Generalized Discontinuity

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Abstract. We define and study a calculus of discontinuity, a version of displacement calculus, which is a logic of segmented strings in exactly the same sense that the Lambek calculus is a logic of strings. Like the Lambek calculus, the displacement calculus is a sequence logic free of structural rules, and enjoys Cut-elimination and its corollaries: the subformula property, decidability, and the finite reading property. The foci of this paper are a formulation with a finite number of connectives, and consideration of how to extend the calculus with defined connectives while preserving its good properties.

1 Introduction: architecture of logical grammar

An *argument* in logic comprises some premises and a conclusion; for example:[1]

(1) a. $\dfrac{\text{All men are mortal.} \quad \text{Socrates is a man.}}{\text{Socrates is mortal.}}$ b. $\dfrac{\text{All men are mortal.} \quad \text{Socrates is mortal.}}{\text{Socrates is a man.}}$

If in an argument the truth of the premises guarantees the truth of the conclusion, the argument is *logical*. If the truth of the premises does not guarantee the truth of the conclusion, the argument is *not logical*. The argument (1a) is logical: independently of the facts of the real world, who Socrates is, etc., if the premises are true then the conclusion must be true. The argument (1b) is not logical: again disregarding how the world actually is, it is possible for the premises to be true but the conclusion false.

In a logical theory premises and conclusions are represented by *formulas*, and we then call an argument a *sequent*. For example, corresponding to (1) there are the sequents:

(2) a. $\forall x(Hx \to Mx), Hs \Rightarrow Ms$
b. $\forall x(Hx \to Mx), Ms \Rightarrow Hs$

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If a sequent $\Gamma \Rightarrow A$ is logical we call it a *theorem* and write $\vdash \Gamma \Rightarrow A$. If it is not logical it is not a theorem and we write $\not\vdash \Gamma \Rightarrow A$. Thus a logical theory takes the form shown in Figure 1.

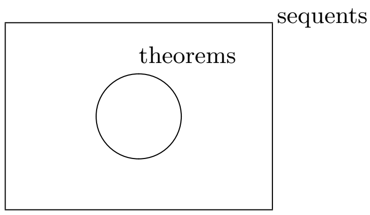

Fig. 1. Logic

A sentence comprises a string of words. Some strings of words are well-formed as sentences and we say they are grammatical, for example *John walks*; others are not well-formed as sentences and we say they are ungrammatical, for example **walks John*. Thus grammar takes the form shown in Figure 2.

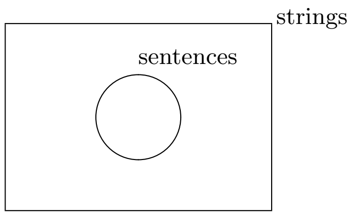

Fig. 2. Grammar

Given a subset of a domain, such as the subset of sequents that are theorems or the subset of strings that are sentences, there is the associated computational decision problem of determining whether an element of the domain belongs to the subset.

A *reduction* of one problem to another is an answer-preserving mapping from the domain of the first problem to the domain of the second problem. Thus a reduction sends members to members and nonmembers to nonmembers as shown in Figure 3. The existence of a reduction from one problem to a second means that the first problem can be solved by the composition of an algorithm for the second problem with an algorithm computing the reduction.

Logical grammar is a reduction of grammar to logic: a string is a sentence if and only if an associated sequent (or one of a set of associated sequents) is a theorem, as shown in Figure 4. Hence in logical grammar, determining grammatical properties is reduced to theorem-proving.

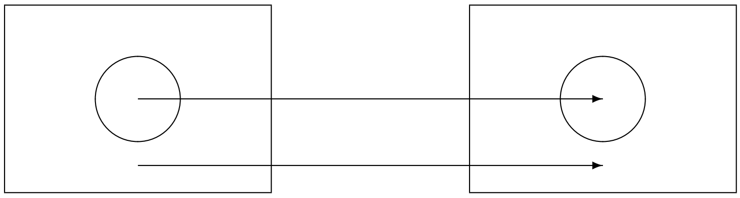

Fig. 3. Reduction

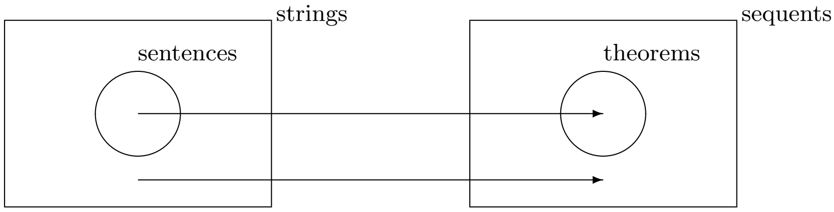

Fig. 4. Logical grammar

2 Logic of strings: the Lambek calculus L

Logic of strings is provided by the calculus of Lambek (1958)[4]. We consider a variant **L** which is multiplicative intuitionistic noncommutative linear logic.

The types $\mathcal{F}$ of **L** are defined and interpreted as subsets of the set of strings over a vocabulary as follows, where 0 is the empty string:

$$
(3)\quad \begin{array}{lll}
\mathcal{F} ::= \mathcal{F}\backslash\mathcal{F} & [A\backslash C] = \{s_2 |\ \forall s_1 \in [A], s_1{+}s_2 \in [C]\} & \text{under} \\
\mathcal{F} ::= \mathcal{F}/\mathcal{F} & [C/B] = \{s_1 |\ \forall s_2 \in [B], s_1{+}s_2 \in [C]\} & \text{over} \\
\mathcal{F} ::= \mathcal{F}{\bullet}\mathcal{F} & [A{\bullet}B] = \{s_1{+}s_2 |\ s_1 \in [A]\ \&\ s_2 \in [B]\} & \text{product} \\
\mathcal{F} ::= I & [I] = \{0\} & \text{product unit}
\end{array}
$$

The set $\mathcal{O}$ of *configurations* is defined as follows, where Λ is the empty string:[2]

$$(4)\quad \mathcal{O} ::= \Lambda \mid \mathcal{F} \mid \mathcal{O}, \mathcal{O}$$

A *sequent* $\Gamma \Rightarrow A$ comprises an antecedent configuration Γ and a succedent type A. The sequent calculus of **L** is as shown in Figure 5, where $\Delta(\Gamma)$ signifies a configuration Δ with a distinguished subconfiguration Γ.

