Proof syntax of discontinuity

Glyn Morrill
Universitat Politècnica de Catalunya

The syntactic calculus of Lambek (1958) provides a logical model of language which presents formulas-as-categories and proofs-as-derivations. The calculus, now recognizable as essentially the multiplicative fragment of non-commutative intuitionistic linear logic (Girard 1987), has a sequent calculus with no structural rules, and a proof net syntax which is more geometrical than that of linear logic, for the proof nets are *planar* (Roorda 1991).

Computationally, the proof nets provide the essential structure of derivations. They support, for example, parsing to normal form semantic output without on-line β-reduction (Morrill 1997: 25–30), and memoisation (Morrill 1996), something prohibitive under the shifting premises of hypothetical reasoning in other forms of proof syntax. The proof nets are for categorial grammar what parse trees are for CFG (furthermore incorporating semantics), adding to our paradigmatic slogans: proof nets-as-syntactic structures.

Still, from a linguistic point of view the possibilities of the Lambek calculus are extremely limited since it is a logic of only *concatenation*; works that have aimed at formulating corresponding logic of discontinuity include Moortgat (1988 pt. 3.3, 1990, 1991/96, 1996), Solias (1992), Morrill and Solias (1993), Morrill (1994 chs. 4–5, 1995), Moortgat and Oehrle (1994), Calcagno (1995), Hendriks (1995), and Morrill and Merenciano (1996).

Let us recall the (associative) Lambek calculus **L**. The category formulas $\mathcal{F}$ are given in terms of primitive category formulas $\mathcal{A}$ as follows.

(1) $\mathcal{F} ::= \mathcal{A} \mid \mathcal{F}\bullet\mathcal{F} \mid \mathcal{F}\backslash\mathcal{F} \mid \mathcal{F}/\mathcal{F}$

We interpret category formulas as subsets of the set L of all strings over some vocabulary V. Given an interpretation $[\![P]\!]$ for each primitive category formula P, each category formula A receives an interpretation $[\![A]\!]$ thus:

(2)
$$\begin{array}{lcl} [\![A\backslash B]\!] & = & \{s \mid \forall s' \in [\![A]\!], s'+s \in [\![B]\!]\} \\ [\![B/A]\!] & = & \{s \mid \forall s' \in [\![A]\!], s+s' \in [\![B]\!]\} \\ [\![A\bullet B]\!] & = & \{s_1+s_2 \mid s_1 \in [\![A]\!] \ \&\ s_2 \in [\![B]\!]\} \end{array}$$

A sequent $\Gamma \Rightarrow A$ comprises a succedent category formula A and an antecedent configuration Γ which is a sequence of category formulas. It asserts that in all interpretations, the ordered concatenation of strings in the antecedent category formulas yields a string in the succedent category formula. The valid sequents are those generated by the following sequent calculus.

(3)

a. $$A \Rightarrow A \ \ \text{id} \qquad \frac{\Gamma \Rightarrow A \quad \Delta(A) \Rightarrow B}{\Delta(\Gamma) \Rightarrow B}\text{Cut}$$

b. $$\frac{\Gamma \Rightarrow A \quad \Delta(B) \Rightarrow C}{\Delta(\Gamma, A\backslash B) \Rightarrow C}\backslash\text{L} \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A\backslash B}\backslash\text{R}$$

c. $$\frac{\Gamma \Rightarrow A \quad \Delta(B) \Rightarrow C}{\Delta(B/A, \Gamma) \Rightarrow C}/\text{L} \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B/A}/\text{R}$$

d. $\dfrac{\Gamma(A,B) \Rightarrow C}{\Gamma(A\bullet B) \Rightarrow C}\bullet L \qquad \dfrac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow B}{[\Gamma,\Delta] \Rightarrow A\bullet B}\bullet R$

The calculus of Lambek (1958) excludes the empty string ε, and the empty configuration Λ, but they are included here, and we add the product unit I. The definition (1) of category formulas becomes (4).

