Mathematical Logic and Linguistics

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BGSMath Course
Class 9

Multiplicative-Additive Focusing for Parsing as Deduction

Multiplicative-Additive Focusing for Parsing as Deduction

- Displacement calculus with additives **DA**

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- Displacement calculus with additives **DA**
- Parsing as deduction

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- Displacement calculus with additives **DA**
- Parsing as deduction
- Spurious ambiguity

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- Parsing as deduction
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- Focusing

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Displacement calculus **D** is an intuitionistic sublinear logic which is a conservative extension of Lambek calculus **L**, but **D** is not *just* an extension of **L** because it requires a whole new machinery to deal with semantically unitary expressions which are separated in more than one piece.

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DA is displacement calculus extended with additives (for polymorphism).

Parsing as deduction

Parsing as deduction

In type logical categorial grammar, grammaticality is reduced to theoremhood in a sublinear logic: a string is grammatical if and only if an associated sequent is a theorem. It follows that parsing is deduction.

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For example

$$
\dfrac{CN \Rightarrow CN \qquad \dfrac{\dfrac{\dfrac{N \Rightarrow N \qquad \dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\,\backslash L}{N, (N\backslash S)/N, N \Rightarrow S}\,/L}{(N\backslash S)/N, N \Rightarrow N\backslash S}\,\backslash R \qquad S \Rightarrow S}{S/(N\backslash S), (N\backslash S)/N, N \Rightarrow S}\,/L}{(S/(N\backslash S))/CN, CN, (N\backslash S)/N, N \Rightarrow S}\,/L
$$

$$
\dfrac{N \Rightarrow N \qquad \dfrac{CN \Rightarrow CN \qquad \dfrac{\dfrac{\dfrac{N \Rightarrow N \qquad S \Rightarrow S}{N, N\backslash S \Rightarrow S}\,\backslash L}{N\backslash S \Rightarrow N\backslash S}\,\backslash R \qquad S \Rightarrow S}{S/(N\backslash S), N\backslash S \Rightarrow S}\,/L}{(S/(N\backslash S))/CN, CN, N\backslash S \Rightarrow S}\,/L}{(S/(N\backslash S))/CN, CN, (N\backslash S)/N, N \Rightarrow S}\,/L
$$

Outline

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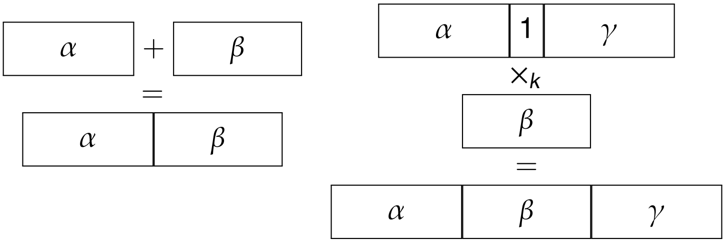
- Definition of **DA**
- Definition of $\mathbf{DA_{foc}}$ (weakly focalised) and $\mathbf{DA_{Foc}}$ (strongly focalised) displacement calculus with additives
- Structure of the reasoning of completeness
- Proof of completeness

Definition of **DA**

Displacement calculus is a logic of discontinuous strings — strings punctuated by a *separator* 1 and subject to operations of append and plug:



append $+ : L_i, L_j \to L_{i+j}$ plug $\times_k : L_{i+1}, L_j \to L_{i+j}$

Definition of types of DA

$$
\begin{array}{rcll}
\mathcal{F}_i & ::= & \mathcal{F}_{i+j}/\mathcal{F}_j & \\
\mathcal{F}_j & ::= & \mathcal{F}_i\backslash\mathcal{F}_{i+j} & \\
\mathcal{F}_{i+j} & ::= & \mathcal{F}_i\bullet\mathcal{F}_j & \\
\mathcal{F}_0 & ::= & I & \\
\mathcal{F}_{i+1} & ::= & \mathcal{F}_{i+j}\uparrow_k\mathcal{F}_j & 1 \leq k \leq i+1 \\
\mathcal{F}_j & ::= & \mathcal{F}_{i+1}\downarrow_k\mathcal{F}_{i+j} & 1 \leq k \leq i+1 \\
\mathcal{F}_{i+j} & ::= & \mathcal{F}_{i+1}\odot_k\mathcal{F}_j & 1 \leq k \leq i+1 \\
\mathcal{F}_1 & ::= & J & \\
\mathcal{F}_i & ::= & \mathcal{F}_i\&\mathcal{F}_i & \\
\mathcal{F}_i & ::= & \mathcal{F}_i\oplus\mathcal{F}_i &
\end{array}
$$