The *Cut-elimination* property of a logic is that every theorem has a Cut-free proof. Lambek (1958)[4] proved Cut-elimination for **L** without the product unit I; Lambek (1969)[5] showed that there is also Cut-elimination when the product unit is included. Cut-elimination has a series of good consequences.

Firstly, Cut-elimination means that the calculus has the *subformula property*: that every theorem has a proof containing only its subformulas. This is so because

[2] Note that this grammar is ambiguous, but that this does not matter because the product is associative.

$$\frac{}{A \Rightarrow A}\,id \qquad \frac{\Gamma \Rightarrow A \qquad \Delta(A) \Rightarrow B}{\Delta(\Gamma) \Rightarrow B}\,Cut$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta(C) \Rightarrow D}{\Delta(\Gamma, A\backslash C) \Rightarrow D}\,\backslash L \qquad \frac{A, \Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C}\,\backslash R$$

$$\frac{\Gamma \Rightarrow B \qquad \Delta(C) \Rightarrow D}{\Delta(C/B, \Gamma) \Rightarrow D}\,/L \qquad \frac{\Gamma, B \Rightarrow C}{\Gamma \Rightarrow C/B}\,/R$$

$$\frac{\Delta(A, B) \Rightarrow D}{\Delta(A\bullet B) \Rightarrow D}\,\bullet L \qquad \frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A\bullet B}\,\bullet R$$

$$\frac{\Delta(\Lambda) \Rightarrow A}{\Delta(I) \Rightarrow A}\,IL \qquad \frac{}{\Lambda \Rightarrow I}\,IR$$

Fig. 5. Sequent calculus for **L**

every rule except Cut has the property that every type in the premises is either the same as, or is an immediate subtype of, a type in the conclusion. Thus every Cut-free proof has the subformula property, and by Cut-elimination every theorem has a Cut-free proof.

Secondly, Cut-elimination means that the calculus is *decidable*. Cut-elimination does not always have this consequence, for example full propositional linear logic enjoys Cut-elimination but is not decidable. But it follows in the present case because of the finiteness of the Cut-free search space without contraction. Every rule except Cut has the property that when a sequent is matched against the conclusions of the rule, there are only a finite number of premises from which it could have been inferred by the rule. The space of Cut-free backward chaining sequent proof search is finite. Thus, whether a sequent has a Cut-free proof can be determined in finite time, and by Cut-elimination a sequent is a theorem if and only if it has a Cut-free proof.

Thirdly, Cut-elimination means that the calculus has the *finite reading property*. Again, this does not always hold, for example intuitionistic propositional logic enjoys Cut-elimination but not the finite reading property. But here there is no contraction. Curry-Howard categorial semantics compositionally associates each proof with a derivational semantics which is its homomorphic image as an intuitionistic proof or typed lambda term. Equivalence of such semantic readings is preserved by Cut-elimination. Since the Cut-free sequent proof search space is finite, every sequent can have only a finite number of nonequivalent proofs, and hence only a finite number of semantic readings.

The Lambek calculus **L** thus has good proof-theoretic properties as a logic of strings, but as is well known, logical syntax and semantics developed on this basis does not accommodate non-peripheral discontinuities. For example, a relative pronoun type $R/(S/N)$ will produce unboundedly long-distance extraction from clause-final positions, but not clause-medial extraction such as *man who John saw today*. And a quantifier phrase type $S/(N\backslash S)$ will produce subject position quantification, and a further quantifier phrase type $(S/N)\backslash S$ will produce in addition sentence-final quantification, but neither of these types will produce sentence-medial quantification such as *John introduced everyone to Mary*.

Overall, the Lambek calculus cannot accommodate the syntax and semantics of:

(5) Discontinuous idioms (*Mary gave the man the cold shoulder*). Quantification (*John gave every book to Mary*; *Mary thinks someone left*; *Everyone loves someone*). VP ellipsis (*John slept before Mary did*; *John slept and Mary did too*). Medial extraction (*dog that Mary saw today*). Pied-piping (*mountain the painting of which by Cezanne John sold for $10,000,000.* Appositive relativization (*John, who jogs, sneezed*). Parentheticals (*Fortunately, John has perseverance*; *John, fortunately, has perseverance*; *John has, fortunately, perseverance*; *John has perseverance, fortunately*). Gapping (*John studies logic, and Charles, phonetics*). Comparative subdeletion (*John ate more donuts than Mary bought bagels*). Reflexivization (*John sent himself flowers*).

Furthermore, since the Lambek calculus is context-free in generative power (Pentus 1992)[15] it cannot generate cross-serial dependencies as in Dutch and Swiss-German (Sheiber 1985[16]).

3 Logic of segmented strings: the displacement calculus D

By *segmented strings* we mean strings over a vocabulary containing a distinguished symbol 1 which we call the *separator*. We define the *sort* of a segmented string as the number of separators it contains. Henceforth, by 'string' we shall mean 'segmented string'.

Morrill and Valentín (2010)[11] defines displacement calculus with k-ary wrapping, $k > 0$, meaning wrapping around the kth separator. Here we consider a variant **D** which is a logic of segmented strings which has continuous connectives $\{\backslash, /, \bullet\}$ for concatenation and discontinuous connectives $\{\downarrow_k, \uparrow_k, \odot_k\}_{k\in\{>,<\}}$ for left and right wrapping. The characteristic feature of this variant is that it has only a finite number of connectives. We consider also here some defined connectives for which rules are compiled.