(4) $\mathcal{F} ::= \mathcal{A} \mid \mathcal{F}\bullet\mathcal{F} \mid \mathcal{F}\backslash\mathcal{F} \mid \mathcal{F}/\mathcal{F} \mid I$

The product unit is interpreted as the set comprising the empty string:

(5) $[\![I]\!] \;=\; \{\varepsilon\}$

The sequent rules are those of (6).

(6) $\Rightarrow I \quad IR \qquad \dfrac{\Gamma_1,\Gamma_2 \Rightarrow A}{\Gamma_1, I, \Gamma_2 \Rightarrow A}IL$

By way of examples of discontinuity beyond the reach of **L** we consider relativisation and in situ binding. In (7) the relative pronoun binds a position which is medial in the relative clause.

(7) (the man) that probably won

Defining the relative pronoun as R/(N\S) or R/(S/N) (where R is CN\CN) allows it to bind only left or right peripheral positions: (7) is not generated. To deal with such cases, Moortgat (1988: 110) defines as follows a binary operator which we write $\uparrow_e$:

(8) $[\![B\uparrow_e A]\!] = \{s_1{+}s_2 | \forall s \in [\![A]\!], s_1{+}s{+}s_2 \in [\![B]\!]\}$

Assigning the relative pronoun to category R/(S$\uparrow_e$N) allows both medial and (assuming the ε) peripheral extraction, via the introduction rule (9).

(9) $\dfrac{\Gamma_1, A, \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B\uparrow_e A}\uparrow_e R$

A satisfactory elimination rule, on the other hand, cannot be formulated, as observed by Moortgat (121–2). Morrill (1992: 13–14) notes that such a treatment potentially accommodates obligatory extraction valencies:

(10) a. (the man) that John assured Mary to be reliable
b. *John assured Mary Bill to be reliable.

If the extraction valency of "assured" is marked by $\uparrow_e$, a sequent corresponding to (10a) is valid while that for (10b) is invalid. However, as pointed out by I. Sag (p.c.), in the absence of an elimination rule it is impossible to actually derive all cases like (10a).

In (11) the quantifier phrase and reflexive are in situ binders, taking scope respectively at the sentence and verb phrase levels.

(11) a. John bought someone Fido.
b. John bought himself Fido.

Moortgat (1991/96) introduces a ternary operator Q for which Morrill (1992: 15) offers the interpretation:

(12) $[\![\mathrm{Q}(B,A,C)]\!] = \{s | \forall s_1, s_3, [\forall s_2 \in [\![A]\!], s_1{+}s_2{+}s_3 \in [\![B]\!]] \rightarrow s_1{+}s_2{+}s_3 \in [\![C]\!]\}$

Moortgat categorises quantifier phrases and reflexives as sentence and verb phrase in situ binders Q(S, N, S) and Q(N\S, N, N\S) respectively. Cases such as (11) are generated by means of the elimination rule (13).

(13) $$\frac{\Gamma(A) \Rightarrow B \qquad \Delta(C) \Rightarrow D}{\Delta(\Gamma(Q(B,A,C))) \Rightarrow D}\mathrm{QL}$$

However, this time no satisfactory introduction rule can be given. Therefore, as pointed out by H. Hendriks (p.c.), a valid sequent such as Q(S, N, S) $\Rightarrow$ Q(N\S, N, N\S), showing that a sentence in situ binder is also a verb phrase in situ binder, cannot actually be derived.

Based on considerations in Morrill and Solias (1993), Morrill (1994, chs. 4–5; 1995) presents an (unsorted) discontinuity calculus and Morrill (1995, app.) and Morrill and Merenciano (1996) a sorted discontinuity calculus. The former has a sequent calculus with an extraction elimination inference, but does not solve the problems alluded to above. The latter has a *labelled* sequent calculus, and does solve these problems, treating $\uparrow_e$ and Q as defined operators. In a labelled sequent calculus a wider class of sequents is generated by rules for formulas which is then filtered by conditions on labels. However, it would be even more satisfactory to have a one-stage characterisation in the spirit of pure sequent calculus.

