phi\ \psi)} \Uparrow E \qquad \frac{\begin{array}{c}\vdots\\ \alpha \|_k \gamma: A: \phi\end{array} \quad \begin{array}{c}\vdots\\ \beta: A\Downarrow_{k,l}C: \psi\end{array}}{\alpha |_k (\beta |_l \gamma): C: (\psi\ \phi)} \Downarrow E$$

- Elimination rule for product

$$\frac{\begin{array}{c}\vdots\\ \gamma: A\circledcirc_{k,l}B: \chi\end{array} \quad \begin{array}{c}\overline{\overrightarrow{a} \|_k \overrightarrow{c}: A: x}^{\,n} \quad \overline{\overrightarrow{b}: B: y}^{\,n}\\ \vdots\\ \delta(\overrightarrow{a} |_k (\overrightarrow{b} |_l \overrightarrow{c})): D: \omega(x, y)\end{array}}{\delta(\gamma): D: \omega(\pi_1\chi, \pi_2\chi)} \circledcirc E^n$$

- Introduction rules for implications

$$\frac{\begin{array}{c}\overline{\overrightarrow{b}: B: y}^{\,n}\\ \vdots\\ \alpha |_k (\overrightarrow{b} |_l \gamma): C: \chi\end{array}}{\alpha \|_k \gamma: C\Uparrow_{k,l}B: \lambda y\chi} \Uparrow I^n \qquad \frac{\begin{array}{c}\overline{\overrightarrow{a} |_l \overrightarrow{c}: A: x}^{\,n}\\ \vdots\\ \overrightarrow{a} |_k (\beta |_l \overrightarrow{c}): C: \chi\end{array}}{\beta: A\Downarrow_{k,l}C: \lambda x\chi} \Downarrow I^n$$

- Introduction rules for product and product unit

$$\frac{\begin{array}{c}\vdots\\ \alpha \|_k \gamma: A: \phi\end{array} \quad \begin{array}{c}\vdots\\ \beta: B: \psi\end{array}}{\alpha |_k (\beta |_l \gamma): A\circledcirc_{k,l}B: (\phi, \psi)} \bullet E \qquad \frac{}{1{+}1{+}1: K: 0} IK$$

We adopt the convention that when subscripts k and l are omitted they are 1, i.e. they default to 1.

By way of example, the following auxiliary derivation shows that a subject followed by an object has type $S\Uparrow(VP\uparrow N)$:

(390) $$\frac{\dfrac{\mathbf{sbj}: N: sbj \quad \dfrac{\overline{a_0{+}1{+}a_1: VP\uparrow N: x}^{\,i} \quad \mathbf{obj}: N: obj}{a_0{+}\mathbf{obj}{+}a_1: VP: (x\ obj)} \uparrow E}{\mathbf{sbj}{+}a_0{+}\mathbf{obj}{+}a_1: S: ((x\ obj)\ sbj)} \backslash E}{\mathbf{sbj}{+}1{+}\mathbf{obj}{+}1: S\Uparrow(VP\uparrow N): \lambda x((x\ obj)\ sbj)} \Uparrow I^i$$

Analyses

The account of gapping consists in an assignment to the coordinator of the following type:

(391) **and**: $(X\backslash X)/(X\circledcirc J): \lambda x\lambda y\lambda z[(y\ z) \wedge (\pi_1 x\ z)]$
where $X = S\Uparrow(VP\uparrow N)$

Consider example (368a) of simple gapping: **Leslie met Sandy and Robin Bill**. Then there is the following derivation of (368a) using (390):

$$\frac{\text{Leslie Sandy}}{\mathbf{L}{+}1{+}\mathbf{S}{+}1\colon S{\Uparrow}(VP{\uparrow}N)\colon \lambda x((x\ s)\ l)}$$

$$\frac{\dfrac{\text{Robin Bill}}{\mathbf{R}{+}1{+}\mathbf{B}{+}1\colon S{\Uparrow}(VP{\uparrow}N)\colon \lambda x((x\ b)\ r)} \quad \dfrac{}{1\colon J\colon 0}JR}{\mathbf{R}{+}\mathbf{B}\colon (S{\uparrow}(VP/N)){\circledcirc}J\colon (\lambda x((x\ b)\ r), 0)}{\circledcirc}I$$