The concatenation of a string of sort i with a string of sort j is a string of sort $i+j$. But in addition to concatenation, we define on (segmented) strings two operations of intercalation or 'wrap'. Where α and β are segmented strings and the sort of α is at least 1, we define the *left wrap* of α around β, $\alpha \times_> \beta$ as the result of replacing the leftmost separator in α by β, and we define the *right wrap* of α around β, $\alpha \times_< \beta$ as the result of replacing the rightmost separator in α by β. For example:

(6) **before**+1+**left**+1+**slept** $\times_<$ **the**+**man** = **before**+1+**left**+**the**+**man**+ **slept**

The types of **D** are sorted into types $\mathcal{F}_i$ of sort i interpreted as sets of strings of sort i as shown in Figure 6 where $k \in \{>, <\}$; the left hand column displays the definition of the types in Backus-Naur form, and $[A]$ where A is a type represents the natural syntactical interpretation of a type in terms of (separated) strings. The set $\mathcal{O}$ of *configurations* is defined as follows, where $[\,]$ is the metalinguistic

$\mathcal{F}_j ::= \mathcal{F}_i \backslash \mathcal{F}_{i+j}$	$[A\backslash C] = \{s_2 \mid \forall s_1 \in [A], s_1{+}s_2 \in [C]\}$	under
$\mathcal{F}_i ::= \mathcal{F}_{i+j} / \mathcal{F}_j$	$[C/B] = \{s_1 \mid \forall s_2 \in [B], s_1{+}s_2 \in [C]\}$	over
$\mathcal{F}_{i+j} ::= \mathcal{F}_i \bullet \mathcal{F}_j$	$[A \bullet B] = \{s_1{+}s_2 \mid s_1 \in [A]\ \&\ s_2 \in [B]\}$	product
$\mathcal{F}_0 ::= I$	$[I] = \{0\}$	product unit
$\mathcal{F}_j ::= \mathcal{F}_{i+1} \downarrow_k \mathcal{F}_{i+j}$	$[A \downarrow_k C] = \{s_2 \mid \forall s_1 \in [A], s_1 \times_k s_2 \in [C]\}$	infix
$\mathcal{F}_{i+1} ::= \mathcal{F}_{i+j} \uparrow_k \mathcal{F}_j$	$[C \uparrow_k B] = \{s_1 \mid \forall s_2 \in [B], s_1 \times_k s_2 \in [C]\}$	extract
$\mathcal{F}_{i+j} ::= \mathcal{F}_{i+1} \odot_k \mathcal{F}_j$	$[A \odot_k B] = \{s_1 \times_k s_2 \mid s_1 \in [A]\ \&\ s_2 \in [B]\}$	disc. product
$\mathcal{F}_1 ::= J$	$[J] = \{1\}$	disc. prod. unit

Fig. 6. Types of the displacement calculus **D** and their interpretation

separator:

(7) $\mathcal{O} ::= \Lambda \mid [\,] \mid \mathcal{F}_0 \mid \mathcal{F}_{i+1}\{\underbrace{\mathcal{O} : \ldots : \mathcal{O}}_{i+1\ \mathcal{O}'s}\} \mid \mathcal{O}, \mathcal{O}$

$A\{\Delta_1 : \ldots : \Delta_n\}$ interpreted syntactically is formed by strings $\alpha_0{+}\beta_1{+}\alpha_1{+}\cdots{+}\alpha_{n-1}{+}\beta_n{+}\alpha_n$ where $\alpha_0{+}1{+}\alpha_1{+}\cdots{+}\alpha_{n-1}{+}1{+}\alpha_n \in A$ and $\beta_1 \in \Delta_1, \ldots, \beta_n \in \Delta_n$. Where A is a type we call its sort sA. The *figure* $\overrightarrow{A}$ of a type A is defined by:

(8) $\overrightarrow{A} = \begin{cases} A & \text{if } sA = 0 \\ A\{\underbrace{[\,] : \ldots : [\,]}_{sA\ [\,]\text{'}s}\} & \text{if } sA > 0 \end{cases}$

The sort of a configuration is the number of metalinguistic separators it contains. Where Γ and Φ are configurations and the sort of Γ is at least 1, $\Gamma|_{>}\Phi$ signifies the configuration which is the result of replacing the leftmost separator in Γ by Φ, and $\Gamma|_{<}\Phi$ signifies the configuration which is the result of replacing the rightmost separator in Γ by Φ. Where Γ is a configuration of sort i and $\Phi_1, \ldots, \Phi_i$ are configurations, the *generalized wrap* $\Gamma \otimes \langle \Phi_1, \ldots, \Phi_i \rangle$ is the result of simultaneously replacing the successive separators in Γ by $\Phi_1, \ldots, \Phi_i$ respectively. $\Delta\langle\Gamma\rangle$ abbreviates $\Delta_0(\Gamma \otimes \langle \Delta_1, \ldots, \Delta_i \rangle)$. Thus where the usual distinguished occurrence notation $\Delta(\Gamma)$ represents a subconfiguration Γ with an *external* context Δ, our distinguished hyperconfiguration notatation $\Delta\langle\Gamma\rangle$ represents a subconfiguration Γ with *external* context Δ_0 and also *internal* contexts $\Delta_1, \ldots, \Delta_i$. A *sequent* $\Gamma \Rightarrow A$ comprises an antecedent configuration Γ of sort i and a succedent type A of sort i. The sequent calculus for the calculus of displacement **D** is as shown in Figure 7 where $k \in \{>, <\}$. Like **L**, **D** has no structural rules.