In this article we provide such a pure sequent calculus for sorted discontinuity and show how the issues raised above are resolved. We then show how to give proof nets for the operators $\uparrow_e$ and Q treated as units. We hope to present proof nets for the full sorted discontinuity calculus in a longer version of the paper.

In the sorted discontinuity calculus, category formulas fall into two sorts: those $\mathcal{F}$ of sort string, interpreted as subsets of L, and those $\mathcal{F}^2$ of sort split string, interpreted as subsets of L^2. Our definition (4) of category formulas becomes (14).

(14) $$\begin{array}{lcl} \mathcal{F} & ::= & A \mid \mathcal{F}{\bullet}\mathcal{F} \mid \mathcal{F}\backslash\mathcal{F} \mid \mathcal{F}/\mathcal{F} \mid I \mid \mathcal{F}^2{\odot}\mathcal{F} \mid \mathcal{F}^2{\downarrow}\mathcal{F} \\ \mathcal{F}^2 & ::= & \mathcal{F}{\uparrow}\mathcal{F} \end{array}$$

The discontinuity operators are interpreted by "residuation" with respect to the interpolation adjunction W of functionality $L^2, L \rightarrow L$, defined by $\langle s_1, s_2\rangle W s = s_1{+}s{+}s_2$, in exactly the same way that the continuity operators are interpreted by residuation with respect to the concatenation adjunction + of functionality $L, L \rightarrow L$:

(15) $$\begin{array}{lcl} [\![A{\downarrow}B]\!] & = & \{s |\ \forall \langle s_1, s_2\rangle \in [\![A]\!], s_1{+}s{+}s_2 \in [\![B]\!]\} \\ [\![B{\uparrow}A]\!] & = & \{\langle s_1, s_2\rangle |\ \forall s \in [\![A]\!], s_1{+}s{+}s_2 \in [\![B]\!]\} \\ [\![A{\odot}B]\!] & = & \{s_1{+}s{+}s_2 |\ \langle s_1, s_2\rangle \in [\![A]\!]\ \&\ s \in [\![B]\!]\} \end{array}$$

We have, then, $B{\uparrow_e}A = (B{\uparrow}A){\odot}I$ and $\mathrm{Q}(B,A,C) = (B{\uparrow}A){\downarrow}C$.

We have already noted the problem of giving sequent rules for categories of the variety $B{\uparrow}A$: a category occurrence $B{\uparrow}A$ in an antecedent would fail to indicate where one is meant to interpolate. Our analysis is that in the sequent calculus of **L** a category occurrence signals two things: a resource, and the location of that resource with respect to others. This double service can be maintained in view of the continuity of concatenation, but discontinuity requires a distinction between signaling a resource, and its locations of action, which may be multiple. In particular, $B{\uparrow}A$ has two discontinuous components. Our solution is for a split string category formula to appear *twice* in a sequent, at its two loci of action. To mark that the two components are to be taken together as a resource, the occurrences are punctuated as roots, $\sqrt{\ }$.

Sequents come in two kinds, those Σ with sort string succedents, which have string antecedent configurations $\mathcal{O}$, and those Σ^2 with sort split string succedents, which have split string antecedent configurations $\mathcal{O}^2$:

(16)
$$\begin{array}{lcl} \Sigma & ::= & \mathcal{O} \Rightarrow \mathcal{F} \\ \Sigma^2 & ::= & \mathcal{O}^2 \Rightarrow \sqrt{\mathcal{F}^2} \\ \mathcal{O} & ::= & \Lambda \mid \mathcal{F}, \mathcal{O} \mid \sqrt[1]{\mathcal{F}^2}, \mathcal{O}, \sqrt[2]{\mathcal{F}^2} \\ \mathcal{O}^2 & ::= & \mathcal{O}, \sqrt{\mathcal{F}^2}, \mathcal{O} \mid \mathcal{O}, \sqrt[1]{\mathcal{F}^2}, \mathcal{O}^2, \sqrt[2]{\mathcal{F}^2}, \mathcal{O} \end{array}$$