$$\frac{\mathbf{and}\colon ((S{\Uparrow}(VP{\uparrow}N))\backslash(S{\Uparrow}(VP{\uparrow}N)))/((S{\Uparrow}(VP{\uparrow}N)){\circledcirc}J)\colon \lambda x\lambda y\lambda z[(y\ z)\wedge(\pi_1 x\ z)] \quad \mathbf{R}{+}\mathbf{B}\colon (S{\uparrow}(VP/N)){\circledcirc}J\colon (\lambda x((x\ b)\ r), 0)}{\mathbf{and}{+}\mathbf{R}{+}\mathbf{B}\colon (S{\Uparrow}(VP{\uparrow}N))\backslash(S{\Uparrow}(VP{\uparrow}N))\colon \lambda y\lambda z[(y\ z)\wedge((z\ b)\ r)]}/E$$

$$\frac{\mathbf{L}{+}1{+}\mathbf{S}{+}1\colon S{\Uparrow}(VP{\uparrow}N)\colon \lambda x((x\ s)\ l) \quad \mathbf{and}{+}\mathbf{R}{+}\mathbf{B}\colon (S{\Uparrow}(VP{\uparrow}N))\backslash(S{\Uparrow}(VP{\uparrow}N))\colon \lambda y\lambda z[(y\ z)\wedge((z\ b)\ r)]}{\mathbf{L}{+}1{+}\mathbf{S}{+}1{+}\mathbf{and}{+}\mathbf{R}{+}\mathbf{B}\colon S{\Uparrow}(VP{\uparrow}N)\colon \lambda z[((z\ s)\ l)\wedge((z\ b)\ r)]}\backslash E$$

$$\frac{\dfrac{\mathbf{met}\colon VP/N\colon meet \quad \dfrac{}{a\colon N\colon x}i}{\mathbf{met}{+}a\colon VP\colon (meet\ x)}/E}{\mathbf{met}{+}1\colon VP{\uparrow}N\colon \lambda x(meet\ x)}{\uparrow}I^i$$

$$\frac{\mathbf{L}{+}1{+}\mathbf{S}{+}1{+}\mathbf{and}{+}\mathbf{R}{+}\mathbf{B}\colon S{\Uparrow}(VP{\uparrow}N)\colon \lambda z[((z\ s)\ l)\wedge((z\ b)\ r)] \quad \mathbf{met}{+}1\colon VP{\uparrow}N\colon \lambda x(meet\ x)}{\mathbf{L}{+}\mathbf{met}{+}\mathbf{S}{+}\mathbf{and}{+}\mathbf{R}{+}\mathbf{B}\colon S\colon [((meet\ s)\ l)\wedge((meet\ b)\ r)]}{\Uparrow}E$$

And from the same coordinator type assignment (391) there is the following derivation of the discontinuous gapping (368b) **John wants Watford to win and Daniel Chelsea**:

$$\frac{\text{John Watford}}{\mathbf{J}{+}1{+}\mathbf{W}{+}1\colon S{\Uparrow}(VP{\uparrow}N)\colon \lambda x((x\ w)\ j)}$$

$$\frac{\dfrac{\text{Daniel Chelsea}}{\mathbf{D}{+}1{+}\mathbf{S}{+}1\colon S{\Uparrow}(VP{\uparrow}N)\colon \lambda x((x\ s)\ d)} \quad \dfrac{}{1\colon J\colon 0}JR}{\mathbf{D}{+}\mathbf{C}\colon (S{\uparrow}(VP/N)){\circledcirc}J\colon (\lambda x((x\ c)\ d), 0)}{\circledcirc}I$$

$$\frac{\mathbf{and}\colon ((S{\Uparrow}(VP{\uparrow}N))\backslash(S{\Uparrow}(VP{\uparrow}N)))/((S{\Uparrow}(VP{\uparrow}N)){\circledcirc}J)\colon \lambda x\lambda y\lambda z[(y\ z)\wedge(\pi_1 x\ z)] \quad \mathbf{D}{+}\mathbf{C}\colon (S{\uparrow}(VP/N)){\circledcirc}J\colon (\lambda x((x\ c)\ d), 0)}{\mathbf{and}{+}\mathbf{D}{+}\mathbf{C}\colon (S{\Uparrow}(VP{\uparrow}N))\backslash(S{\Uparrow}(VP{\uparrow}N))\colon \lambda y\lambda z[(y\ z)\wedge((z\ s)\ d)]}/E$$