$$\frac{}{\overrightarrow{A} \Rightarrow A}\,id \qquad \frac{\Gamma \Rightarrow A \qquad \Delta\langle \overrightarrow{A} \rangle \Rightarrow B}{\Delta\langle \Gamma \rangle \Rightarrow B}\,Cut$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \Gamma, \overrightarrow{A\backslash C} \rangle \Rightarrow D}\,\backslash L \qquad \frac{\overrightarrow{A}, \Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C}\,\backslash R$$

$$\frac{\Gamma \Rightarrow B \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{C/B}, \Gamma \rangle \Rightarrow D}\,/L \qquad \frac{\Gamma, \overrightarrow{B} \Rightarrow C}{\Gamma \Rightarrow C/B}\,/R$$

$$\frac{\Delta\langle \overrightarrow{A}, \overrightarrow{B} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{A\bullet B} \rangle \Rightarrow D}\,\bullet L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A\bullet B}\,\bullet R$$

$$\frac{\Delta\langle \Lambda \rangle \Rightarrow A}{\Delta\langle \overrightarrow{I} \rangle \Rightarrow A}\,IL \qquad \frac{}{\Lambda \Rightarrow I}\,IR$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \Gamma|_k \overrightarrow{A{\downarrow_k}C} \rangle \Rightarrow D}\,{\downarrow_k}L \qquad \frac{\overrightarrow{A}|_k \Gamma \Rightarrow C}{\Gamma \Rightarrow A{\downarrow_k}C}\,{\downarrow_k}R$$

$$\frac{\Gamma \Rightarrow B \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{C{\uparrow_k}B}|_k \Gamma \rangle \Rightarrow D}\,{\uparrow_k}L \qquad \frac{\Gamma|_k \overrightarrow{B} \Rightarrow C}{\Gamma \Rightarrow C{\uparrow_k}B}\,{\uparrow_k}R$$

$$\frac{\Delta\langle \overrightarrow{A}|_k \overrightarrow{B} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{A{\odot_k}B} \rangle \Rightarrow D}\,{\odot_k}L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1|_k \Gamma_2 \Rightarrow A{\odot_k}B}\,{\odot_k}R$$

$$\frac{\Delta\langle [\,] \rangle \Rightarrow A}{\Delta\langle \overrightarrow{J} \rangle \Rightarrow A}\,JL \qquad \frac{}{[\,] \Rightarrow J}\,JR$$

Fig. 7. Sequent calculus for **D**

Morrill and Valentín (2010)[11] proves Cut-elimination for the k-ary displacement calculus, $k > 0$, and the variant **D** considered here enjoys Cut-elimination by the same reasoning since left wrap is the same as first wrap, and right wrap is k-ary wrap with k the corresponding maximal sort; see Morrill, Valentín and Fadda (forthcoming, appendix)[13]. As a consequence **D**, like **L**, enjoys in addition the subformula property, decidability, and the finite reading property. The calculus of displacement provides basic analyses of all of the phenomena itemized in (5) (Morrill and Valentín 2010[11], Morrill 2010 chapter 6[14], Morrill, Valentín and Fadda forthcoming[13]). Furthermore it analyses verb raising and cross-serial dependencies (Morrill, Valentín and Fadda 2009)[12].

4 Examples

When s is of sort 1, $s \times_> s' = s \times_< s'$ which we may write $s \times s'$. Hence, when $sA = 1$, $A{\downarrow_>}C \Leftrightarrow A{\downarrow_<}C$, which we abbreviate $A{\downarrow}C$; and when $sC - sB = 1$, $C{\uparrow_>}B \Leftrightarrow C{\uparrow_<}B$, which we may abbreviate $C{\uparrow}B$; and when $sA = 1$, $A{\odot_>}B \Leftrightarrow A{\odot_<}B$, which we may write $A{\odot}B$.

Our first example is of a discontinuous idiom, where the lexicon has to assign *give ... the cold shoulder* a non-compositional meaning 'shun':

(9) **mary+gave+the+man+the+cold+shoulder** : S

Lexical insertion yields the following sequent, which is labelled with the lexical semantics:

(10) $N : m, (N\backslash S){\uparrow}N\{N/CN : \iota, CN : man\} : shunned \ \Rightarrow\ S$

This has a proof as follows.

(11) $$\dfrac{\dfrac{\dfrac{}{CN \Rightarrow CN} \quad \dfrac{}{N \Rightarrow N}}{N/CN, CN \Rightarrow N}/L \qquad \dfrac{\dfrac{}{N \Rightarrow N} \quad \dfrac{}{S \Rightarrow S}}{N, N\backslash S \Rightarrow S}\backslash L}{N, (N\backslash S){\uparrow}N\{N/CN, CN\} \Rightarrow S}{\uparrow}L$$

This delivers the semantics:

(12) $((shunned\ (\iota\ man))\ m)$

Consider medial extraction:

(13) **dog+that+mary+saw+today** : CN

An associated semantically annotated sequent may be as follows:

(14) $CN : dog, (CN\backslash CN)/((S{\uparrow}N){\odot}I) : \lambda A\lambda B\lambda C[(B\ C) \wedge (\pi_1 A\ C)], N : m, (N\backslash S)/N : saw, (N\backslash S)\backslash(N\backslash S) : \lambda A\lambda B(today\ (A\ B)) \ \Rightarrow\ CN$

This has the sequent derivation given in Figure 8. This yields semantics:

(15) $\lambda C[(dog\ C) \wedge (today\ ((saw\ C)\ m))]$

$$
\dfrac{\dfrac{}{N \Rightarrow N} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{N \Rightarrow N} \quad \dfrac{}{S \Rightarrow S}}{N, N\backslash S \Rightarrow S}\backslash L}{N\backslash S \Rightarrow N\backslash S}\backslash R \quad \dfrac{\dfrac{}{N \Rightarrow N} \quad \dfrac{}{S \Rightarrow S}}{N, N\backslash S \Rightarrow S}\backslash L}{N, N\backslash S, (N\backslash S)\backslash(N\backslash S) \Rightarrow S}\backslash L}{\dfrac{N, (N\backslash S)/N, N, (N\backslash S)\backslash(N\backslash S) \Rightarrow S}{\dfrac{N, (N\backslash S)/N, [\,], (N\backslash S)\backslash(N\backslash S) \Rightarrow S{\uparrow}N}{}\uparrow R}}/L
$$

$$
\dfrac{\dfrac{N, (N\backslash S)/N, [\,], (N\backslash S)\backslash(N\backslash S) \Rightarrow S{\uparrow}N \quad \dfrac{}{\Rightarrow I}IR}{N, (N\backslash S)/N, (N\backslash S)\backslash(N\backslash S) \Rightarrow (S{\uparrow}N){\odot}I}\odot R \quad \dfrac{\dfrac{}{CN \Rightarrow CN} \quad \dfrac{}{CN \Rightarrow CN}}{CN, CN\backslash CN \Rightarrow CN}\backslash L}{CN, (CN\backslash CN)/((S{\uparrow}N){\odot}I), N, (N\backslash S)/N, (N\backslash S)\backslash(N\backslash S) \Rightarrow CN}/L
$$