Sort $s(A)$ = the i s.t. $A \in \mathcal{F}_i$
For example, where $s(N) = s(S) = 0$,
$s((S\uparrow_1 N)\uparrow_2 N) = s((S\uparrow_1 N)\uparrow_1 N) = 2$

Interpretation of multiplicative types

$$
\begin{array}{rcl}
[[C/B]] & = & \{s_1 \mid \forall s_2 \in [[B]], s_1{+}s_2 \in [[C]]\} \\
[[A\backslash C]] & = & \{s_2 \mid \forall s_1 \in [[A]], s_1{+}s_2 \in [[C]]\} \\
[[A{\bullet}B]] & = & \{s_1{+}s_2 \mid s_1 \in [[A]] \;\&\; s_2 \in [[B]]\} \\
[[I]] & = & \{0\}
\end{array}
$$

$$
\begin{array}{rcl}
[[C{\uparrow_k}B]] & = & \{s_1 \mid \forall s_2 \in [[B]], s_1{\times_k}s_2 \in [[C]]\} \\
[[A{\downarrow_k}C]] & = & \{s_2 \mid \forall s_1 \in [[A]], s_1{\times_k}s_2 \in [[C]]\} \\
[[A{\odot_k}B]] & = & \{s_1{\times_k}s_2 \mid s_1 \in [[A]] \;\&\; s_2 \in [[B]]\} \\
[[J]] & = & \{1\}
\end{array}
$$

Definition of configurations and sequents of **DA**

$$\begin{array}{llll} \text{Configurations} & \mathcal{O} & ::= & \Lambda \mid \mathcal{T}, \mathcal{O} \\ & \mathcal{T} & ::= & 1 \mid \mathcal{F}_0 \mid \mathcal{F}_{i>0}\{\underbrace{\mathcal{O} : \ldots : \mathcal{O}}_{i\,\mathcal{O}\text{'s}}\} \end{array}$$

For example, there is the configuration
$(S{\uparrow_1}N){\uparrow_2}N\{N, 1 : S{\uparrow_1}N\{S\}\}, 1, N, 1$

Where a type A of sort $i > 0$ includes
$\alpha_0{+}1{+}\alpha_1{+}\cdots{+}\alpha_{i-1}{+}1{+}\alpha_i$ and $\beta_1 \in \Delta_1, \ldots, \beta_i \in \Delta_i$,
$A\{\Delta_1 : \ldots : \Delta_i\}$ contains $\alpha_0{+}\beta_1{+}\alpha_1{+}\cdots{+}\alpha_{i-1}{+}\beta_i{+}\alpha_i$.

Sort $s(\mathcal{O}) = |\mathcal{O}|_1$

For example $s((S{\uparrow_1}N){\uparrow_2}N\{N, 1 : S{\uparrow_1}N\{S\}\}, 1, N, 1) = 3$

Sequents $\Sigma ::= \mathcal{O} \Rightarrow A$ s.t. $s(\mathcal{O}) = s(A)$

Figure of a type

The figure $\overrightarrow{A}$ of a type A is defined by:

$$\overrightarrow{A} = \begin{cases} A & \text{if } s(A) = 0 \\ A\{\underbrace{1 : \ldots : 1}_{s(A)\ 1\text{'s}}\} & \text{if } s(A) > 0 \end{cases}$$

Fold

Where Γ is a configuration of sort i and $\Delta_1, \ldots, \Delta_i$ are configurations, the *fold* $\Gamma \otimes \langle \Delta_1 : \ldots : \Delta_i \rangle$ is the result of replacing the successive 1's in Γ by $\Delta_1, \ldots, \Delta_i$ respectively.

Metalinguistic wrap

Where Δ is a configuration of sort $i > 0$ and Γ is a configuration, the kth metalinguistic wrap $\Delta\,|_k\,\Gamma$, $1 \leq k \leq i$, is given by

$$\Delta\,|_k\,\Gamma =_{df} \Delta \otimes \langle \underbrace{1 : \ldots : 1}_{k-1\ 1\text{'s}} : \Gamma : \underbrace{1 : \ldots : 1}_{i-k\ 1\text{'s}} \rangle$$

I.e. $\Delta\,|_k\,\Gamma$ is the configuration resulting from replacing by Γ the kth separator in Δ.