$$\frac{\mathbf{J}{+}1{+}\mathbf{W}{+}1\colon S{\Uparrow}(VP{\uparrow}N)\colon \lambda x((x\ w)\ j) \quad \mathbf{and}{+}\mathbf{D}{+}\mathbf{C}\colon (S{\Uparrow}(VP{\uparrow}N))\backslash(S{\Uparrow}(VP{\uparrow}N))\colon \lambda y\lambda z[(y\ z)\wedge((z\ s)\ d)]}{\mathbf{J}{+}1{+}\mathbf{W}{+}1{+}\mathbf{and}{+}\mathbf{D}{+}\mathbf{C}\colon S{\Uparrow}(VP{\uparrow}N)\colon \lambda z[((z\ w)\ j)\wedge((z\ c)\ d)]}\backslash E$$

$$\frac{\dfrac{\dfrac{\mathbf{wants}\colon (VP/VP)/N\colon want \quad \dfrac{}{a\colon N\colon x}i}{\mathbf{wants}{+}a\colon VP/VP\colon (want\ x)}/E \quad \dfrac{\text{to win}}{\mathbf{to}{+}\mathbf{win}\colon VP\colon win}}{\mathbf{wants}{+}a{+}\mathbf{to}{+}\mathbf{win}\colon VP\colon ((want\ x)\ win)}/E}{\mathbf{wants}{+}1{+}\mathbf{to}{+}\mathbf{win}\colon VP{\uparrow}N\colon \lambda x((want\ x)\ win)}{\uparrow}I^i$$

$$\frac{\mathbf{J}{+}1{+}\mathbf{W}{+}1{+}\mathbf{and}{+}\mathbf{D}{+}\mathbf{C}\colon S{\Uparrow}(VP{\uparrow}N)\colon \lambda z[((z\ w)\ j)\wedge((z\ c)\ d)] \quad \mathbf{wants}{+}1{+}\mathbf{to}{+}\mathbf{win}\colon VP{\uparrow}N\colon \lambda x((want\ x)\ win)}{\mathbf{J}{+}\mathbf{wants}{+}\mathbf{W}{+}\mathbf{to}{+}\mathbf{win}{+}\mathbf{and}{+}\mathbf{D}{+}\mathbf{C}\colon S\colon [(((want\ w)\ win)\ j)\wedge(((want\ c)\ win)\ d)]}{\Uparrow}E$$

Observe how in the last step of both of the above derivations adjunction combines a string of sort 2 and a string of sort 1. But, in the first, simplex, case the separator of the second operand is right peripheral, whereas in the second, complex, case the separator of the second operand is medial. This is how the account unifies simplex and complex gapping under a single coordinator type.

The account of determiner gapping consists in an assignment to the coordinator of the following type (Q is $((S{\uparrow}N){\downarrow}S)/CN$):

(392) $\mathbf{and}\colon (X\backslash X)/((X{\circledcirc}J){\odot}I)\colon \lambda x\lambda y\lambda z\lambda w[((y\ z)\ w)\wedge((\pi_1\pi_1 x\ z)\ w)]$
where $X = (S{\Uparrow}(VP{\uparrow}N)){\uparrow}Q$

Consider example (369a) of simplex determiner gapping: **Some dogs like Whiskas and cats Alpo**. We use the following auxiliary derivation showing that a common noun followed by an object has type $(S{\Uparrow}(VP{\uparrow}N)){\uparrow}Q$:

(393)

$$\frac{\dfrac{}{a\colon N\colon x}i \quad \dfrac{\dfrac{}{b_0{+}1{+}b_1\colon VP{\uparrow}N\colon y}j \quad \mathbf{obj}\colon N\colon obj}{b_0{+}\mathbf{obj}{+}b_1\colon VP\colon (y\ obj)}{\uparrow}E}{a{+}b_0{+}\mathbf{obj}{+}b_1\colon S\colon ((y\ obj)\ x)}\backslash E$$

$$\frac{a{+}b_0{+}\mathbf{obj}{+}b_1\colon S\colon ((y\ obj)\ x)}{1{+}b_0{+}\mathbf{obj}{+}b_1\colon S{\uparrow}N\colon \lambda x((y\ obj)\ x)}{\uparrow}I^i \qquad \frac{\dfrac{}{c\colon Q\colon z}k \quad \mathbf{cn}\colon CN\colon cn}{c{+}\mathbf{cn}\colon (S{\uparrow}N){\downarrow}S\colon (z\ cn)}/E$$