Fig. 8. Derivation of medial extraction

Consider medial quantification:

(16) **john+gave+every+book+to+mary** : S

An associated semantically annotated sequent may be as follows:

(17) $N : j, (N\backslash S)/(N\bullet PP) : \lambda A((gave\ \pi_2 A)\ \pi_1 A), ((S{\uparrow}N){\downarrow}S)/CN :$
$\lambda A\lambda B\forall C[(A\ C) \rightarrow (B\ C)], CN : book, PP/N : \lambda AA, N : m \Rightarrow S$

This has the sequent derivation given in Figure 9. This yields semantics:

$$
\dfrac{\dfrac{}{CN \Rightarrow CN} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{N \Rightarrow N} \quad \dfrac{\dfrac{}{N \Rightarrow N} \quad \dfrac{}{PP \Rightarrow PP}}{PP/N, N \Rightarrow PP}/L}{N, PP/N, N \Rightarrow N\bullet PP}\bullet R \quad \dfrac{\dfrac{}{N \Rightarrow N} \quad \dfrac{}{S \Rightarrow S}}{N, N\backslash S \Rightarrow S}\backslash L}{N, (N\backslash S)/(N\bullet PP), N, PP/N, N \Rightarrow S}/L}{N, (N\backslash S)/(N\bullet PP), [\,], PP/N, N \Rightarrow S{\uparrow}N}\uparrow R \quad \dfrac{}{S \Rightarrow S}}{N, (N\backslash S)/(N\bullet PP), (S{\uparrow}N){\downarrow}S, PP/N, N \Rightarrow S}\downarrow L}{N, (N\backslash S)/(N\bullet PP), ((S{\uparrow}N){\downarrow}S)/CN, CN, PP/N, N \Rightarrow S}/L
$$

Fig. 9. Derivation of medial quantification

(18) $\forall C[(book\ C) \rightarrow (((gave\ m)\ C)\ j)]$

5 Defined nondeterministic continuous and discontinuous connectives

Let us consider a categorial displacement calculus including additives (Girard 1987)[3] which we call displacement calculus with additives (**DA**):

(19) $\mathcal{F}_i := \mathcal{F}_i \& \mathcal{F}_i \mid \mathcal{F}_i \oplus \mathcal{F}_i$

(20) $\dfrac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\&B}\rangle \Rightarrow C}\&L_1 \qquad \dfrac{\Gamma\langle\overrightarrow{B}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\&B}\rangle \Rightarrow C}\&L_2$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A\&B}\&R$$

$$\frac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow C \qquad \Gamma\langle\overrightarrow{B}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\oplus B}\rangle \Rightarrow C}\oplus L$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A\oplus B}\oplus L_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A\oplus B}\oplus L_2$$

Then we may define nondeterministic continuous and discontinuous connectives as follows, where $+(s_1, s_2, s_3)$ if and only if $s_3 = s_1{+}s_2$ or $s_3 = s_2{+}s_1$, and $\times(s_1, s_2, s_3)$ if and only if $s_3 = s_1 \times_> s_2$ or $s_3 = s_2 \times_< s_1$.

(21)
$\frac{B}{A} =_{df} (A\backslash B)\&(B/A) \qquad \{s \mid \forall s' \in A, s_3, +(s, s', s_3) \;\Rightarrow\; s_3 \in B\}$
nondeterministic concatenation
$A \otimes B =_{df} (A{\bullet}B)\oplus(B{\bullet}A) \qquad \{s_3 \mid \exists s_1 \in A, s_2 \in B, +(s_1, s_2, s_3)\}$
nondeterministic product
$A{\Downarrow}C =_{df} (A{\downarrow_>}C)\&(A{\downarrow_<}C) \quad \{s_2 \mid \forall s_1 \in A, s_3, \times(s_1, s_2, s_3) \;\Rightarrow\; s_3 \in C\}$
nondeterministic infix
$C{\Uparrow}B =_{df} (C{\uparrow_>}B)\&(C{\uparrow_<}B) \quad \{s_1 \mid \forall s_2 \in B, s_3, \times(s_1, s_2, s_3) \;\Rightarrow\; s_3 \in C\}$
nondeterministic extract
$A{\circledcirc}B =_{df} (A{\odot_>}B)\oplus(A{\odot_<}B) \; \{s_3 \mid \exists s_1 \in A, s_2 \in B, \times(s_1, s_2, s_3)\}$
nondeterministic disc. product

These have the derived rules shown in Figure 10 where $k \in \{>, <\}$. We call the displacement calculus extended with nondeterministic connectives the nondeterministic displacement calculus **ND**.

Concerning Cut-elimination for the nondeterministic rules, the usual Lambek-style reasoning applies. For example, using the method and definition of Cut-degree in Morrill and Valentín (2010)[11], here we mention how the nondeterministic extract and discontinuous product behave in the Cut elimination steps. We show one case of principal Cut and one case of permutation conversion. Observe that in the last conversion the logical rule and the Cut rule are permuted by two Cuts and one logical rule, contrary to what is standard, but as required both Cut-degrees are lower.