Observe that configurations have balanced occurrences of parenthesising punctuation $\sqrt[1]{\ }$ and $\sqrt[2]{\ }$. These mark the two components of split antecedent categories. In a sequent with a split succedent category there is a $\sqrt{\ }$ in the antecedent marking the split point, and around which the parenthesising is balanced. The sequent rules are thus:

(17) a.
$$\dfrac{\Gamma(\sqrt{A}) \Rightarrow \sqrt{A} \qquad \Delta(B) \Rightarrow C}{\Delta(\Gamma(A{\downarrow}B)) \Rightarrow C}\,{\downarrow}\mathrm{L} \qquad \dfrac{\sqrt[1]{A}, \Gamma, \sqrt[2]{A} \Rightarrow B}{\Gamma \Rightarrow A{\downarrow}B}\,{\downarrow}\mathrm{R}$$

b.
$$\dfrac{\Gamma \Rightarrow A \qquad \Delta(B) \Rightarrow C}{\Delta(\sqrt[1]{B{\uparrow}A}, \Gamma, \sqrt[2]{B{\uparrow}A}) \Rightarrow C}\,{\uparrow}\mathrm{L} \qquad \dfrac{\Gamma(A) \Rightarrow B}{\Gamma(\sqrt{B{\uparrow}A}) \Rightarrow \sqrt{B{\uparrow}A}}\,{\uparrow}\mathrm{R}$$

c.
$$\dfrac{\Gamma(\sqrt[1]{A}, B, \sqrt[2]{A}) \Rightarrow C}{\Gamma(A{\odot}B) \Rightarrow C}\,{\odot}\mathrm{L} \qquad \dfrac{\Gamma(\sqrt{A}) \Rightarrow \sqrt{A} \qquad \Delta \Rightarrow B}{\Gamma(\Delta) \Rightarrow A{\odot}B}\,{\odot}\mathrm{R}$$

By way of example, the medial relativisation (7) is treated as follows.

(18)
$$\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{S/S,\ N,\ N\backslash S \Rightarrow S}}{\mathrm{S/S}, \sqrt{\mathrm{S{\uparrow}N}}, \mathrm{N\backslash S} \Rightarrow \sqrt{\mathrm{S{\uparrow}N}}}\,{\uparrow}\mathrm{R} \qquad \Rightarrow I}{\mathrm{S/S,\ N\backslash S} \Rightarrow (\mathrm{S{\uparrow}N}){\odot}I}\,{\odot}\mathrm{R} \qquad \mathrm{R \Rightarrow R}}{\mathrm{R}/((\mathrm{S{\uparrow}N}){\odot}I), \mathrm{S/S,\ N\backslash S \Rightarrow R}}\,/\mathrm{L}}{\textbf{that+probably+won}\text{: R}}$$

Turning to proof nets for **L** (Roorda 1991), the proof frame for a sequent $A_1, \ldots, A_n \Rightarrow A$ is obtained by recursively unfolding the cyclically ordered polar formulas $A_1^-, \ldots, A_n^-, A^+$ up to atomic literals as follows.

(19) a.
$$\dfrac{A^+ \quad \text{ii} \quad B^-}{A\backslash B^-} \qquad \dfrac{B^+ \quad \text{i} \quad A^-}{A\backslash B^+}$$
b.
$$\dfrac{B^- \quad \text{ii} \quad A^+}{B/A^-} \qquad \dfrac{A^- \quad \text{i} \quad B^+}{B/A^+}$$

$$\text{c.}\quad \frac{A^- \quad \text{i} \quad B^-}{A{\bullet}B^-} \qquad \frac{B^+ \quad \text{ii} \quad A^-}{A{\bullet}B^+}$$

The unfolding defines a cyclic total order (chain) $>$ on the literals. A proof structure is a graph of polar formulas that is the result of connecting with an axiom link each literal to exactly one other with the same atom and opposite polarity. A proof structure is a proof net iff it satisfies planarity (Roorda) and the long trip condition (Girard).[1]

Morrill (1996) offers a correctness criterion in terms of unifiability. Here we employ a graph theoretic statement of this criterion. From an axiom linking, construct the graph on i- and ii-vertices (we use numerals and letters respectively below) which has, for each axiom link, an edge between the two vertices immediately inside and between the two vertices immediately outside the link. The correctness criterion is that in the resulting graph no i-vertex be connected to any other i-vertex, or to any ii-vertex from which it is a descendant in the proof frame.