$$\frac{\dfrac{\dfrac{1{+}b_0{+}\mathbf{obj}{+}b_1\colon S{\uparrow}N\colon \lambda x((y\ obj)\ x) \quad c{+}\mathbf{cn}\colon (S{\uparrow}N){\downarrow}S\colon (z\ cn)}{c{+}\mathbf{cn}{+}b_0{+}\mathbf{obj}{+}b_1\colon S\colon ((z\ cn)\ \lambda x((y\ obj)\ x))}{\downarrow}E}{c{+}\mathbf{cn}{+}1{+}\mathbf{obj}{+}1\colon S{\Uparrow}(VP{\uparrow}N)\colon \lambda y((z\ cn)\ \lambda x((y\ obj)\ x))}{\Uparrow}I^j}{1{+}\mathbf{cn}{+}1{+}\mathbf{obj}{+}1\colon (S{\Uparrow}(VP{\uparrow}N)){\uparrow}Q\colon \lambda z\lambda y((z\ cn)\ \lambda x((y\ obj)\ x))}{\uparrow}I^k$$

The derivation of (369a) is given in Figure 17.1. Finally, from the same coordinator type assignment (392)

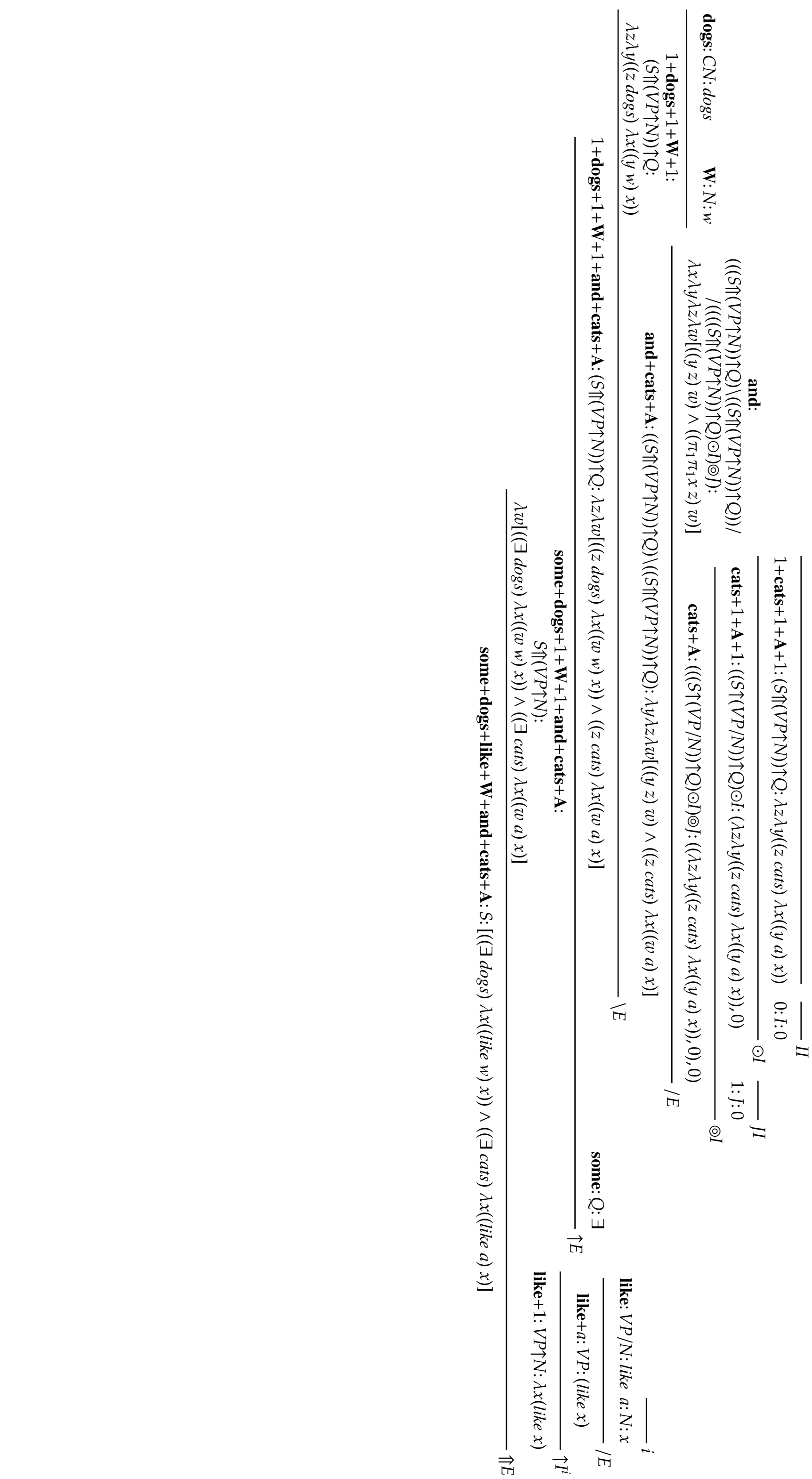


Figure 17.1: Determiner gapping

we can derive the case of discontinuous determiner gapping (369b):

Every cook wants Barçato win and waiter Madrid

in Figure 17.2.