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{C}\rangle \Rightarrow D}{\Delta\langle\Gamma, \frac{\overrightarrow{C}}{A}\rangle \Rightarrow D}\,-L_1 \qquad \frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{C}\rangle \Rightarrow D}{\Delta\langle\frac{\overrightarrow{C}}{A}, \Gamma\rangle \Rightarrow D}\,-L_2$$

$$\frac{\overrightarrow{A}, \Gamma \Rightarrow C \qquad \Gamma, \overrightarrow{A} \Rightarrow C}{\Gamma \Rightarrow \frac{C}{A}}\,-R$$

$$\frac{\Delta\langle\overrightarrow{A}, \overrightarrow{B}\rangle \Rightarrow D \qquad \Delta\langle\overrightarrow{B}, \overrightarrow{A}\rangle \Rightarrow D}{\Delta\langle\overrightarrow{A \otimes B}\rangle \Rightarrow D}\,\otimes L$$

$$\frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \otimes B}\,\otimes R_1 \qquad \frac{\Gamma_1 \Rightarrow B \qquad \Gamma_2 \Rightarrow A}{\Gamma_1, \Gamma_2 \Rightarrow A \otimes B}\,\otimes R_2$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{C}\rangle \Rightarrow D}{\Delta\langle\Gamma|_k \overrightarrow{A \Downarrow C}\rangle \Rightarrow D}\,\Downarrow L \qquad \frac{\overrightarrow{A}|_> \Gamma \Rightarrow C \qquad \overrightarrow{A}|_< \Gamma \Rightarrow C}{\Gamma \Rightarrow A \Downarrow C}\,\Downarrow R$$

$$\frac{\Gamma \Rightarrow B \qquad \Delta\langle\overrightarrow{C}\rangle \Rightarrow D}{\Delta\langle\overrightarrow{C \Uparrow B}|_k \Gamma\rangle \Rightarrow D}\,\Uparrow L \qquad \frac{\Gamma|_> \overrightarrow{B} \Rightarrow C \qquad \Gamma|_< \overrightarrow{B} \Rightarrow C}{\Gamma \Rightarrow C \Uparrow B}\,\Uparrow R$$

$$\frac{\Delta\langle\overrightarrow{A}|_> \overrightarrow{B}\rangle \Rightarrow D \qquad \Delta\langle\overrightarrow{A}|_< \overrightarrow{B}\rangle \Rightarrow D}{\Delta\langle\overrightarrow{A \circledcirc B}\rangle \Rightarrow D}\,\circledcirc L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1|_k \Gamma_2 \Rightarrow A \circledcirc B}\,\circledcirc R$$

Fig. 10. Derived rules for the defined nondeterministic continuous and discontinuous connectives of **ND**

- $\Uparrow$ principal cut case:

$$\frac{\dfrac{\Delta|_> \overrightarrow{A} \Rightarrow B \qquad \Delta|_< \overrightarrow{A} \Rightarrow B}{\Delta \Rightarrow B \Uparrow A}\,\Uparrow R \qquad \dfrac{\Gamma \Rightarrow A \qquad \Theta\langle\overrightarrow{B}\rangle}{\Theta\langle\overrightarrow{B \Uparrow A}|_k \Gamma\rangle \Rightarrow C}\,\Uparrow L}{\Theta\langle\Delta|_k \Gamma\rangle \Rightarrow C}\,Cut \quad \rightsquigarrow$$

$$\frac{\Gamma \Rightarrow A \qquad \dfrac{\Delta|_k \overrightarrow{A} \Rightarrow B \qquad \Theta\langle\overrightarrow{B}\rangle \Rightarrow C}{\Theta\langle\Delta|_k \overrightarrow{A}\rangle \Rightarrow C}\,Cut}{\Theta\langle\Delta|_k \Gamma\rangle \Rightarrow C}\,Cut$$

- $\circledcirc$ permutation conversion case:

$$\frac{\dfrac{\Delta\langle\overrightarrow{B}|_> \overrightarrow{C}\rangle \Rightarrow A \qquad \Delta\langle\overrightarrow{B}|_< \overrightarrow{C}\rangle \Rightarrow A}{\Delta\langle\overrightarrow{B \circledcirc C}\rangle \Rightarrow A}\,\circledcirc L \qquad \Theta\langle\overrightarrow{A}\rangle \Rightarrow D}{\Theta\langle\Delta\langle\overrightarrow{B \circledcirc C}\rangle\rangle \Rightarrow D}\,Cut \quad \rightsquigarrow$$

$$\frac{\dfrac{\Delta\langle\overrightarrow{B}|_> \overrightarrow{C}\rangle \Rightarrow A \qquad \Theta\langle\overrightarrow{A}\rangle \Rightarrow D}{\Theta\langle\Delta\langle\overrightarrow{B}|_> \overrightarrow{C}\rangle\rangle \Rightarrow D}\,Cut \qquad \dfrac{\Delta\langle\overrightarrow{B}|_< \overrightarrow{C}\rangle \Rightarrow A \qquad \Theta\langle\overrightarrow{A}\rangle \Rightarrow D}{\Theta\langle\Delta\langle\overrightarrow{B}|_< \overrightarrow{C}\rangle\rangle \Rightarrow D}\,Cut}{\Theta\langle\Delta\langle\overrightarrow{B \circledcirc C}\rangle\rangle \Rightarrow D}\,\circledcirc L$$

By way of linguistic applications, a functor of type $\frac{B}{A}$ can concatenate with its argument A either to the left or to the right to form a B. For example, in Catalan subjects can appear either preverbally or clause-finally (*Barcelona creix* or *Creix Barcelona* "Barcelona expands/grows"). This generalization may be captured by assigning a verb phrase such as *creix* the type $\frac{S}{N}$. And a nondeterministic concatenation product type $A \otimes B$ comprises an A concatenated with a B or a B concatenated with an A. For example, in English two prepositional complements may appear in either order (*talks to John about Mary* or *talks about Mary to John*). This generalization may be captured by assigning a verb such as *talks* the type $VP/(PP_{\text{to}} \otimes PP_{\text{about}})$.