Rules of **DA**

The rules of the displacement calculus with additives are as follows, where $\Delta\langle\Gamma\rangle$ abbreviates $\Delta_0(\Gamma \otimes \langle\Delta_1 : \ldots : \Delta_i\rangle)$:

$$\frac{}{P \Rightarrow P}\,\text{id} \qquad \frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{A}\rangle \Rightarrow B}{\Delta\langle\Gamma\rangle \Rightarrow B}\,\text{Cut}$$

$$\frac{\Gamma \Rightarrow B \qquad \Delta\langle\overrightarrow{C}\rangle \Rightarrow D}{\Delta\langle\overrightarrow{C/B}, \Gamma\rangle \Rightarrow D}\,/L \qquad \frac{\Gamma, \overrightarrow{B} \Rightarrow C}{\Gamma \Rightarrow C/B}\,/R$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{C}\rangle \Rightarrow D}{\Delta\langle\Gamma, \overrightarrow{A\backslash C}\rangle \Rightarrow D}\,\backslash L \qquad \frac{\overrightarrow{A}, \Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C}\,\backslash R$$

$$\frac{\Delta\langle\overrightarrow{A}, \overrightarrow{B}\rangle \Rightarrow D}{\Delta\langle\overrightarrow{A\bullet B}\rangle \Rightarrow D}\,\bullet L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A\bullet B}\,\bullet R$$

$$\frac{\Delta\langle\Lambda\rangle \Rightarrow A}{\Delta\langle\overrightarrow{I}\rangle \Rightarrow A}\,IL \qquad \frac{}{\Lambda \Rightarrow I}\,IR$$

$$\frac{\Gamma \Rightarrow B \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{C{\uparrow_k}B}\,|_k\,\Gamma \rangle \Rightarrow D}\,{\uparrow_k}L \qquad \frac{\Gamma\,|_k\,\overrightarrow{B} \Rightarrow C}{\Gamma \Rightarrow C{\uparrow_k}B}\,{\uparrow_k}R$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \Gamma\,|_k\,\overrightarrow{A{\downarrow_k}C} \rangle \Rightarrow D}\,{\downarrow_k}L \qquad \frac{\overrightarrow{A}\,|_k\,\Gamma \Rightarrow C}{\Gamma \Rightarrow A{\downarrow_k}C}\,{\downarrow_k}R$$

$$\frac{\Delta\langle \overrightarrow{A}\,|_k\,\overrightarrow{B} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{A{\odot_k}B} \rangle \Rightarrow D}\,{\odot_k}L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1\,|_k\,\Gamma_2 \Rightarrow A{\odot_k}B}\,{\odot_k}R$$

$$\frac{\Delta\langle 1 \rangle \Rightarrow A}{\Delta\langle \overrightarrow{J} \rangle \Rightarrow A}\,JL \qquad \frac{}{1 \Rightarrow J}\,JR$$

$$\frac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\&B}\rangle \Rightarrow C}\,\&L_1 \qquad \frac{\Gamma\langle\overrightarrow{B}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\&B}\rangle \Rightarrow C}\,\&L_2$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A\&B}\,\&R$$

$$\frac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow C \qquad \Gamma\langle\overrightarrow{B}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\oplus B}\rangle \Rightarrow C}\,\oplus L$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A\oplus B}\,\oplus R_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A\oplus B}\,\oplus R_2$$

Definition of weakly focalised **DA**, $\mathbf{DA_{foc}}$

Where atomic types are partitioned into those with positive (At^+) and negative (At^-) bias, situated (input $^\bullet$/output $^\circ$) polar types are classified into positive and negative according as their rules is not or is invertible respectively as follows:

$$\begin{array}{llll}
\text{Pos. out., neg. in.} & P, M & ::= & \mathrm{At}^+ \mid A{\bullet}B \mid I \mid A{\odot_k}B \mid J \mid A{\oplus}B \\
\text{Pos. in., neg. out.} & Q, N & ::= & \mathrm{At}^- \mid C/B \mid A\backslash C \mid C{\uparrow_k}B \mid A{\downarrow_k}C \mid A\&B
\end{array}$$

Rules of **DA**$_{\mathbf{foc}}$

A sequent is either unfocused and as before, or else focused and has exactly one type boxed.

$$\frac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w A}{\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow_w A}\, foc \qquad \frac{\Delta \Longrightarrow_w \boxed{P}}{\Delta \Longrightarrow_w P}\, foc$$