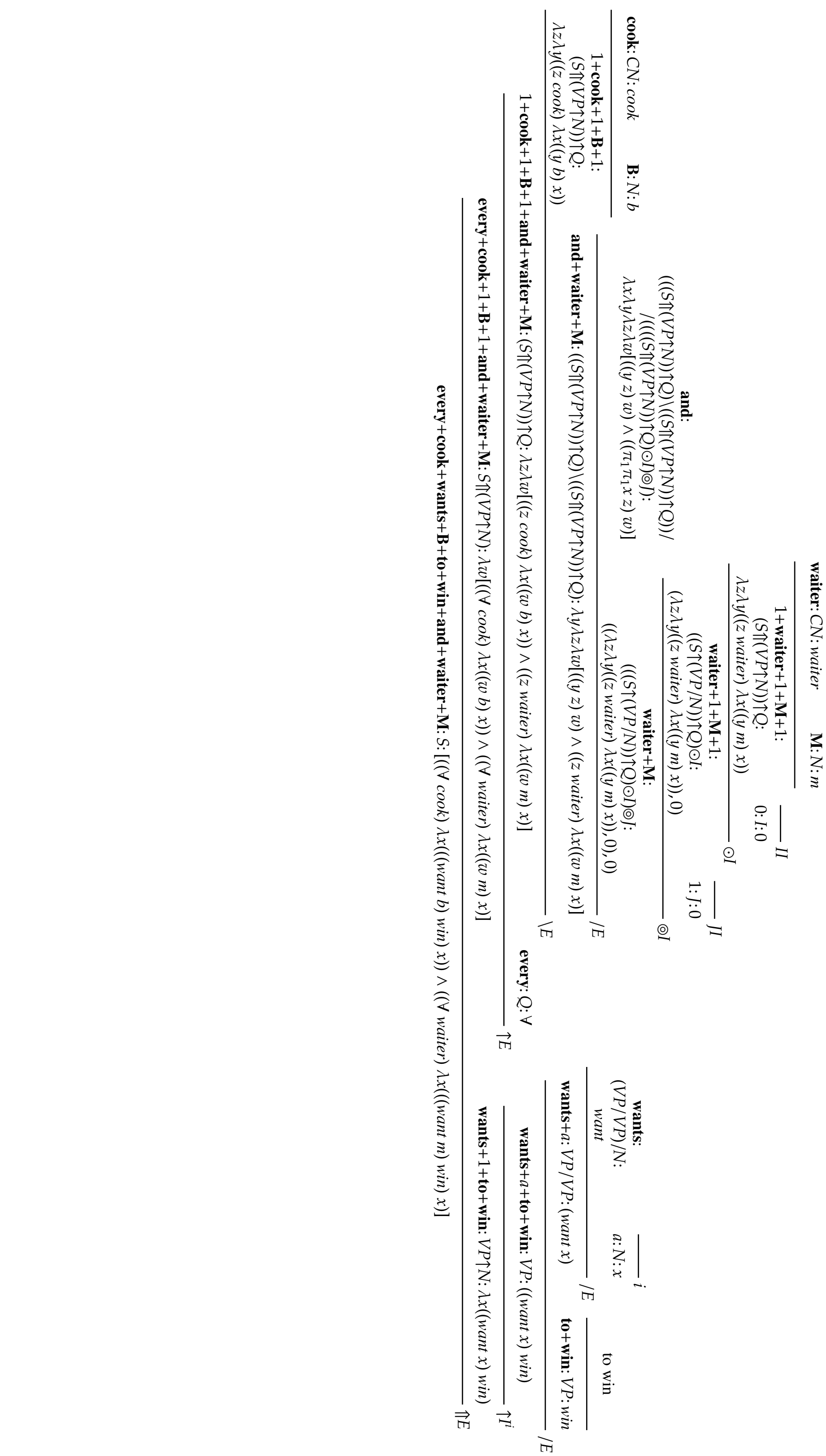


Figure 17.2: Discontinuous determiner gapping

17.4 Summary

The categorial analysis of gapping as like-type coordination was established in Steedman (1990[87]) and Hendriks (1995[26]). In the framework of HTLG Kubota and Levine (2012[38]) go further in that they provide like-type coordination for discontinuous gapping. Our analysis is inspired by that of Kubota and Levine (2012[38]; 2013[39]). However, our analysis of gapping represents an improvement on the HTLG analysis because we do not require two types for simple and discontinuous gapping: a single type suffices.

Finally, we have noted that the HTLG account of gapping suffers from determiner-transitive verb order inconsistency overgeneration; the same problem would arise for HTLG in relation to discontinuous determiner gapping:

(394) a. ∗Some boy wants Everton to win and Mary (wants) (some) London club (to win)
b. ∗Mary wants some London club to win and (some) boy (wants) Everton (to win)

In addition to capturing simplex gapping (and determiner gapping) as a special case of complex gapping (and determiner gapping), our DTLG account does not have the order inconsistency overgeneration problem of determiner gapping and discontinuous determiner gapping.

Continuous subcategorisation	$\backslash \quad \bullet \quad / \quad I$	cont. mult.
Disontinuous subcategorisation	$\uparrow \quad \odot \quad \downarrow \quad J$	disc. mult.
Weak polymorphism	$\& \quad \oplus$	add.
Features	$\wedge \quad \vee$	qu.
