

Displacement Calculus

GLYN MORRILL AND ORIOL VALENTÍN
Universitat Politècnica de Catalunya
and
Universitat Pompeu Fabra

Abstract

The Lambek calculus \mathbf{L} provides a foundation for categorial grammar in the form of a logic of concatenation. But natural language is characterized by dependencies which may also be discontinuous. In this paper we introduce the displacement calculus \mathbf{D} , a generalization of Lambek calculus, which preserves the good proof-theoretic properties of the latter while embracing discontinuity and subsuming \mathbf{L} . We illustrate linguistic applications and prove Cut-elimination, the subformula property, and decidability

1. Introduction

Lambek (1958) applied mathematical logic to linguistics in such a way that the analysis of a sentence is a proof.¹ This was the genesis of logical syntax, a decade before the advent of logical semantics. Once these applications of logic are born they take on a life of their own, for by comparison the rest seems ... illogical. The Lambek calculus is a sequence logic without structural rules which enjoys Cut-elimination, the subformula property, and decidability. It is intuitionistic, hence the standard Curry-Howard categorial semantics. It is sound and complete with respect to interpretation by residuation in free semigroups. But for all its elegance, as a logic of concatenation, the Lambek calculus can only analyse displacement when the dependencies happen to be peripheral. As a consequence it cannot account for the syntax and semantics of, for example:

- (1) • Discontinuous idioms (*Mary gave the man the cold shoulder*).
- Quantification (*John gave every book to Mary; Mary thinks someone left; Everyone loves someone*).

¹The research reported in the present paper was supported by DGICYT project SESAAME-BAR (TIN2008-06582-C03-01).

- VP ellipsis (*John slept before Mary did; John slept and Mary did too*).
- Medial extraction (*dog that Mary saw today*).
- Pied-piping (*mountain the painting of which by Cezanne John sold for \$10,000,000.*)
- Appositive relativization (*John, who jogs, sneezed*).
- Parentheticals (*Fortunately, John has perseverance; John, fortunately, has perseverance; John has, fortunately, perseverance; John has perseverance, fortunately*).
- Gapping (*John studies logic, and Charles, phonetics*).
- Comparative subdeletion (*John ate more donuts than Mary bought bagels*).
- Reflexivization (*John sent himself flowers*).

In the decade of the 90s it seemed that a general methodology for obtaining more adequate categorial grammars might be to introduce families of residuated connectives for multiple modes of composition related by structural rules (Moortgat, 1997): so-called multimodal categorial grammar. But this paper marks a return to unimodal categorial grammar like the Lambek calculus, in that there is a single primitive mode of binary composition, namely concatenation; the modes of composition with respect to which the other connectives are specified are defined. Indeed, we present displacement calculus which, like the Lambek calculus, is a sequence logic without structural rules which, as we shall show here, enjoys Cut-elimination, the subformula property, and decidability. Moreover, like the Lambek calculus it is intuitionistic, and so supports the standard categorial Curry-Howard type-logical semantics. We shall show how it provides basic analyses of all of the phenomena itemized in (1).

In Section 2 we define the calculus of displacement. In Section 3 we give linguistic applications. In Section 4 we prove Cut-elimination, and we conclude in Section 5.

2. Displacement Calculus

The types of the calculus of displacement \mathbf{D} classify strings over a vocabulary including a distinguished placeholder 1 called the *separator*. The sort $i \in \mathcal{N}$ of a (discontinuous) string is the number of

separators it contains and these punctuate it into $i + 1$ maximal continuous substrings or *segments*. The types of \mathbf{D} are sorted into types \mathcal{F}_i of sort i by mutual recursion as follows:

$$\begin{array}{ll}
(2) \quad \mathcal{F}_j & := \mathcal{F}_i \setminus \mathcal{F}_{i+j} & \text{under} \\
\mathcal{F}_i & := \mathcal{F}_{i+j} / \mathcal{F}_j & \text{over} \\
\mathcal{F}_{i+j} & := \mathcal{F}_i \bullet \mathcal{F}_j & \text{product} \\
\mathcal{F}_0 & := I & \text{product unit} \\
\mathcal{F}_j & := \mathcal{F}_{i+1} \downarrow_k \mathcal{F}_{i+j}, 1 \leq k \leq i+1 & \text{infix} \\
\mathcal{F}_{i+1} & := \mathcal{F}_{i+j} \uparrow_k \mathcal{F}_j, 1 \leq k \leq i+1 & \text{extract} \\
\mathcal{F}_{i+j} & := \mathcal{F}_{i+1} \odot_k \mathcal{F}_j, 1 \leq k \leq i+1 & \text{disc. product} \\
\mathcal{F}_1 & := J & \text{disc. prod. unit}
\end{array}$$

Where A is a type we call its sort sA . The set \mathcal{O} of *configurations* is defined as follows, where Λ is the empty string and $[]$ is the metalinguistic separator:

$$(3) \quad \mathcal{O} ::= \Lambda \mid [] \mid \mathcal{F}_0 \mid \underbrace{\mathcal{F}_{i+1} \{ \mathcal{O} : \dots : \mathcal{O} \}}_{i+1 \text{ } \mathcal{O}'s} \mid \mathcal{O}, \mathcal{O}$$

Note that the configurations are of a new kind in which some type formulas, namely the type formulas of sort greater than one, label mother nodes rather than leaves, and have a number of immediate subconfigurations equal to their sort. This signifies a discontinuous type intercalated by these subconfigurations. Thus $A\{\Delta_1 : \dots : \Delta_n\}$ interpreted syntactically is formed by strings $\alpha_0 + \beta_1 + \dots + \beta_n + \alpha_n$ where $\alpha_0 + 1 + \dots + 1 + \alpha_n \in A$ and $\beta_1 \in \Delta_1, \dots, \beta_n \in \Delta_n$. We call these types *hyperleaves* since in multimodal calculus they would be leaves. We call these new configurations *hyperconfigurations*. The sort of a (hyper)configuration is the number of separators it contains. A *hypersequent* $\Gamma \Rightarrow A$ comprises an antecedent hyperconfiguration Γ of sort i and a succedent type A of sort i . The *vector* \vec{A} of a type A is defined by:

$$(4) \quad \vec{A} = \begin{cases} A & \text{if } sA = 0 \\ A\{\underbrace{[] : \dots : []}_{sA \text{ } []'s}\} & \text{if } sA > 0 \end{cases}$$

Where Δ is a configuration of sort at least k and Γ is a configuration, the *k-ary wrap* $\Delta|_k\Gamma$ signifies the configuration which is the result of replacing by Γ the k th separator in Δ . Where Δ is a configuration of sort i and $\Gamma_1, \dots, \Gamma_i$ are configurations, the *generalized wrap*

$\Delta \langle \Gamma_1, \dots, \Gamma_i \rangle$ is the result of simultaneously replacing the successive separators in Δ by $\Gamma_1, \dots, \Gamma_i$ respectively. In the hypersequent calculus we use a discontinuous distinguished hyperoccurrence notation $\Delta \langle \Gamma \rangle$ to refer to a configuration Δ and continuous subconfigurations $\Delta_1, \dots, \Delta_i$ and a discontinuous subconfiguration Γ of sort i such that $\Gamma \otimes \langle \Delta_1, \dots, \Delta_i \rangle$ is a continuous subconfiguration. That is, where Γ is of sort i , $\Delta \langle \Gamma \rangle$ abbreviates $\Delta(\Gamma \otimes \langle \Delta_1, \dots, \Delta_i \rangle)$ where $\Delta(\dots)$ is the usual distinguished occurrence notation. Technically, whereas the usual distinguished occurrence notation $\Delta(\Gamma)$ refers to a context containing a *hole* which is a leaf, in hypersequent calculus the distinguished hyperoccurrence notation $\Delta \langle \Gamma \rangle$ refers to a context containing a hole which may be a hyperleaf, a *hyperhole*.

The hypersequent calculus for the calculus of displacement is given in Figure 1. Observe that the rules for both the concatenating connectives $\backslash, \bullet, /$ and the wrapping connectives $\downarrow_k, \odot_k, \uparrow_k$ are just like the rules for Lambek calculus except for the vectorial notation and hyperoccurrence notation; the former are specified in relation to the primitive concatenation represented by the sequent comma and the latter are specified in relation to the defined operations of k -ary wrap.