Identity group

$$\frac{}{\overrightarrow{P} \Longrightarrow_w \boxed{\mathrm{P}}} \text{ id, if } P \in \mathrm{At}^+ \qquad \frac{}{\boxed{\overrightarrow{\mathrm{Q}}} \Longrightarrow_w Q} \text{ id, if } Q \in \mathrm{At}^-$$

$$\frac{\Gamma \Longrightarrow_w \boxed{P} \qquad \Delta\langle \overrightarrow{P} \rangle \Longrightarrow_w C \diamond \text{foc}}{\Delta\langle \Gamma \rangle \Longrightarrow_w C \diamond \text{foc}} \textit{p-Cut}_1 \qquad \frac{\Gamma \Longrightarrow_w N \diamond \text{foc} \qquad \Delta\langle \overrightarrow{\boxed{N}} \rangle \Longrightarrow_w C}{\Delta\langle \Gamma \rangle \Longrightarrow_w C \diamond \text{foc}} \textit{p-Cut}_2$$

$$\frac{\Gamma \Longrightarrow_w P \diamond \text{foc} \qquad \Delta\langle \overrightarrow{P} \rangle \Longrightarrow_w C}{\Delta\langle \Gamma \rangle \Longrightarrow_w C \diamond \text{foc}} \textit{n-Cut}_1 \qquad \frac{\Gamma \Longrightarrow_w N \qquad \Delta\langle \overrightarrow{N} \rangle \Longrightarrow_w C \diamond \text{foc}}{\Delta\langle \Gamma \rangle \Longrightarrow_w C \diamond \text{foc}} \textit{n-Cut}_2$$

Logical rules of $\mathbf{DA}_{\mathbf{foc}}$

The focalised logical rules are as follows including Curry-Howard categorial semantic labelling.

Asynchronous multiplicative rules

$$\frac{\overrightarrow{A}: x, \Gamma \Rightarrow C: \chi}{\Gamma \Rightarrow A\backslash C: \lambda x\chi}\backslash R \qquad \frac{\Gamma, \overrightarrow{B}: y \Rightarrow C: \chi}{\Gamma \Rightarrow C/B: \lambda y\chi}/R$$

$$\frac{\Delta\langle\overrightarrow{A}: x, \overrightarrow{B}: y\rangle \Rightarrow D: \omega}{\Delta\langle\overrightarrow{A\bullet B}: z\rangle \Rightarrow D: \omega\{\pi_1 z/x, \pi_2 z/y\}}\bullet L \qquad \frac{\Delta\langle\Lambda\rangle \Rightarrow A: \phi}{\Delta\langle\overrightarrow{I}: x\rangle \Rightarrow A: \phi}IL$$

$$\frac{\overrightarrow{A}: x \,|_k\, \Gamma \Rightarrow C: \chi}{\Gamma \Rightarrow A \,|_k\, C: \lambda x\chi}\downarrow_k R \qquad \frac{\Gamma \,|_k\, \overrightarrow{B}: y \Rightarrow C: \chi}{\Gamma \Rightarrow C\uparrow_k B: \lambda y\chi}\uparrow_k R$$

$$\frac{\Delta\langle\overrightarrow{A}: x \,|_k\, \overrightarrow{B}: y\rangle \Rightarrow D: \omega}{\Delta\langle\overrightarrow{A\odot_k B}: z\rangle \Rightarrow D: \omega\{\pi_1 z/x, \pi_2 z/y\}}\odot_k L \qquad \frac{\Delta\langle 1\rangle \Rightarrow A: \phi}{\Delta\langle\overrightarrow{J}: x\rangle \Rightarrow A: \phi}JL$$

Asynchronous additive rules

$$\frac{\Gamma \Rightarrow A\colon \phi \qquad \Gamma \Rightarrow B\colon \psi}{\Gamma \Rightarrow A\&B\colon (\phi, \psi)}\,\&R$$

$$\frac{\Gamma\langle \overrightarrow{A}\colon x\rangle \Rightarrow C\colon \chi_1 \qquad \Gamma\langle \overrightarrow{B}\colon y\rangle \Rightarrow C\colon \chi_2}{\Gamma\langle \overrightarrow{A\oplus B}\colon z\rangle \Rightarrow C\colon z \to x.\chi_1; y.\chi_2}\,\oplus L$$