We give the unfolding for $\uparrow_e$ and Q:

(20) a.

$$\begin{array}{cc} \text{i} \quad A^+ \quad \text{i} & \text{ii} \quad A^- \quad \text{ii} \\[2ex] \dfrac{B^-}{B{\uparrow_e}A^-} & \dfrac{B^+}{B{\uparrow_e}A^+} \end{array}$$

b.

$$\begin{array}{cc} \text{ii} \quad B^+ \quad \text{ii}' \quad C^- \quad \text{ii} & \text{i} \quad B^- \quad \text{i}' \quad C^+ \quad \text{i} \\[2ex] \dfrac{A^-}{Q(B,A,C)^-} & \dfrac{A^+}{Q(B,A,C)^+} \end{array}$$

Vertically stacked literals are unordered with respect to one another; the upper literals enter into a separate chain which is cyclic: the peripheral vertices are the same. But linking must still be planar in the chains of the partial order.

Consider the proof frame for the medial extraction (7).

(21)

$$\cfrac{\begin{array}{ccccccc} & k \quad \mathrm{N}^- \quad k & & & & & \\[2ex] 0 & \dfrac{\mathrm{R}^- \quad j \quad \dfrac{\mathrm{S}^+}{\mathrm{S}{\uparrow_e}\mathrm{N}^+}}{\mathrm{R}/(\mathrm{S}{\uparrow_e}\mathrm{N})^-} & 1 & \dfrac{\mathrm{S}^- \quad l \quad \mathrm{S}^+}{\mathrm{S}/\mathrm{S}^-} & 2 & \dfrac{\mathrm{N}^+ \quad m \quad \mathrm{S}^-}{\mathrm{N}\backslash\mathrm{S}^-} & 3 \quad \mathrm{R}^+ \quad 0 \end{array}}{\mathrm{R}/(\mathrm{S}{\uparrow_e}\mathrm{N}),\ \mathrm{S}/\mathrm{S},\ \mathrm{N}\backslash\mathrm{S} \Rightarrow \mathrm{R}}$$

The partial order on literals comprises the cyclic chains $0\mathrm{R}^-j\mathrm{S}^+1\mathrm{S}^-l\mathrm{S}^+2\mathrm{N}^+$-$m\mathrm{S}^-3\mathrm{R}^+0$ and $k\mathrm{N}^-k$. We can add the axiom linkings $0\mathrm{R}^-j{=}3\mathrm{R}^+0$, $j\mathrm{S}^+1{=}1\mathrm{S}^-l$, $k\mathrm{N}^-k{=}2\mathrm{N}^+m$, and $l\mathrm{S}^+2{=}m\mathrm{S}^-3$, which are planar in this partial order. The

[1] A crossing is an elementary path in the literal ordering $l_1 > \ldots > l_2 > \ldots > l_1' > \ldots > l_2'$ where l_1 and l_1' and l_2 and l_2' are linked. A proof net satisfies planarity iff there is no crossing. A circularity is a cycle in the proof structure which does not traverse the premisses of any i-node. A proof net satisfies the long trip condition iff there is no circularity.

vertex graph is $0-0, j-3, j-l, 1-1, k-m, k-2, l-3, 2-m$, and satisfies the connectedness constraints.[2]

Bibliography

Calcagno, M.: 1995, 'A Sign-Based Extension to the Lambek Calculus for Discontinuous Constituency', *Bulletin of the Interest Group in Pure and Applied Logics* **3**, 555–578.