3. Linguistic Applications

A parser/theorem-prover for the displacement calculus has been implemented in Prolog. In this section we give the analyses it produces for the examples of (1). These are examples from Chapter 6 of Morrill (2010). There a very similar system called discontinuous Lambek calculus is used with unary bridge and split operators and no nullary product units. Here we use the displacement calculus which has the continuous and discontinuous product units I and J instead of unary operators. The lexicon for the analyses is as follows; we abbreviate \downarrow_1, \odot_1 and \uparrow_1 as \downarrow, \odot and \uparrow respectively.

- (5) **\$10,000,000** : N : *tenmilliondollars*
and : $(S \backslash S) / S$: $\lambda A \lambda B [B \wedge A]$
and :
 $((S \uparrow ((N \backslash S) / N)) \backslash (S \uparrow ((N \backslash S) / N))) / ((S \uparrow ((N \backslash S) / N)) \odot I)$:
 $\lambda A \lambda B \lambda C [(B \ C) \wedge (\pi_1 A \ C)]$
ate : $(N \backslash S) / N$: *ate*

$$\begin{array}{c}
\frac{}{\vec{A} \Rightarrow A} id \quad \frac{\Gamma \Rightarrow A \quad \Delta\langle \vec{A} \rangle \Rightarrow B}{\Delta\langle \Gamma \rangle \Rightarrow B} Cut \\
\\
\frac{\Gamma \Rightarrow A \quad \Delta\langle \vec{C} \rangle \Rightarrow D}{\Delta\langle \Gamma, \vec{A} \setminus \vec{C} \rangle \Rightarrow D} \setminus L \quad \frac{\vec{A}, \Gamma \Rightarrow C}{\Gamma \Rightarrow A \setminus C} \setminus R \\
\\
\frac{\Gamma \Rightarrow B \quad \Delta\langle \vec{C} \rangle \Rightarrow D}{\Delta\langle \vec{C} / \vec{B}, \Gamma \rangle \Rightarrow D} /L \quad \frac{\Gamma, \vec{B} \Rightarrow C}{\Gamma \Rightarrow C / B} /R \\
\\
\frac{\Delta\langle \vec{A}, \vec{B} \rangle \Rightarrow D}{\Delta\langle \vec{A} \bullet \vec{B} \rangle \Rightarrow D} \bullet L \quad \frac{\Gamma_1 \Rightarrow A \quad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \bullet B} \bullet R \\
\\
\frac{\Delta\langle \Lambda \rangle \Rightarrow A}{\Delta\langle \vec{T} \rangle \Rightarrow A} IL \quad \frac{}{\Lambda \Rightarrow I} IR \\
\\
\frac{\Gamma \Rightarrow A \quad \Delta\langle \vec{C} \rangle \Rightarrow D}{\Delta\langle \Gamma |_k \vec{A} |_k \vec{C} \rangle \Rightarrow D} \downarrow_k L \quad \frac{\vec{A} |_k \Gamma \Rightarrow C}{\Gamma \Rightarrow A |_k C} \downarrow_k R \\
\\
\frac{\Gamma \Rightarrow B \quad \Delta\langle \vec{C} \rangle \Rightarrow D}{\Delta\langle \vec{C} \uparrow_k \vec{B} |_k \Gamma \rangle \Rightarrow D} \uparrow_k L \quad \frac{\Gamma |_k \vec{B} \Rightarrow C}{\Gamma \Rightarrow C \uparrow_k B} \uparrow_k R \\
\\
\frac{\Delta\langle \vec{A} |_k \vec{B} \rangle \Rightarrow D}{\Delta\langle \vec{A} \odot_k \vec{B} \rangle \Rightarrow D} \odot_k L \quad \frac{\Gamma_1 \Rightarrow A \quad \Gamma_2 \Rightarrow B}{\Gamma_1 |_k \Gamma_2 \Rightarrow A \odot_k B} \odot_k R \\
\\
\frac{\Delta\langle [] \rangle \Rightarrow A}{\Delta\langle \vec{J} \rangle \Rightarrow A} JL \quad \frac{}{[] \Rightarrow J} JR
\end{array}$$

Figure 1: Calculus of displacement **D**

bagels : CN : *bagels*
before : $((N \setminus S) \setminus (N \setminus S)) / S$: $\lambda A \lambda B \lambda C ((before\ A)\ (B\ C))$
book : CN : *book*
bought : $(N \setminus S) / N$: *bought*
by : $(CN \setminus CN) / N$: *by*
cezanne : N : *cezanne*
charles : N : *c*
did : $((((N \setminus S) \uparrow (N \setminus S)) / (N \setminus S)) \setminus ((N \setminus S) \uparrow (N \setminus S)))$:
 $\lambda A \lambda B ((A\ B)\ B)$
did+too : $((((N \setminus S) \uparrow (N \setminus S)) / (N \setminus S)) \setminus ((N \setminus S) \uparrow (N \setminus S)))$:
 $\lambda A \lambda B ((A\ B)\ B)$
dog : CN : *dog*
donuts : CN : *donuts*
every : $((S \uparrow N) \downarrow S) / CN$: $\lambda A \lambda B \forall C [(A\ C) \rightarrow (B\ C)]$
everyone : $(S \uparrow N) \downarrow S$: $\lambda A \forall B [(person\ B) \rightarrow (A\ B)]$
flowers : N : *flowers*
for : PP / N : λAA
fortunately : $(S \uparrow I) \downarrow S$: $\lambda A (fortunately\ (A\ d))$
john : N : *j*
gave : $(N \setminus S) / (N \bullet PP)$: $\lambda A ((gave\ \pi_2 A)\ \pi_1 A)$
gave+1+the+cold+shoulder : $(N \setminus S) \uparrow N$: *shunned*
has : $(N \setminus S) / N$: *has*
himself : $((N \setminus S) \uparrow N) \downarrow (N \setminus S)$: $\lambda A \lambda B ((A\ B)\ B)$
jogs : $N \setminus S$: *jogs*
left : $N \setminus S$: *left*
logic : N : *logic*
loves : $(N \setminus S) / N$: *love*
man : CN : *man*
mary : N : *m*
more :
 $(S \uparrow (((S \uparrow N) \downarrow S) / CN)) \downarrow (S / ((CP \uparrow (((S \uparrow N) \downarrow S) / CN)) \odot I))$:
 $\lambda A \lambda B [|\lambda C (A\ \lambda D \lambda E [(D\ C) \wedge (E\ C)])| > |\lambda C (\pi_1 B\ \lambda D \lambda E$
 $[(D\ C) \wedge (E\ C)])|]$
mountain : CN : *mountain*
painting : CN : *painting*
perseverance : N : *perseverance*
phonetics : N : *phonetics*
of : $(CN \setminus CN) / N$: *of*
slept : $N \setminus S$: *slept*
saw : $(N \setminus S) / N$: *saw*

sent : $(N \setminus S) / (N \bullet N) : \lambda A((sent \pi_1 A) \pi_2 A)$
sneezed : $N \setminus S : sneezed$
sold : $(N \setminus S) / (N \bullet PP) : \lambda A((sold \pi_2 A) \pi_1 A)$
someone : $(S \uparrow N) \downarrow S : \lambda A \exists B[(person B) \wedge (A B)]$
studies : $(N \setminus S) / N : studies$
than : $CP / S : \lambda AA$
that : $(CN \setminus CN) / ((S \uparrow N) \odot I) : \lambda A \lambda B \lambda C[(B C) \wedge (\pi_1 A C)]$
the : $N / CN : \iota$
thinks : $(N \setminus S) / S : thinks$
to : $PP / N : \lambda AA$
today : $(N \setminus S) \setminus (N \setminus S) : \lambda A \lambda B(today (A B))$
which : $(N \uparrow N) \downarrow ((CN \setminus CN) / ((S \uparrow N) \odot I)) :$
 $\lambda A \lambda B \lambda C \lambda D[(C D) \wedge (\pi_1 B (A D))]$
who : $(N \setminus ((S \uparrow N) \downarrow S)) / ((S \uparrow N) \odot I)$
 $\lambda A \lambda B \lambda C[(\pi_1 A B) \wedge (C B)]$

The phenomena itemized in (1) are considered in the following subsections.