5.1 Embedding translation between ND and DA

We propose the following embedding translation $(\cdot)^\natural : \mathbf{ND} \longrightarrow \mathbf{DA}$ which we define recursively:[3]

$$\begin{array}{ll}
(A)^\natural & = A \text{ for atomic types } A \\
(B \Uparrow A)^\natural & = B^\natural \uparrow_> A^\natural \& B^\natural \uparrow_< A^\natural \\
(A \Downarrow B)^\natural & = A^\natural \downarrow_> B^\natural \& A^\natural \downarrow_< B^\natural \\
(A \circledcirc B)^\natural & = A^\natural \odot_> B^\natural \oplus A^\natural \odot_< B^\natural \\
(\frac{B}{A})^\natural & = A^\natural \backslash B^\natural \& B^\natural / A^\natural \\
(A \otimes B)^\natural & = A^\natural \bullet B^\natural \oplus B^\natural \bullet A^\natural \\
(A \star B)^\natural & = A^\natural \star B^\natural \text{ where } \star \text{ is any other binary connective}
\end{array}$$

We have the following interesting result:

Lemma 1. *The $(\cdot)^\natural$ embedding is faithful.*

Proof. From **ND** to **DA**, hypersequent derivations translate without any trouble while preserving provability. Let us suppose now that we have a **DA** provable hypersequent which corresponds to the image by $(\cdot)^\natural$ of a **ND** hypersequent, i.e. $\Delta^\natural \Rightarrow A^\natural$ where Δ and A are in the language of **ND**. We want to prove that if $\Delta^\natural \Rightarrow A^\natural$ is **DA**-provable then $\Delta \Rightarrow A$ is **ND** provable. Since the Cut rule is admissible in **DA**, we can assume only **DA** Cut-free provable hypersequents $\Delta^\natural \Rightarrow A^\natural$. The proof is by induction on the length (or height) of Cut-free **DA** derivations. If the length is 0 there is nothing to prove. If the end-hypersequent is derived by a multiplicative inference there is no problem. We analyze then the cases where the last rule is an additive rule:[4]

- Left rules:

– Case where the additive active formula corresponds to $(A \Uparrow B)^\natural = A^\natural \uparrow_> B^\natural \ \& \ A^\natural \uparrow_< B^\natural$:[5]

$$\frac{\Delta^\natural \langle \overrightarrow{A^\natural \uparrow_< B^\natural} \rangle \Rightarrow C^\natural}{\Delta^\natural \langle \overrightarrow{A^\natural \uparrow_> B^\natural \ \& \ A^\natural \uparrow_< B^\natural} \rangle \Rightarrow C^\natural} \&L_2$$

By induction hypothesis (i.h.), $\Delta \langle \overrightarrow{A \uparrow_< B} \rangle \Rightarrow C$ is derivable in the system without additives. Since $\overrightarrow{A \Uparrow B} \Rightarrow A \uparrow_< B$ is **ND**-provable, we can apply the Cut rule as follows:

[3] We assume a convention of precedence whereby the multiplicative connectives take higher priority than the additives.

[4] By way of example we only consider some cases of nondeterministic discontinuous rules: nondeterministic $\Downarrow$ and continuous connectives are similar.

[5] The other case of the & left rule, i.e. $\&L_1$, is completely similar.

$$\frac{\overrightarrow{A \Uparrow B} \Rightarrow A \uparrow_{<} B \qquad \Delta \langle \overrightarrow{A \uparrow_{<} B} \rangle \Rightarrow C}{\Delta \langle \overrightarrow{A \Uparrow B} \rangle \Rightarrow C} Cut$$

– Case where the additive active formula corresponds to $(A \circledcirc B)^\natural = A^\natural \odot_> B^\natural \oplus A^\natural \odot_< B^\natural$:

$$\frac{\Delta^\natural \langle A \odot_> B \rangle \Rightarrow C^\natural \qquad \Delta^\natural \langle A \odot_< B \rangle \Rightarrow C^\natural}{\Delta^\natural \langle A \odot_> B \oplus A \odot_< B \rangle \Rightarrow C^\natural} \circledcirc L$$

By i.h. $\Delta \langle A \odot_> B \rangle \Rightarrow C$ and $\Delta \langle A \odot_< B \rangle \Rightarrow C$ are **ND**-provable. Moreover, the hypersequents $\overrightarrow{A}|_k \overrightarrow{B} \Rightarrow A \odot_k B$ for $k \in \{>, <\}$ are **ND**-provable. By Cut we have $\Delta \langle \overrightarrow{A}|_> \overrightarrow{B} \rangle \Rightarrow C$ and $\Delta \langle \overrightarrow{A}|_< \overrightarrow{B} \rangle \Rightarrow C$. Applying then the left $\circledcirc$ rule we have:

$$\frac{\Delta \langle \overrightarrow{A}|_> \overrightarrow{B} \rangle \Rightarrow C \qquad \Delta \langle \overrightarrow{A}|_< \overrightarrow{B} \rangle \Rightarrow C}{\Delta \langle \overrightarrow{A \circledcirc B} \rangle \Rightarrow C} \circledcirc L$$

- Right rules:

– Case where the additive formula corresponds to $(A \Uparrow B)^\natural = A^\natural \uparrow_> B^\natural \& A^\natural \uparrow_< B^\natural$:

$$\frac{\Delta^\natural \Rightarrow A^\natural \uparrow_> B^\natural \qquad \Delta^\natural \Rightarrow A^\natural \uparrow_< B^\natural}{\Delta^\natural \Rightarrow A^\natural \uparrow_> B^\natural \& A^\natural \uparrow_< B} \& R$$

By i.h. we have that the hypersequents $\Delta \Rightarrow A \uparrow_> B$ and $\Delta \Rightarrow A \uparrow_< B$ are **ND**-provable. We then apply the right $\Uparrow$ rule:

$$\frac{\Delta \Rightarrow A \uparrow_> B \qquad \Delta \Rightarrow A \uparrow_< B}{\Delta \Rightarrow A \Uparrow B} \Uparrow R$$

– Case where the additive active formula corresponds to $(A \circledcirc B)^\natural = A^\natural \odot_> B^\natural \oplus A^\natural \odot_< B^\natural$:[6]

$$\frac{\Delta^\natural \Rightarrow A^\natural \odot_< B^\natural}{\Delta^\natural \Rightarrow A^\natural \odot_> B^\natural \oplus A^\natural \odot_< B^\natural} \oplus R_2$$

By i.h. $\Delta \Rightarrow A \odot_< B$. Now it is **ND**-provable that $\overrightarrow{A \odot_< B} \Rightarrow A \circledcirc B$. Then by Cut:

$$\frac{\Delta \Rightarrow A \odot_< B \qquad \overrightarrow{A \odot_< B} \Rightarrow A \circledcirc B}{\Delta \Rightarrow A \circledcirc B} Cut$$

□

[6] Without loss of generality we suppose that the instance of the last right $\oplus$ rule is $\oplus R_2$.