Intensionality	$\Box \quad \Diamond$	norm. mod.
Syntactic domains	$[\,]^{-1} \quad \langle\rangle$	brack. mod.
Nonlinearity	$! \quad ?$	exp.
Anaphora & words as types	$\vert \quad +$	limited contr. & limited expan.
Exceptions	$-$	diff.

	cont. mult.	disc. mult.	secon. disc. mult.	add.	qu.	norm. mod.	brack. mod.	exp.	limited contr. & weak.
primitive	$/_1$ $\bullet_3$ I_4 $\backslash_2$	$\upharpoonleft_5$ $\odot_7$ J_8 $\downharpoonleft_6$	$\Uparrow_9$ $\circledcirc_{11}$ K_{12} $\Downarrow_{10}$	$\&_{13}$ $\oplus_{14}$	$\bigwedge_{15}$ $\bigvee_{16}$	$\Box_{17}$ $\Diamond_{18}$	$[\,]^{-1}_{19}$ $\langle\rangle_{20}$	$!_{21}$ $?_{22}$	$\vert_{23}$ W_{24}
sem. inactive variants	$\bullet\!\!-_{25}$ $-\!\!\circ_{26}$ $\circ\!\!-_{28}$ $-\!\!\bullet_{27}$ $◐_{29}$ $◑_{30}$	$\upharpoonleft^{\bullet}_{31}$ $\downharpoonleft_{\circ\,32}$ $\upharpoonleft^{\circ}_{34}$ $\downharpoonleft_{\bullet\,35}$ $◓_{35}$ $◒_{36}$	$\Uparrow^{\bullet}_{37}$ $\Downarrow_{\circ\,38}$ $\Uparrow^{\circ}_{40}$ $\Downarrow_{\bullet\,41}$ $\underline{\underline{◓}}_{41}$ $\underline{\underline{◒}}_{42}$	$\sqcap_{43}$ $\sqcup_{44}$	$\forall_{45}$ $\exists_{46}$	$\blacksquare_{47}$ $\blacklozenge_{48}$			
det. synth.	$\triangleleft^{-1}_{49}$ $\triangleright^{-1}_{50}$ $\triangleleft_{51}$ $\triangleright_{52}$	$\check{}_{53}$ $\hat{}_{54}$	$\check{\check{}}_{55}$ $\hat{\hat{}}_{56}$						diff.
non – det. synth.	$\div_{57}$ $\circ_{58}$	$\hat{=}_{59}$ $\check{=}_{60}$ $\diamond_{61}$							$-_{62}$