3.1. Discontinuous Idioms

Our first example is of a discontinuous idiom, where the lexicon has to assign *give ... the cold shoulder* a non-compositional meaning ‘shun’:

(6) **mary+gave+the+man+the+cold+shoulder** : S

Lexical insertion yields the following sequent, which is labelled with the lexical semantics:

(7) $N : m, (N \setminus S) \uparrow N \{N / CN : \iota, CN : man\} : shunned \Rightarrow S$

This has a proof as follows.

$$(8) \frac{\frac{\overline{CN \Rightarrow CN} \quad \overline{N \Rightarrow N}}{N / CN, CN \Rightarrow N} /L \quad \frac{\overline{N \Rightarrow N} \quad \overline{S \Rightarrow S}}{N, N \setminus S \Rightarrow S} \setminus L}{N, (N \setminus S) \uparrow N \{N / CN, CN\} \Rightarrow S} \uparrow L$$

This delivers the semantics:

(9) $((shunned (\iota man)) m)$

3.2. Quantification

Lambek categorial grammar can analyse a subject quantifier phrase by assigning it type $S/(N \setminus S)$. To obtain an object quantifier phrase it requires another type $(S/N) \setminus S$. But to analyse an example as follows with a medial quantifier phrase would require still another type.

(10) **john+gave+every+book+to+mary** : S

Our treatment on the other hand requires just a single type $(S \uparrow N) \downarrow S$ for all quantifier phrase positions. Lexical insertion for this example yields the following semantically labelled sequent:

(11) $N : j, (N \setminus S)/(N \bullet PP) : \lambda A((gave \pi_2 A) \pi_1 A), ((S \uparrow N) \downarrow S)/CN : \lambda A \lambda B \forall C[(A C) \rightarrow (B C)], CN : book, PP/N : \lambda AA, N : m \Rightarrow S$

This is proved as follows:

$$(12) \frac{\frac{\frac{\frac{N \Rightarrow N \quad PP \Rightarrow PP}{N \Rightarrow N \quad PP/N, N \Rightarrow PP} /L \quad \frac{N \Rightarrow N \quad S \Rightarrow S}{N, N \setminus S \Rightarrow S} \setminus L}{N, PP/N, N \Rightarrow N \bullet PP} \bullet R}{N, (N \setminus S)/(N \bullet PP), N, PP/N, N \Rightarrow S} /L \quad \frac{S \Rightarrow S}{S \Rightarrow S} \downarrow L}{\frac{CN \Rightarrow CN \quad N, (N \setminus S)/(N \bullet PP), (S \uparrow N) \downarrow S, PP/N, N \Rightarrow S}{N, (N \setminus S)/(N \bullet PP), (S \uparrow N) \downarrow S, CN, PP/N, N \Rightarrow S} /L} \uparrow R$$

The semantics is thus:

(13) $\forall C[(book C) \rightarrow (((gave m) C) j)]$

The next example exhibits de re/de dicto ambiguity:

(14) **mary+thinks+someone+left** : S

Mary's thoughts could be specifically directed towards a particular person, or concern a non-specific person. Lexical lookup yields the following:

(15) $N : m, (N \setminus S)/S : thinks, (S \uparrow N) \downarrow S : \lambda A \exists B[(person B) \wedge (A B)], N \setminus S : left \Rightarrow S$

The non-specific derivation and semantics are thus:

$$(16) \frac{\frac{\frac{\overline{N \Rightarrow N} \quad \overline{S \Rightarrow S}}{N, N \setminus S \Rightarrow S} \setminus L}{[\], N \setminus S \Rightarrow S \uparrow N} \uparrow R \quad \frac{\overline{S \Rightarrow S}}{S \Rightarrow S} \downarrow L \quad \frac{\overline{N \Rightarrow N} \quad \overline{S \Rightarrow S}}{N, N \setminus S \Rightarrow S} \setminus L}{\frac{(S \uparrow N) \downarrow S, N \setminus S \Rightarrow S}{N, (N \setminus S) / S, (S \uparrow N) \downarrow S, N \setminus S \Rightarrow S} / L} / L$$

$$(17) ((\text{thinks } \exists B[(\text{person } B) \wedge (\text{left } B)]) m)$$

The specific derivation and semantics are:

$$(18) \frac{\frac{\frac{\overline{N \Rightarrow N} \quad \overline{S \Rightarrow S}}{N, N \setminus S \Rightarrow S} \setminus L \quad \frac{\overline{N \Rightarrow N} \quad \overline{S \Rightarrow S}}{N, N \setminus S \Rightarrow S} \setminus L}{N, (N \setminus S) / S, N, N \setminus S \Rightarrow S} / L}{\frac{N, (N \setminus S) / S, [\], N \setminus S \Rightarrow S \uparrow N} \uparrow R \quad \frac{\overline{S \Rightarrow S}}{S \Rightarrow S} \downarrow L}{N, (N \setminus S) / S, (S \uparrow N) \downarrow S, N \setminus S \Rightarrow S} \downarrow L} / L$$

$$(19) \exists B[(\text{person } B) \wedge ((\text{thinks } (\text{left } B)) m)]$$

Consider the classic example of quantifier scope ambiguity:

$$(20) \text{everyone+loves+someone} : S$$

Lexical lookup yields:

$$(21) (S \uparrow N) \downarrow S : \lambda A \forall B[(\text{person } B) \rightarrow (A B)], (N \setminus S) / N : \text{love}, \\ (S \uparrow N) \downarrow S : \lambda A \exists B[(\text{person } B) \wedge (A B)] \Rightarrow S$$

In the object wide scope analysis the object quantifier phrase is processed first top-down:

$$(22) \frac{\frac{\frac{\overline{N \Rightarrow N} \quad \overline{S \Rightarrow S}}{N, N \setminus S \Rightarrow S} \setminus L}{N \Rightarrow N \quad N, N \setminus S \Rightarrow S} / L}{\frac{N, (N \setminus S) / N, N \Rightarrow S}{[\], (N \setminus S) / N, N \Rightarrow S \uparrow N} \uparrow R \quad \frac{\overline{S \Rightarrow S}}{S \Rightarrow S} \downarrow L}{\frac{(S \uparrow N) \downarrow S, (N \setminus S) / N, N \Rightarrow S}{(S \uparrow N) \downarrow S, (N \setminus S) / N, [\] \Rightarrow S \uparrow N} \uparrow R \quad \frac{\overline{S \Rightarrow S}}{S \Rightarrow S} \downarrow L}{(S \uparrow N) \downarrow S, (N \setminus S) / N, (S \uparrow N) \downarrow S \Rightarrow S} \downarrow L} / L$$

(23) $\exists B[(person B) \wedge \forall E[(person E) \rightarrow ((love B) E)]]$

In the subject wide scope analysis the subject quantifier phrase is processed first top-down:

$$\begin{array}{c}
 \frac{}{N \Rightarrow N} \quad \frac{}{S \Rightarrow S} \\
 \hline
 \frac{}{N, N \setminus S \Rightarrow S} \setminus L \\
 \hline
 \frac{}{N, (N \setminus S) / N, N \Rightarrow S} / L \\
 \hline
 \frac{}{N, (N \setminus S) / N, [] \Rightarrow S \uparrow N} \uparrow R \quad \frac{}{S \Rightarrow S} \\
 \hline
 \frac{}{N, (N \setminus S) / N, (S \uparrow N) \downarrow S \Rightarrow S} \downarrow L \\
 \hline
 \frac{}{[], (N \setminus S) / N, (S \uparrow N) \downarrow S \Rightarrow S \uparrow N} \uparrow R \quad \frac{}{S \Rightarrow S} \\
 \hline
 \frac{}{(S \uparrow N) \downarrow S, (N \setminus S) / N, (S \uparrow N) \downarrow S \Rightarrow S} \downarrow L
 \end{array}$$

(25) $\forall B[(person B) \rightarrow \exists E[(person E) \wedge ((love E) B)]]$

3.3. VP Ellipsis

In VP ellipsis an auxiliary such as *did* receives its interpretation from an antecedent verb phrase:

(26) **john+slept+before+mary+did** : *S*

Lexical lookup for this example yields the following labelled sequent.