6 Defined unary connectives

We may define unary connectives as follows:

(22)
$$
\begin{array}{rll}
\rhd^{-1}A =_{df} J\backslash A & \{s|\ 1{+}s \in A\} \\
& \text{right projection} \\
\lhd^{-1}A =_{df} A/J & \{s|s{+}1 \in A\} \\
& \text{left projection} \\
\rhd A =_{df} J\bullet A & \{1{+}s|\ s \in A\} \\
& \text{right injection} \\
\lhd A =_{df} A\bullet J & \{s{+}1|\ s \in A\} \\
& \text{left injection} \\
\check{}^{>}A =_{df} A\uparrow_{>}I & \{s|\ s \times_{>} 0 \in A\} \\
& \text{first split} \\
\check{}^{<}A =_{df} A\uparrow_{<}I & \{s|\ s \times_{<} 0 \in A\} \\
& \text{last split} \\
\hat{}^{>}A =_{df} A\odot_{>}I & \{s \times_{>} 0|\ s \in A\} \\
& \text{first bridge} \\
\hat{}^{<}A =_{df} A\odot_{<}I & \{s \times_{<} 0|\ s \in A\} \\
& \text{last bridge}
\end{array}
$$

The derived rules of inference can be compiled straightforwardly. Some interdefinabilities are as follows:

(23)
$$
\begin{array}{rcl}
\frac{B}{A} & \Leftrightarrow & \lhd^{-1}\rhd^{-1}((B\uparrow A)\Uparrow I) \text{ when } sB = 1 \\
A \otimes B & \Leftrightarrow & (\lhd\rhd A\circledcirc I)\odot B \\
A\backslash B & \Leftrightarrow & \rhd^{-1}(B\uparrow_{>}A) \\
B/A & \Leftrightarrow & \lhd^{-1}(B\uparrow_{<}A)
\end{array}
$$

When $sA = 0$, $\check{}^{>}A \Leftrightarrow \check{}^{<}A$, which we abbreviate $\check{}A$; and when $sA = 1$, $\hat{}^{>}A \Leftrightarrow \hat{}^{<}A$, which we abbreviate $\hat{}A$. By way of linguistic application, to produce particle shift (*rings up Mary* or *rings Mary up*) we may assign **rings**+1+**up** the type $\lhd^{-1}(\check{}\,VP\Uparrow N)$.

7 Discussion

The defined connectives considered in this paper facilitate more concise lexical entries, but since they are defined they do not in any way increase the expressivity of the displacement calculus (with additives). But in addition, the use of defined connectives with their derived rules can eliminate bureaucracy in sequent derivations in the case of the introduction of the additives. Consider the two following derivations which are *equal* modulo some permutations:

$$
\mathcal{D}_1 \vdash \dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B\{[\,]\} \Rightarrow B}{B\uparrow_{>}A\{A:[\,]\} \Rightarrow B}\uparrow_{>}}{(B\uparrow_{>}A)\&(B\uparrow_{<}A)\{A:[\,]\} \Rightarrow B}\&L \qquad C\{[\,]\} \Rightarrow C}{C/B, (B\uparrow_{>}A)\&(B\uparrow_{<}A)\{A:[\,]\} \Rightarrow C}/L
\qquad
\mathcal{D}_2 \vdash \dfrac{\dfrac{C\{[\,]\} \Rightarrow C \qquad \dfrac{A \Rightarrow A \qquad B\{[\,]\} \Rightarrow B}{B\uparrow_{>}A\{A:[\,]\} \Rightarrow B}\uparrow_{>}}{C/B, (B\uparrow_{>}A)\{A:[\,]\} \Rightarrow C}/L}{C/B, (B\uparrow_{>}A)\&(B\uparrow_{<}A)\{A:[\,]\} \Rightarrow C}\&L
$$

Observe that both derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ are essentially the same. The only (inessencial) difference is the permutations steps of the additive connective & and the forward slash connective /. In $\mathcal{D}_1$ the left rule $\&L$ precedes the left rule $/L$, whereas in $\mathcal{D}_2$ the left rule $/L$ precedes the left rule $\&L$. These would have the same corresponding derivation for defined connectives. It follows then that derived or compiled rules for defined connectives eliminate some undesirable bureaucracy in the derivations.

The displacement calculus has been formulated in this paper in terms of first and last wrap, as opposed to the k-ary wrap, $k > 0$, of Morrill and Valentín (2010)[11], and has a finite rather than an infinite number of connectives. This last version of displacement calculus draws together ideas spanning three decades:

(24)
- Bach (1981, 1984)[1], [2]: the idea of categorial connectives for discontinuity/wrapping; wrap, extract, infix.
- Moortgat (1988)[6]: first type logical account of extract and infix discontinuous connectives (string interpretation and sequent calculus).
- Morrill and Merenciano (1996)[10]: sorts; bridge and split.
- Morrill (2002)[8]: separators; unboundedly many points of discontinuity.
- Morrill, Fadda and Valentín (2007)[9]: nondeterministic discontinuity.
- Morrill, Valentín and Fadda (2009)[12]: projection and injection.
- Morrill and Valentín (2010)[11]: product and discontinuous product units, Cut-elimination.

This road to discontinuity has respected fully intuitionism, residuation, and natural algebraic string models. Further logical and mathematical properties of the resulting system remain to be studied, and it also remains to be seen whether it may be necessary to appeal to continuation semantics or classical (symmetric) calculi (Moortgat 2009)[7].

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