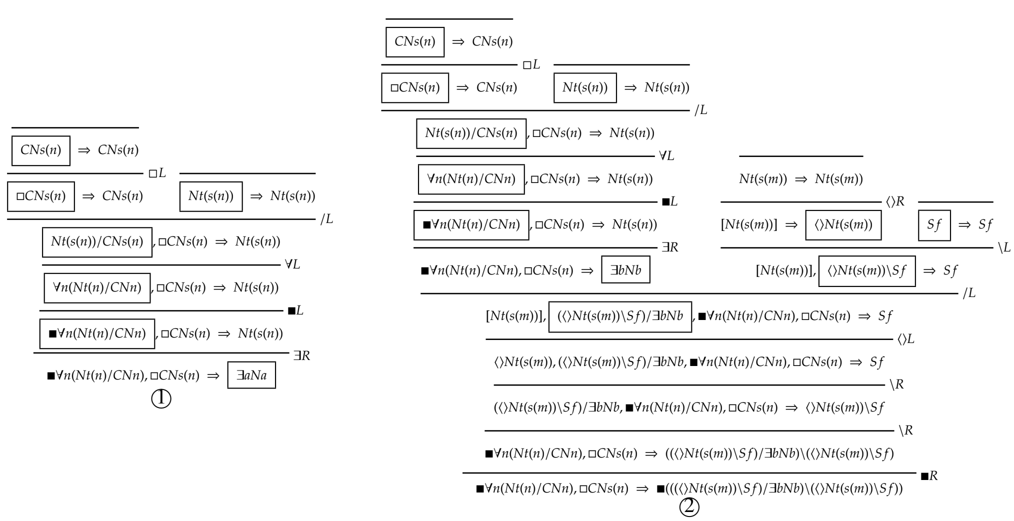
Chapter 18

Scripture

Parsing the first sentence of the Bible with parser/theorem-prover *CatLog3* (available on GitHub; Morrill 2019[67]), the analysis and semantics delivered are:

(395) (gen(one(1))) **in**+**the**+**beginning**+[**God**]+**created**+[[**the**+**heaven**+**and**+**the**+**earth**]] : Sf

(396) $$\Box(\forall f(Sf \div Sf)/\exists aNa) : in, \blacksquare\forall n(Nt(n)/CNn) : \iota, \Box CNs(n) : beginning, [\blacksquare Nt(s(m)) : God], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa) : \hat{}\lambda A\lambda B(Past\ ((\check{}create\ A)\ B)), [[\blacksquare\forall n(Nt(n)/CNn) : \iota, \Box CNs(n) : heaven, \blacksquare\forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))) : (\mathbf{\Phi^{n+}}\ (s\ (s\ 0))\ and), \blacksquare\forall n(Nt(n)/CNn) : \iota, \Box CNs(n) : earth]] \Rightarrow Sf$$



$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\boxed{CNs(n)} \Rightarrow CNs(n)}{\boxed{\Box CNs(n)} \Rightarrow CNs(n)}\Box L \qquad \dfrac{}{\boxed{Nt(s(n))} \Rightarrow Nt(s(n))}}{\boxed{Nt(s(n))/CNs(n)}, \Box CNs(n) \Rightarrow Nt(s(n))}/L}{\boxed{\forall n(Nt(n)/CNn)}, \Box CNs(n) \Rightarrow Nt(s(n))}\forall L}{\boxed{\blacksquare\forall n(Nt(n)/CNn)}, \Box CNs(n) \Rightarrow Nt(s(n))}\blacksquare L}{\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow \boxed{\exists bNb}}\exists R \qquad \dfrac{\dfrac{\dfrac{}{Nt(s(m)) \Rightarrow Nt(s(m))}}{[Nt(s(m))] \Rightarrow \boxed{\langle\rangle Nt(s(m))}}\langle\rangle R \qquad \dfrac{}{\boxed{Sf} \Rightarrow Sf}}{[Nt(s(m))], \boxed{\langle\rangle Nt(s(m))\backslash Sf} \Rightarrow Sf}\backslash L}{[Nt(s(m))], \boxed{(\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb}, \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow Sf}/L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[Nt(s(m))], \boxed{(\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb}, \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow Sf}{\langle\rangle Nt(s(m)), (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb, \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow Sf}\langle\rangle L}{(\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb, \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow \langle\rangle Nt(s(m))\backslash Sf}\backslash R}{\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow ((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf)}\backslash R}{\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow \blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))}\blacksquare R}{\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n) \Rightarrow \boxed{?\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))}}?R$$