$$\begin{array}{l}
 (27) \ N : j, N \setminus S : slept, \\
 \quad ((N \setminus S) \setminus (N \setminus S)) / S : \lambda A \lambda B \lambda C ((before\ A)\ (B\ C)), N : m, \\
 \quad (((N \setminus S) \uparrow (N \setminus S)) / (N \setminus S)) \setminus ((N \setminus S) \uparrow (N \setminus S)) : \lambda A \lambda B ((A\ B)\ B) \\
 \quad \Rightarrow S
 \end{array}$$

This has the proof given in Figure 2. The semantics is:

$$(28) \ ((before\ (slept\ m))\ (slept\ j))$$

By way of a second example consider:

(29) **john+slept+and+mary+did+too** : *S*

Lexical lookup yields:

$$\begin{array}{l}
 (30) \ N : j, N \setminus S : slept, (S \setminus S) / S : \lambda A \lambda B [B \wedge A], N : m, \\
 \quad (((N \setminus S) \uparrow (N \setminus S)) / (N \setminus S)) \setminus ((N \setminus S) \uparrow (N \setminus S)) : \lambda A \lambda B ((A\ B)\ B) \\
 \quad \Rightarrow S
 \end{array}$$

This has the proof given in Figure 3. The semantics is:

$$(31) \ [(slept\ j) \wedge (slept\ m)]$$

$$\begin{array}{c}
\frac{}{N \Rightarrow N \quad S \Rightarrow S} \\
\frac{}{N, N \setminus S \Rightarrow S} \quad \frac{}{N \Rightarrow N \quad S \Rightarrow S} \\
\frac{}{N \setminus S \Rightarrow N \setminus S} \quad \frac{}{N, N \setminus S \Rightarrow S} \\
\frac{}{N, N \setminus S \Rightarrow S} \quad \frac{}{N, N \setminus S, (N \setminus S) \setminus (N \setminus S) \Rightarrow S} \\
\frac{}{N, N \setminus S, ((N \setminus S) \setminus (N \setminus S)) \setminus S, N, N \setminus S \Rightarrow S} \quad \frac{}{N \Rightarrow N \quad S \Rightarrow S} \\
\frac{}{N \setminus S, ((N \setminus S) \setminus (N \setminus S)) \setminus S, N, N \setminus S \Rightarrow N \setminus S} \quad \frac{}{N, N \setminus S \Rightarrow S} \quad \frac{}{N \Rightarrow N \quad S \Rightarrow S} \\
\frac{}{[], ((N \setminus S) \setminus (N \setminus S)) \setminus S, N, N \setminus S \Rightarrow (N \setminus S) \uparrow (N \setminus S)} \quad \frac{}{N \setminus S \Rightarrow N \setminus S} \quad \frac{}{N, N \setminus S \Rightarrow S} \\
\frac{}{[], ((N \setminus S) \setminus (N \setminus S)) \setminus S, N \Rightarrow ((N \setminus S) \uparrow (N \setminus S)) \setminus (N \setminus S)} \quad \frac{}{N, (N \setminus S) \uparrow (N \setminus S) \setminus (N \setminus S) \Rightarrow S} \\
\frac{}{N, N \setminus S, ((N \setminus S) \setminus (N \setminus S)) \setminus S, N, ((N \setminus S) \uparrow (N \setminus S)) \setminus (N \setminus S) \setminus ((N \setminus S) \uparrow (N \setminus S)) \Rightarrow S}
\end{array}$$

Figure 2: *John slept before Mary did*

$$\begin{array}{c}
\frac{}{N \Rightarrow N \quad S \Rightarrow S} \quad \frac{}{S \Rightarrow S \quad S \Rightarrow S} \\
\frac{}{N, N \setminus S \Rightarrow S} \quad \frac{}{S, S \setminus S \Rightarrow S} \\
\frac{}{N \Rightarrow N} \quad \frac{}{S, (S \setminus S) \setminus S, N, N \setminus S \Rightarrow S} \\
\frac{}{N, N \setminus S, (S \setminus S) \setminus S, N, N \setminus S \Rightarrow S} \quad \frac{}{N \Rightarrow N \quad S \Rightarrow S} \\
\frac{}{N \setminus S, (S \setminus S) \setminus S, N, N \setminus S \Rightarrow N \setminus S} \quad \frac{}{N, N \setminus S \Rightarrow S} \quad \frac{}{N \Rightarrow N \quad S \Rightarrow S} \\
\frac{}{[], (S \setminus S) \setminus S, N, N \setminus S \Rightarrow (N \setminus S) \uparrow (N \setminus S)} \quad \frac{}{N \setminus S \Rightarrow N \setminus S} \quad \frac{}{N, N \setminus S \Rightarrow S} \\
\frac{}{[], (S \setminus S) \setminus S, N \Rightarrow ((N \setminus S) \uparrow (N \setminus S)) \setminus (N \setminus S)} \quad \frac{}{N, (N \setminus S) \uparrow (N \setminus S) \setminus (N \setminus S) \Rightarrow S} \\
\frac{}{N, N \setminus S, (S \setminus S) \setminus S, N, ((N \setminus S) \uparrow (N \setminus S)) \setminus (N \setminus S) \setminus ((N \setminus S) \uparrow (N \setminus S)) \Rightarrow S}
\end{array}$$

Figure 3: *John slept and Mary did too*

3.4. Medial Extraction

Lambek categorial grammar can characterize subject relativization with a relative pronoun type $(CN \setminus CN)/(N \setminus S)$ and clause-final object relativization with a relative pronoun type $(CN \setminus CN)/(S/N)$, but neither of these suffice for medial relativization such as the following:

(32) **dog+that+mary+saw+today** : CN

Extraction from all positions is obtained with our displacement calculus type, for which lexical lookup yields:

(33) $CN : dog, (CN \setminus CN)/((S \uparrow N) \circ I) : \lambda A \lambda B \lambda C [(B C) \wedge (\pi_1 A C)],$
 $N : m, (N \setminus S)/N : saw, (N \setminus S) \setminus (N \setminus S) : \lambda A \lambda B (today (A B)) \Rightarrow$
 CN

The proof analysis is:

$$\begin{array}{c}
 \frac{}{N \Rightarrow N} \quad \frac{}{S \Rightarrow S} \\
 \hline
 N, N \setminus S \Rightarrow S \quad \frac{}{N \Rightarrow N} \quad \frac{}{S \Rightarrow S} \\
 \hline
 N \setminus S \Rightarrow N \setminus S \quad \frac{}{N, N \setminus S \Rightarrow S} \\
 \hline
 N, (N \setminus S) \setminus (N, (N \setminus S) \setminus (N \setminus S)) \Rightarrow S \\
 \hline
 N, (N \setminus S) \setminus N, N, (N \setminus S) \setminus (N \setminus S) \Rightarrow S \\
 \hline
 N, (N \setminus S) \setminus N, [1, (N \setminus S) \setminus (N \setminus S)] \Rightarrow S \uparrow N \\
 \hline
 N, (N \setminus S) \setminus N, (N \setminus S) \setminus (N \setminus S) \Rightarrow (S \uparrow N) \circ I \\
 \hline
 CN, (CN \setminus CN) \setminus ((S \uparrow N) \circ I), N, (N \setminus S) \setminus N, (N \setminus S) \setminus (N \setminus S) \Rightarrow CN
 \end{array}$$

This delivers semantics:

(35) $\lambda C [(dog C) \wedge (today ((saw C) m))]$

3.5. Pied-Piping

In pied-piping a relative pronoun is accompanied by further material from the extraction site:

(36) **mountain+the+painting+of+which+by+cezanne+john+sold+for+\$10,000,000** : CN

The type we use for this example subsumes that of the previous subsection since the latter is derivable from the former, so the lexicon only requires the type employed in this lexical lookup:

$$(37) \text{ } CN : \textit{mountain}, N/CN : \iota, CN : \textit{painting}, (CN \setminus CN)/N : \textit{of}, (N \uparrow N) \downarrow ((CN \setminus CN) / ((S \uparrow N) \odot I)) : \lambda A \lambda B \lambda C \lambda D [(C D) \wedge (\pi_1 B (A D))], (CN \setminus CN)/N : \textit{by}, N : \textit{cezanne}, N : j, (N \setminus S) / (N \bullet PP) : \lambda A ((\textit{sold} \pi_2 A) \pi_1 A), PP/N : \lambda AA, N : \textit{tenmilliondollars} \Rightarrow CN$$