Girard, J-Y.: 1987, 'Linear Logic', *Theoretical Computer Science* **50**, 1–102.

Hendriks, P.: 1995, *Comparatives and Categorial Grammar*, Ph.D. dissertation, Rijsuniversiteit Groningen.

Lambek, J.: 1958, 'The mathematics of sentence structure', *American Mathematical Monthly* **65**, 154–170, also in Buszkowski, W., W. Marciszewski, and J. van Benthem (eds.): 1988, *Categorial Grammar*, Linguistic & Literary Studies in Eastern Europe Volume 25, John Benjamins, Amsterdam, 153–172.

Moortgat, M.: 1988, *Categorial Investigations: Logical and Linguistic Aspects of the Lambek Calculus*, Foris, Dordrecht.

Moortgat, M.: 1990, 'The Quantification Calculus: Questions of Axiomatisation', in Deliverable R1.2.A of DYANA Dynamic Interpretation of Natural Language, ESPRIT Basic Research Action BR3175.

Moortgat, M.: 1991/96, 'Generalised Quantification and Discontinuous type constructors', in Bunt and van Horck (eds.) *Discontinuous Constituency*, Mouton de Gruyter, Berlin, 181–208.

Moortgat, M.: 1996, 'In situ binding; a modal analysis', *Proceedings Tenth Amsterdam Colloquium*, Institute for Logic, Language and Computation, Amsterdam.

Moortgat, M. and R.T Oehrle: 1994, 'Adjacency, dependency and order', *Proceedings Ninth Amsterdam Colloquium*, Institute for Logic, Language and Computation, Amsterdam, 447–466.

Morrill, G.: 1992, 'Categorial formalisation of relativisation: pied piping, islands, and extraction sites', Report de Recerca LSI-92-23-R, Departament de Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya.

Morrill, G.: 1994, *Type Logical Grammar: Categorial Logic of Signs*, Kluwer Academic Publishers, Dordrecht.

Morrill, G.: 1995, 'Discontinuity in Categorial Grammar', *Linguistics and Philosophy* **18**, 175–219.

Morrill, G.: 1996, 'Memoisation of Categorial Proof Nets: Parallelism in Categorial Processing', Report de Recerca LSI-96-24-R, Departament de Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya. Also in M. Abrusci and C. Casadio (eds.) *Proofs and Linguistic Categories*, CLUEB, Bologna, 157–169.

Morrill, G.: 1997, 'Geometry of Language', Report de Recerca LSI-97-45-R, Departament de Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya.

Morrill, G. and Merenciano, J.M.: 1996, 'Generalising Discontinuity', *Traitement Automatique des Langues* **37**, *2*, 119–143.

Morrill, G. and T. Solias: 1993, 'Tuples, Discontinuity and Gapping', Proceedings *Meeting of the European Chapter of the Association for Computational Linguistics*, Utrecht, 287–297.

Roorda, Dirk: 1991, *Resource Logics: proof-theoretical investigations*, Ph.D. dissertation, Universiteit van Amsterdam.

Solias, T.: 1992, *Gramáticas Categoriales, Coordinación Generalizada y Elisión*, Ph.D. dissertation, Universidad Autónoma de Madrid.

[2] In view of space limitations the reader is invited to check the sequent calculus and proof net derivability of $R/((S{\uparrow}N)\odot I)$, N, $((((N\backslash S)/VP){\uparrow}N)\odot I)/N$, N, VP $\Rightarrow$ R (and underivability of N, $((((N\backslash S)/VP){\uparrow}N)\odot I)/N$, N, N, VP $\Rightarrow$ S) corresponding to the obligatory extraction (10), of N, $((N\backslash S)/N)/N$, $\{Q(S, N, S)|Q(N\backslash S, N, N\backslash S)\}$, N $\Rightarrow$ S corresponding to the quantification and reflexivisation (11), and of $Q(S, N, S) \Rightarrow Q(N\backslash S, N, N\backslash S)$.