③

$$Nt(s(m)) \Rightarrow Nt(s(m))$$

$$Nt(s(m)) \Rightarrow \exists aNa \quad \exists R$$

$$[Nt(s(m))] \Rightarrow \langle\rangle\exists aNa \quad \langle\rangle R \qquad Sf \Rightarrow Sf$$

$$[Nt(s(m))], \langle\rangle\exists aNa\backslash Sf \Rightarrow Sf \quad \backslash L$$

$$N7 \Rightarrow N7$$

$$N7 \Rightarrow \exists aNa \quad \exists R$$

$$[Nt(s(m))], (\langle\rangle\exists aNa\backslash Sf)/\exists aNa, N7 \Rightarrow Sf \quad /L$$

$$[Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), N7 \Rightarrow Sf \quad \Box L$$

$$[Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf \quad \exists L$$

$$\langle\rangle Nt(s(m)), \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), \exists bNb \Rightarrow Sf \quad \langle\rangle L$$

$$\Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), \exists bNb \Rightarrow \langle\rangle Nt(s(m))\backslash Sf \quad \backslash R$$

$$\Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa) \Rightarrow (\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb \quad /R$$

$$Nt(s(m)) \Rightarrow Nt(s(m))$$

$$\blacksquare Nt(s(m)) \Rightarrow Nt(s(m)) \quad \blacksquare L$$

$$[\blacksquare Nt(s(m))] \Rightarrow \langle\rangle Nt(s(m)) \quad \langle\rangle R \qquad Sf \Rightarrow Sf$$

$$[\blacksquare Nt(s(m))], \langle\rangle Nt(s(m))\backslash Sf \Rightarrow Sf \quad \backslash L$$

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), ((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf) \Rightarrow Sf \quad \backslash L$$

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), [[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))] \Rightarrow Sf \quad []^{-1}L$$

③

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), [[[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))]] \Rightarrow Sf \quad []^{-1}L$$

②

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), ?\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))]] \Rightarrow Sf \quad \backslash L$$

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), (?\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf)))/\blacksquare(((\langle\rangle Nt(s(m))\backslash Sf)/\exists bNb)\backslash(\langle\rangle Nt(s(m))\backslash Sf)), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n)]] \Rightarrow Sf \quad /L$$

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n)]] \Rightarrow Sf \quad \forall L$$

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n)]] \Rightarrow Sf \quad \forall L$$

$$[\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare\forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n)]] \Rightarrow Sf \quad \blacksquare L$$

$$Sf \Rightarrow Sf$$

$$Sf\div Sf, [\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare\forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n)]] \Rightarrow Sf \quad \div L$$

①

$$\forall f(Sf\div Sf), [\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare\forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n)]] \Rightarrow Sf \quad \forall L$$

$$\forall f(Sf\div Sf)/\exists aNa, \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), [\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare\forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n)]] \Rightarrow Sf \quad /L$$

$$\Box(\forall f(Sf\div Sf)/\exists aNa), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), [\blacksquare Nt(s(m))], \Box((\langle\rangle\exists aNa\backslash Sf)/\exists aNa), [[\blacksquare\forall n(Nt(n)/CNn), \Box CNs(n), \blacksquare\forall f\forall a((?\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))\backslash[]^{-1}[]^{-1}(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf)))/\blacksquare(((\langle\rangle Na\backslash Sf)/\exists bNb)\backslash(\langle\rangle Na\backslash Sf))), \blacksquare\forall n(Nt(n)/CNn), \Box CNs(n)]] \Rightarrow Sf \quad \Box L$$

(397) $((\breve{}\mathit{in}\ (\iota\ \check{}\mathit{beginning}))\ [(\mathit{Past}\ ((\breve{}\mathit{create}\ (\iota\ \check{}\mathit{heaven}))\ \mathit{God}))\wedge(\mathit{Past}\ ((\breve{}\mathit{create}\ (\iota\ \check{}\mathit{earth}))\ \mathit{God}))])$

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