The derivation is given in Figure 4. This assigns semantics:

$$(38) \lambda D [(mountain D) \wedge (((\textit{sold} \textit{tenmilliondollars}) (\iota ((\textit{by} \textit{cezanne}) ((\textit{of} D) \textit{painting})))))) j]$$

3.6. Appositive Relativization

In appositive relativization the head modified by the relative clause is a noun phrase and the predication of the body of the relative clause to this head is conjoined to the propositional content of the head interpreted in the embedding sentence. Our example is:

$$(39) \textbf{john+who+jogs+sneezed} : S$$

Lexical lookup yields:

$$(40) N : j, (N \setminus ((S \uparrow N) \downarrow S)) / ((S \uparrow N) \odot I) : \lambda A \lambda B \lambda C [(\pi_1 A B) \wedge (C B)], N \setminus S : \textit{jogs}, N \setminus S : \textit{sneezed} \Rightarrow S$$

The grammaticality proof is:

$$(41) \frac{\frac{\frac{\frac{\frac{\frac{\frac{N \Rightarrow N \quad S \Rightarrow S}{\downarrow L} \quad \frac{\frac{\frac{N \Rightarrow N \quad S \Rightarrow S}{\downarrow L} \quad \frac{N, N \setminus S \Rightarrow S}{\downarrow L}}{\uparrow R}}{[\cdot], N \setminus S \Rightarrow S \uparrow N \quad S \Rightarrow S}{\downarrow L}}{\uparrow R} \quad \frac{N \Rightarrow N \quad (S \uparrow N) \downarrow S, N \setminus S \Rightarrow S}{\downarrow L}}{\uparrow R} \quad \frac{N \setminus S \Rightarrow (S \uparrow N) \odot I \quad N, N \setminus ((S \uparrow N) \downarrow S), N \setminus S \Rightarrow S}{\downarrow L}}{\circ R} \quad \frac{N \setminus S \Rightarrow (S \uparrow N) \odot I \quad N, N \setminus ((S \uparrow N) \downarrow S), N \setminus S \Rightarrow S}{\downarrow L}}{\downarrow L}}{N, (N \setminus ((S \uparrow N) \downarrow S)) / ((S \uparrow N) \odot I), N \setminus S, N \setminus S \Rightarrow S}$$

This yields semantics:

$$(42) [(jogs j) \wedge (\textit{sneezed} j)]$$

$$\begin{array}{c}
\frac{\frac{CN \Rightarrow CN \quad CN \Rightarrow CN}{N \Rightarrow N \quad CN, CN \setminus CN \Rightarrow CN} \setminus L}{CN \Rightarrow CN \quad CN, (CN \setminus CN) / N, N \Rightarrow CN} \setminus L \\
\frac{N \Rightarrow N \quad CN, CN \setminus (CN \setminus CN) / N, N \Rightarrow CN}{CN, (CN \setminus CN) / N, N, (CN \setminus CN) / N, N \Rightarrow CN} \setminus L \\
\frac{N \Rightarrow N \quad CN, (CN \setminus CN) / N, (CN \setminus CN) / N, N \Rightarrow N}{N / CN, CN, (CN \setminus CN) / N, [], (CN \setminus CN) / N, N \Rightarrow N \uparrow R} \uparrow R \\
\frac{N / CN, CN, (CN \setminus CN) / N, N \Rightarrow N \quad N \Rightarrow N}{N / CN, CN, (CN \setminus CN) / N, [], (CN \setminus CN) / N, N \Rightarrow N \uparrow N} \uparrow R \\
\frac{N, (N \setminus S) / (N \bullet PP), N, PP / N, N \Rightarrow S}{N, (N \setminus S) / (N \bullet PP), [], PP / N, N \Rightarrow S \uparrow N} \uparrow R \\
\frac{N, PP / N, N \Rightarrow N \bullet PP \quad N, N \setminus S \Rightarrow S}{N, (N \setminus S) / (N \bullet PP), PP / N, N \Rightarrow (S \uparrow N) \circ I} \circ R \\
\frac{N, (N \setminus S) / (N \bullet PP), PP / N, N \Rightarrow S \quad CN, CN \setminus CN \Rightarrow CN}{CN, (CN \setminus CN) / ((S \uparrow N) \circ I), N, (N \setminus S) / (N \bullet PP), PP / N, N \Rightarrow CN} \setminus L \\
\frac{N, (N \setminus S) / (N \bullet PP), PP / N, N \Rightarrow (S \uparrow N) \circ I \quad CN, CN \setminus CN \Rightarrow CN}{CN, (CN \setminus CN) / ((S \uparrow N) \circ I), N, (N \setminus S) / (N \bullet PP), PP / N, N \Rightarrow CN} \setminus L \\
\frac{N, (N \setminus S) / (N \bullet PP), [], PP / N, N \Rightarrow S \uparrow N \quad \Rightarrow I \quad CN \Rightarrow CN \quad CN \Rightarrow CN}{CN, N / CN, CN, (CN \setminus CN) / N, (N \uparrow N) \downarrow ((CN \setminus CN) / ((S \uparrow N) \circ I)), (CN \setminus CN) / N, N, N, (N \setminus S) / (N \bullet PP), PP / N, N \Rightarrow CN} \setminus L
\end{array}$$

Figure 4: mountain the painting of which by Cezanne John sold for \$10,000,000

3.7. Parentheticals

We make the simplifying assumption that a parenthetical adverbial such as *fortunately* can appear freely. Then our lexical assignment yields the following series of examples and analyses.

(43) **fortunately+john+has+perseverance** : S

(44) $(S \uparrow I) \downarrow S$: $\lambda A(\text{fortunately } (A \ d)), N : j, (N \setminus S) / N : \text{has}, N : \text{perseverance} \Rightarrow S$

$$(45) \frac{\frac{\frac{N \Rightarrow N \quad S \Rightarrow S}{N, N \setminus S \Rightarrow S} \setminus L}{N \Rightarrow N} / L}{\frac{N, (N \setminus S) / N, N \Rightarrow S}{I, N, (N \setminus S) / N, N \Rightarrow S} IL} \uparrow R \quad \frac{S \Rightarrow S}{[], N, (N \setminus S) / N, N \Rightarrow S \uparrow I} \downarrow L$$

(46) $(\text{fortunately } ((\text{has perseverance}) . j))$

(47) **john+fortunately+has+perseverance** : S

(48) $N : j, (S \uparrow I) \downarrow S$: $\lambda A(\text{fortunately } (A \ d)), (N \setminus S) / N : \text{has}, N : \text{perseverance} \Rightarrow S$

$$(49) \frac{\frac{\frac{N \Rightarrow N \quad S \Rightarrow S}{N, N \setminus S \Rightarrow S} \setminus L}{N \Rightarrow N} / L}{\frac{N, (N \setminus S) / N, N \Rightarrow S}{N, I, (N \setminus S) / N, N \Rightarrow S} IL} \uparrow R \quad \frac{S \Rightarrow S}{N, [], (N \setminus S) / N, N \Rightarrow S \uparrow I} \downarrow L$$

(50) $(\text{fortunately } ((\text{has perseverance}) . j))$

(51) **john+has+fortunately+perseverance** : S

(52) $N : j, (N \setminus S) / N : \text{has}, (S \uparrow I) \downarrow S$: $\lambda A(\text{fortunately } (A \ d)), N : \text{perseverance} \Rightarrow S$

$$(53) \frac{\frac{\frac{N \Rightarrow N}{I, N \Rightarrow N} IL \quad \frac{\frac{N \Rightarrow N \quad S \Rightarrow S}{N, N \setminus S \Rightarrow S} \setminus L}{N, (N \setminus S) / N, I, N \Rightarrow S} /L}{\frac{N, (N \setminus S) / N, [\] \Rightarrow S \uparrow I \quad \frac{S \Rightarrow S}{S \Rightarrow S}}{\frac{N, (N \setminus S) / N, (S \uparrow I) \downarrow S, N \Rightarrow S}{N, (N \setminus S) / N, (S \uparrow I) \downarrow S, N \Rightarrow S} \downarrow L} \uparrow R$$

(54) *(fortunately ((has perseverance) j))*

(55) **john+has+perseverance+fortunately** : *S*

(56) *N* : *j, (N \setminus S) / N* : *has, N* : *perseverance, (S \uparrow I) \downarrow S* :
 $\lambda A(\text{fortunately } (A \ d)) \Rightarrow S$

$$(57) \frac{\frac{\frac{\frac{N \Rightarrow N}{N \Rightarrow N} \quad \frac{\frac{S \Rightarrow S}{S, I \Rightarrow S} IL}{N, N \setminus S, I \Rightarrow S} \setminus L}{N, (N \setminus S) / N, N, I \Rightarrow S} /L}{\frac{N, (N \setminus S) / N, N, [\] \Rightarrow S \uparrow I \quad \frac{S \Rightarrow S}{S \Rightarrow S}}{\frac{N, (N \setminus S) / N, N, (S \uparrow I) \downarrow S \Rightarrow S}{N, (N \setminus S) / N, N, (S \uparrow I) \downarrow S \Rightarrow S} \downarrow L} \uparrow R$$

(58) *(fortunately ((has perseverance) j))*

3.8. Gapping

In gapping coordination a verb in the left conjunct is understood in the right conjunct:

(59) **john+studies+logic+and+charles+phonetics** : *S*

Lexical lookup for the gapping coordinator type yields:

(60) *N* : *j, (N \setminus S) / N* : *studies, N* : *logic, ((S \uparrow ((N \setminus S) / N)) \setminus (S \uparrow ((N \setminus S) / N))) / ((S \uparrow ((N \setminus S) / N)) \odot I)* : $\lambda A \lambda B \lambda C [(B \ C) \wedge (\pi_1 A \ C)], N$: *c, N* : *phonetics* $\Rightarrow S$

The derivation is as shown in Figure 5. This yields semantics:

(61) $[(\text{studies logic } j) \wedge (\text{studies phonetics } c)]$

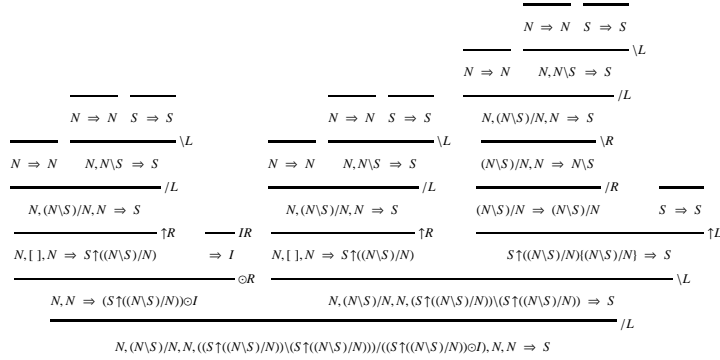


Figure 5: *John studies logic, and Charles, phonetics*

3.9. Comparative Subdeletion

In comparative subdeletion a clause containing a comparative determiner such as *more* is compared to a *than*-clause from which a determiner is missing, with the comparative semantics:

(62) **john+ate+more+donuts+than+mary+bought+bagels : S**

For this example lexical lookup of our assignments yields:

(63) $N : j, (N \setminus S) / N : ate, (S \uparrow(((S \uparrow N) \downarrow S) / CN)) \downarrow (S / ((CP \uparrow$
 $((S \uparrow N) \downarrow S) / CN) \circ I)) : \lambda A \lambda B [| \lambda C (A \lambda D \lambda E [(D C) \wedge$
 $(E C)]] > | \lambda C (\pi_1 B \lambda D \lambda E [(D C) \wedge (E C)])]], CN : donuts,$
 $CP / S : \lambda AA, N : m, (N \setminus S) / N : bought, CN : bagels \Rightarrow S$

A sequent proof derivation is given in Figure 6. This yields semantics:

(64) $[| \lambda C [(donuts C) \wedge ((ate C) j)]] > | \lambda C [(bagels C) \wedge$
 $((bought C) m)]]]$

3.10. Reflexivization

In our example the reflexive receives its interpretation from the subject:

(65) **john+sent+himself+flowers : S**

$$\begin{array}{c}
\frac{\frac{\frac{\frac{N \Rightarrow N \quad S \Rightarrow S}{N \Rightarrow N} \quad \frac{N \Rightarrow N \quad S \Rightarrow S}{N, N \setminus S \Rightarrow S}}{N, (N \setminus S)/N, N \Rightarrow S} /L}{\frac{N, (N \setminus S)/N, [] \Rightarrow S \uparrow N} \uparrow R} \quad \frac{S \Rightarrow S}{S \Rightarrow S}}{CN \Rightarrow CN} \quad \frac{N, (N \setminus S)/N, (S \uparrow N) \downarrow S \Rightarrow S}{N, (N \setminus S)/N, (S \uparrow N) \downarrow S \Rightarrow S} \downarrow L}{\frac{N, (N \setminus S)/N, ((S \uparrow N) \downarrow S)/CN, CN \Rightarrow S}{N, (N \setminus S)/N, ((S \uparrow N) \downarrow S)/CN, CN \Rightarrow S} /L} \quad \frac{CP \Rightarrow CP}{CP \Rightarrow CP} /L \\
\frac{CP/S, N, (N \setminus S)/N, ((S \uparrow N) \downarrow S)/CN, CN \Rightarrow CP}{CP/S, N, (N \setminus S)/N, [], CN \Rightarrow CP \uparrow ((S \uparrow N) \downarrow S)/CN} \uparrow R \quad \frac{IR}{\Rightarrow I} \\
\frac{CP/S, N, (N \setminus S)/N, CN \Rightarrow (CP \uparrow ((S \uparrow N) \downarrow S)/CN) \circ I}{S / ((CP \uparrow ((S \uparrow N) \downarrow S)/CN) \circ I), CP/S, N, (N \setminus S)/N, CN \Rightarrow S} \circ R \quad \frac{S \Rightarrow S}{S \Rightarrow S} /L \\
\frac{N, (N \setminus S)/N, (S \uparrow ((S \uparrow N) \downarrow S)/CN) \downarrow S / ((CP \uparrow ((S \uparrow N) \downarrow S)/CN) \circ I), CN, CP/S, N, (N \setminus S)/N, CN \Rightarrow S}{N, (N \setminus S)/N, (S \uparrow ((S \uparrow N) \downarrow S)/CN) \downarrow S / ((CP \uparrow ((S \uparrow N) \downarrow S)/CN) \circ I), CN, CP/S, N, (N \setminus S)/N, CN \Rightarrow S} \downarrow L
\end{array}$$

Figure 6: John ate more donuts than Mary bought bagels

Lexical lookup yields:

$$(66) \quad N : j, (N \setminus S) / (N \bullet N) : \lambda A ((sent \pi_1 A) \pi_2 A), \\ ((N \setminus S) \uparrow N) \downarrow (N \setminus S) : \lambda A \lambda B ((A B) B), N : flowers \Rightarrow S$$

This has derivation:

$$(67) \quad \frac{\frac{\frac{N \Rightarrow N \quad N \Rightarrow N}{N, N \Rightarrow N \bullet N} \bullet R \quad \frac{\frac{N \Rightarrow N \quad S \Rightarrow S}{N, N \setminus S \Rightarrow S} \setminus L}{N, (N \setminus S) / (N \bullet N), N, N \Rightarrow S} / L}{\frac{\frac{N, (N \setminus S) / (N \bullet N), N, N \Rightarrow S}{(N \setminus S) / (N \bullet N), N, N \Rightarrow N \setminus S} \setminus R \quad \frac{N \Rightarrow N \quad S \Rightarrow S}{N, N \setminus S \Rightarrow S} \setminus L}{(N \setminus S) / (N \bullet N), [], N \Rightarrow (N \setminus S) \uparrow N} \uparrow R}{N, (N \setminus S) / (N \bullet N), ((N \setminus S) \uparrow N) \downarrow (N \setminus S), N \Rightarrow S} \downarrow L$$

This delivers semantics:

$$(68) \quad (((sent j) flowers) j)$$

4. Cut-Elimination

Lambek (1958) proved Cut-elimination for the Lambek calculus **L**. Cut-elimination states that every theorem can be proved without the use of Cut. Lambek's proof is simpler than that of Gentzen for standard logic due to the absence of structural rules. It consists of defining a notion of degree of Cut instances and showing how Cuts in a proof can be successively replaced by Cuts of lower degree until they are removed altogether. Thus Lambek's proof provides an algorithm for transforming proofs into Cut-free counterparts. The Cut-elimination theorem has as corollaries the subformula property and decidability.

Here we prove Cut-elimination for the displacement calculus **D**. Like **L**, **D** contains no structural rules (structural properties are built into the sequent calculus notation) and the Cut-elimination is proved following the same strategy as for **L**.

We define the *weight* $|A|$ of a type A as the number of connectives occurrences (including units) that it contains. The weight $|\Gamma|$ of a configuration is the sum of the weights of the types that occur in it, that is, it is defined recursively as follows:

$$\begin{aligned}
(69) \quad |\Delta| &= 0 \\
|[]| &= 0 \\
|A| &= |A| \\
|A\{\Gamma_1 : \dots : \Gamma_{i+1}\}| &= |A| + \sum_{j=1}^{i+1} |\Gamma_j| \\
|\Gamma, \Theta| &= |\Gamma| + |\Theta|
\end{aligned}$$

The weight of a hypercontext is defined similarly with a hole having weight zero.

Consider the Cut rule:

$$(70) \quad \frac{\Gamma \Rightarrow A \quad \Delta\langle \vec{A} \rangle \Rightarrow B}{\Delta\langle \Gamma \rangle \Rightarrow B} \text{Cut } (\star)$$

We define the *degree* $d(\star)$ of an instance \star of the Cut rule as follows:

$$(71) \quad d(\star) = |\Gamma| + |\Delta| + |A| + |B|$$

We call the type A in (70) the *Cut formula*. We call the type which is newly created by a logical rule the *active formula*. Consider a proof which is not Cut-free. Then there is some Cut-instance above which there are no Cuts. We will show that this Cut can either be removed or replaced by one or two Cuts of lower degree. The following three cases are exhaustive:

- (72) • A premise of the Cut is the identity axiom: then the conclusion is identical to the other premise and the Cut as a whole can be removed.
- Both the premises are conclusions of logical rules and it is not the case that the Cut formula is the active formula of both premises: then we apply *permutation conversion* cases.
 - Both the premises are conclusions of logical rules and the Cut formula is the active formula of both premises: then we apply *principal Cut* cases.

There are several cases to consider. We give representative examples.

4.1. Permutation conversion cases

4.1.1. The active formula in the left premise of the Cut rule is not the Cut formula

- The rule applying at the left premise of the Cut rule is $\odot_i L$:

$$\frac{\frac{\Delta\langle\vec{B}|_i\vec{C}\rangle \Rightarrow A}{\Delta\langle B\odot_i\vec{C}\rangle \Rightarrow A} \odot_i L \quad \Gamma\langle\vec{A}\rangle \Rightarrow D}{\Gamma\langle\Delta\langle B\odot_i\vec{C}\rangle\rangle \Rightarrow D} Cut \quad \rightsquigarrow$$

$$\frac{\Delta\langle\vec{B}|_i\vec{C}\rangle \Rightarrow A \quad \Gamma\langle\vec{A}\rangle \Rightarrow D}{\Gamma\langle\Delta\langle\vec{B}|_i\vec{C}\rangle\rangle \Rightarrow A} Cut \quad \odot_i L}{\Gamma\langle\Delta\langle B\odot_i\vec{C}\rangle\rangle \Rightarrow D}$$

- The rule applying at the left premise of the Cut rule is $\uparrow_i L$:

$$\frac{\frac{\Gamma\langle\vec{C}\rangle \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma\langle C\uparrow_i\vec{B}|_i\Delta\rangle \Rightarrow A} \uparrow_i L \quad \Theta\langle\vec{A}\rangle \Rightarrow D}{\Theta\langle\Gamma\langle C\uparrow_i\vec{B}|_i\Delta\rangle\rangle \Rightarrow D} Cut \quad \rightsquigarrow$$

$$\frac{\Gamma\langle\vec{C}\rangle \Rightarrow A \quad \Theta\langle\vec{A}\rangle \Rightarrow D}{\Theta\langle\Gamma\langle\vec{C}\rangle\rangle \Rightarrow D} Cut \quad \Delta \Rightarrow B}{\Theta\langle\Gamma\langle C\uparrow_i\vec{B}|_i\Delta\rangle\rangle \Rightarrow D} \uparrow_i L$$

- The rule applying at the left premise of the Cut rule is JL :

$$\frac{\frac{\Gamma\langle[]\rangle \Rightarrow A}{\Gamma\langle\vec{J}\rangle \Rightarrow A} JL \quad \Delta\langle\vec{A}\rangle \Rightarrow B}{\Delta\langle\Gamma\langle\vec{J}\rangle\rangle \Rightarrow B} Cut \quad \rightsquigarrow$$

$$\frac{\Gamma\langle[]\rangle \Rightarrow A \quad \Delta\langle\vec{A}\rangle \Rightarrow B}{\Delta\langle\Gamma\langle[]\rangle\rangle \Rightarrow A} Cut \quad JL}{\Delta\langle\Gamma\langle\vec{J}\rangle\rangle \Rightarrow B}$$

4.1.2. *The active formula in the right premise of the Cut rule is not the Cut formula*

- The rule applying at the right premise of the Cut rule is $\uparrow_i L$:

$$\frac{\Delta \Rightarrow A \quad \frac{\Gamma\langle \vec{A}; \vec{C} \rangle \Rightarrow D \quad \Theta \Rightarrow B}{\Gamma\langle \vec{A}; C\uparrow_i \vec{B}|_i \Theta \rangle \Rightarrow D} \uparrow_i L}{\Gamma\langle \Delta; \vec{C}\uparrow_i \vec{B}|_i \Theta \rangle \Rightarrow D} Cut$$

\rightsquigarrow

$$\frac{\Delta \Rightarrow A \quad \frac{\Gamma\langle \vec{A}; \vec{C} \rangle \Rightarrow D}{\Gamma\langle \Delta; \vec{C} \rangle \Rightarrow D} Cut \quad \Theta \Rightarrow B}{\Gamma\langle \Delta; C\uparrow_i \vec{B}|_i \Theta \rangle \Rightarrow D} \uparrow_i L$$

- The rule applying at the right premise of the Cut rule is $\uparrow_i R$:

$$\frac{\Delta \Rightarrow A \quad \frac{\Gamma\langle \vec{A} \rangle|_i \vec{B} \Rightarrow C}{\Gamma\langle \vec{A} \rangle \Rightarrow C\uparrow_i B} \uparrow_i R}{\Gamma\langle \Delta \rangle \Rightarrow C\uparrow_i B} Cut$$

\rightsquigarrow

$$\frac{\Delta \Rightarrow A \quad \frac{\Gamma\langle \vec{A} \rangle|_i \vec{B} \Rightarrow C}{\Gamma\langle \Delta \rangle|_i \vec{B} \Rightarrow C} Cut}{\Gamma\langle \Delta \rangle \Rightarrow C\uparrow_i B} \uparrow_i R$$

- The rule applying at the right premise of the Cut rule is $\odot_i L$:

$$\frac{\Delta \Rightarrow A \quad \frac{\Gamma\langle \vec{A}; \vec{B}|_i \vec{C} \rangle \Rightarrow D}{\Gamma\langle \vec{A}; B\odot_i \vec{C} \rangle \Rightarrow D} \odot_i L}{\Gamma\langle \Delta; B\odot_i \vec{C} \rangle \Rightarrow D} Cut$$

\rightsquigarrow

$$\frac{\Delta \Rightarrow A \quad \frac{\Gamma\langle \vec{A}; \vec{B}|_i \vec{C} \rangle \Rightarrow D}{\Gamma\langle \Delta; \vec{B}|_i \vec{C} \rangle \Rightarrow D} Cut}{\Gamma\langle \Delta; B\odot_i \vec{C} \rangle \Rightarrow D} \odot_i L$$

- The rule applying at the right premise of the Cut rule is $\odot_i R$:

$$\frac{\Delta \Rightarrow A \quad \frac{\Gamma \langle \vec{A} \rangle \Rightarrow B \quad \Theta \Rightarrow C}{\Gamma \langle \vec{A} \rangle |_i \Theta \Rightarrow B \odot_i C} \odot_i R}{\Gamma \langle \Delta \rangle |_i \Theta \Rightarrow B \odot_i C} Cut$$

\sim

$$\frac{\Delta \Rightarrow A \quad \frac{\Gamma \langle \vec{A} \rangle \Rightarrow B}{\Gamma \langle \Delta \rangle \Rightarrow B} Cut \quad \Theta \Rightarrow C}{\Gamma \langle \Delta \rangle |_i \Theta \Rightarrow B \odot_i C} \odot_i R$$

4.2. Principal Cut cases

- The rules applying at the left and right premises of the Cut rule are respectively $\odot_i R$ and $\odot_i L$:

$$\frac{\Delta \Rightarrow A \quad \frac{\Gamma \Rightarrow B}{\Delta |_i \Gamma \Rightarrow A \odot_i B} \odot_i R \quad \frac{\Theta \langle \vec{A} |_i \vec{B} \rangle \Rightarrow C}{\Theta \langle \vec{A} \odot_i \vec{B} \rangle \Rightarrow C} \odot_i L}{\Theta \langle \Delta |_i \Gamma \rangle \Rightarrow C} Cut$$

\sim

$$\frac{\Delta \Rightarrow A \quad \frac{\Theta \langle \vec{A} |_i \vec{B} \rangle \Rightarrow C}{\Theta \langle \Delta |_i \vec{B} \rangle \Rightarrow C} Cut}{\Gamma \Rightarrow B \quad \frac{\Theta \langle \Delta |_i \vec{B} \rangle \Rightarrow C}{\Theta \langle \Delta |_i \Gamma \rangle \Rightarrow C} Cut} Cut$$

- The rules applying at the left and right premises of the Cut rule are respectively $\uparrow_i R$ and $\uparrow_i L$:

$$\frac{\frac{\Delta |_i \vec{A} \Rightarrow B}{\Delta \Rightarrow B \uparrow_i A} \uparrow_i R \quad \frac{\Gamma \Rightarrow A \quad \Theta \langle \vec{B} \rangle \Rightarrow C}{\Theta \langle B \uparrow_i A |_i \Gamma \rangle \Rightarrow C} \uparrow_i L}{\Theta \langle \Delta |_i \Gamma \rangle \Rightarrow C} Cut$$

\sim

$$\frac{\Delta \Rightarrow A \quad \frac{\Delta |_i \vec{A} \Rightarrow B \quad \Theta \langle \vec{B} \rangle \Rightarrow C}{\Theta \langle \vec{A} |_i \Gamma \rangle \Rightarrow C} Cut}{\Theta \langle \Delta |_i \Gamma \rangle \Rightarrow C} Cut$$

- The rules applying at the left and right premises of the Cut rule are respectively *IR* and *IL*:

$$\frac{\frac{}{\Lambda \Rightarrow I} IR \quad \frac{\Delta\langle \Lambda \rangle \Rightarrow A}{\Delta\langle I \rangle \Rightarrow A} IL}{\Delta\langle \Lambda \rangle \Rightarrow A} Cut$$

~

$$\Delta\langle \Lambda \rangle \Rightarrow A$$

- The rules applying at the left and right premises of the Cut rule are respectively *JR* and *JL*:

$$\frac{\frac{}{[] \Rightarrow J} JR \quad \frac{\Delta\langle [] \rangle \Rightarrow A}{\Delta\langle \vec{J} \rangle \Rightarrow A} JL}{\Delta\langle [] \rangle \Rightarrow A} Cut$$

~

$$\Delta\langle [] \rangle \Rightarrow A$$

5. Conclusion

The reasoning given in the previous section yields the following properties:

(73) **Theorem** (*Cut-elimination for \mathbf{D}*).

Every theorem of the displacement calculus \mathbf{D} has a Cut-free proof.

Proof. As we have indicated, in every proof which is not Cut-free it is always possible to replace a Cut above which there are no Cuts either by replacing it by one or two Cuts of lower degree or by removing it altogether, conserving the endsequent of the proof. Since the degree of a Cut is always finite and non-negative, repeated application of this procedure will transform every proof into a Cut-free counterpart. \square

(74) **Corollary** (*Subformula property for \mathbf{D}*).

Every theorem of the displacement calculus \mathbf{D} has a proof in which appear only subformulas of the theorem.

Proof. In every rule except Cut every formula in a premise is a subformula of a formula in the conclusion, and Cut itself is eliminable. Hence, every theorem has a proof containing only subformulas of the theorem, namely any one of its Cut-free proofs. \square

(75) **Corollary** (*Decidability of \mathbf{D}*).

It is decidable whether a (hyper)sequent of \mathbf{D} is a theorem.

Proof. In backward chaining Cut-free hypersequent proof search a hypersequent can be matched against a rule only in a finite number of ways and generates only a finite number of subgoals. Hence the backward chaining Cut-free hypersequent proof search space is finite and it is determined in finite time whether a sequent is a theorem. \square

This paper offers an account of generalized discontinuity in the sense anticipated in Morrill and Merenciano (1996) in respect of sorts and in Morrill (2002) in respect of unboundedly many positions of discontinuity. All the applications of Section 3 fall within the fragment with just one point of discontinuity but the full calculus allows arbitrarily many such points.² The program of generalizing categorial grammar in this way goes back to Moortgat (1988) and Bach (1981).

Logically, we have generalized and extended the concatenative multiplicative connectives of Lambek calculus/intuitionistic non-commutative linear logic with families of non-concatenative multiplicative connectives, but concatenation remains the unique primitive mode of composition and the calculus remains free of structural rules. These features contribute to the simplicity of implementation of displacement calculus parsing-as-deduction.

²The sequent notation here employs an improvement over that of Morrill *et al.* (2007) following a suggestion by Sylvain Salvati (p.c.).

Works Cited

1. Bach, Emmon. 1981. Discontinuous constituents in generalized categorial grammars. In *Proceedings of the 11th Annual Meeting of the North Eastern Linguistics Society, New York*, edited by V.A. Burke and J. Pustejovsky, 1–12. Amherst, Massachusetts: GLSA Publications, Department of Linguistics, University of Massachusetts at Amherst.
2. Lambek, Joachim. 1958. The mathematics of sentence structure. *American Mathematical Monthly* 65: 154–170. Reprinted in Buszkowski, Wojciech, Wojciech Marciszewski, and Johan van Benthem, editors, 1988, *Categorial Grammar*, Linguistic & Literary Studies in Eastern Europe volume 25, John Benjamins, Amsterdam, 153–172.
3. Moortgat, Michael. 1988. *Categorial Investigations: Logical and Linguistic Aspects of the Lambek Calculus*. Foris, Dordrecht. PhD thesis, Universiteit van Amsterdam.
4. —. 1997. Categorial Type Logics. In *Handbook of Logic and Language*, edited by Johan van Benthem and Alice ter Meulen, 93–177. Amsterdam and Cambridge, Massachusetts: Elsevier Science B.V. and The MIT Press.
5. Morrill, Glyn. 2002. Towards Generalised Discontinuity. In *Proceedings of the 7th Conference on Formal Grammar*, edited by Gerhard Jäger, Paula Monachesi, Gerald Penn, and Shuly Wintner, 103–111. Trento: ESSLLI.
6. Morrill, Glyn, Mario Fadda, and Oriol Valentín. 2007. Non-deterministic Discontinuous Lambek Calculus. In *Proceedings of the Seventh International Workshop on Computational Semantics, IWCS-7*, edited by Jeroen Geertzen, Elias Thijsse, Harry Bunt, and Amanda Schiffrin, 129–141. Tilburg University.
7. Morrill, Glyn and Josep-Maria Merenciano. 1996. Generalising Discontinuity. *traitement automatique des langues* 37(2): 119–143.
8. Morrill, Glyn V. 2010. *Categorial Grammar: Logical Syntax, Semantics, and Processing*. Oxford University Press.