Left synchronous continuous multiplicative rules

$$\frac{\Gamma \Rightarrow \boxed{P}: \phi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma, \overrightarrow{\boxed{P\backslash Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}} \backslash L \qquad \frac{\Gamma \Rightarrow \boxed{P}: \phi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma, \overrightarrow{\boxed{P\backslash M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}} \backslash L$$

$$\frac{\Gamma \Rightarrow N: \phi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma, \overrightarrow{\boxed{N\backslash Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}} \backslash L \qquad \frac{\Gamma \Rightarrow N: \phi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma, \overrightarrow{\boxed{N\backslash M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}} \backslash L$$

$$\frac{\Gamma \Rightarrow \boxed{P}: \psi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{Q/P}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}} /L \qquad \frac{\Gamma \Rightarrow N: \psi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{Q/N}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}} /L$$

$$\frac{\Gamma \Rightarrow \boxed{P}: \psi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{M/P}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}} /L \qquad \frac{\Gamma \Rightarrow M: \psi \qquad \Delta\langle \overrightarrow{N}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{N/M}}: x, \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}} /L$$

Left synchronous discontinuous multiplicative rules

$$\frac{\Gamma \Rightarrow \boxed{P}: \phi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma |_k \overrightarrow{\boxed{P{\downarrow_k}Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}} \downarrow_k L$$

$$\frac{\Gamma \Rightarrow \boxed{P}: \phi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma |_k \overrightarrow{\boxed{P{\downarrow_k}M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}} \downarrow_k L$$

$$\frac{\Gamma \Rightarrow N: \phi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma |_k \overrightarrow{\boxed{N{\downarrow_k}Q}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}} \downarrow_k L$$

$$\frac{\Gamma \Rightarrow N: \phi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \Gamma |_k \overrightarrow{\boxed{N{\downarrow_k}M}}: y\rangle \Rightarrow D: \omega\{(y\ \phi)/z\}} \downarrow_k L$$

$$\frac{\Gamma \Rightarrow \boxed{P}: \psi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{Q{\uparrow_k}P}}: x |_k \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}} \uparrow_k L$$

$$\frac{\Gamma \Rightarrow N: \psi \qquad \Delta\langle \overrightarrow{\boxed{Q}}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{Q{\uparrow_k}N}}: x |_k \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}} \uparrow_k L$$

$$\frac{\Gamma \Rightarrow \boxed{P}: \psi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{M{\uparrow_k}P}}: x |_k \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}} \uparrow_k L$$

$$\frac{\Gamma \Rightarrow N: \psi \qquad \Delta\langle \overrightarrow{M}: z\rangle \Rightarrow D: \omega}{\Delta\langle \overrightarrow{\boxed{M{\uparrow_k}N}}: x |_k \Gamma\rangle \Rightarrow D: \omega\{(x\ \psi)/z\}} \uparrow_k L$$

Left synchronous additive rules

$$\frac{\Gamma\langle \overrightarrow{\boxed{Q}} : x\rangle \Rightarrow C : \chi}{\Gamma\langle \overrightarrow{\boxed{Q\&B}} : z\rangle \Rightarrow C : \chi\{\pi_1 z/x\}}\,\&L_1 \qquad \frac{\Gamma\langle \overrightarrow{M} : x\rangle \Rightarrow C : \chi}{\Gamma\langle \overrightarrow{\boxed{M\&B}} : z\rangle \Rightarrow C : \chi\{\pi_1 z/x\}}\,\&L_1$$

$$\frac{\Gamma\langle \overrightarrow{\boxed{Q}} : y\rangle \Rightarrow C : \chi}{\Gamma\langle \overrightarrow{\boxed{A\&Q}} : z\rangle \Rightarrow C : \chi\{\pi_2 z/y\}}\,\&L_2 \qquad \frac{\Gamma\langle \overrightarrow{M} : y\rangle \Rightarrow C : \chi}{\Gamma\langle \overrightarrow{\boxed{A\&M}} : z\rangle \Rightarrow C : \chi\{\pi_2 z/y\}}\,\&L_2$$

Right synchronous continuous multiplicative rules

$$\frac{\Gamma_1 \Rightarrow \boxed{P_1}: \phi \qquad \Gamma_2 \Rightarrow \boxed{P_2}: \psi}{\Gamma_1, \Gamma_2 \Rightarrow \boxed{P_1 \bullet P_2}: (\phi, \psi)} \bullet R \qquad \frac{\Gamma_1 \Rightarrow \boxed{P}: \phi \qquad \Gamma_2 \Rightarrow N: \psi}{\Gamma_1, \Gamma_2 \Rightarrow \boxed{P \bullet N}: (\phi, \psi)} \bullet R$$

$$\frac{\Gamma_1 \Rightarrow N: \phi \qquad \Gamma_2 \Rightarrow \boxed{P}: \psi}{\Gamma_1, \Gamma_2 \Rightarrow \boxed{N \bullet P}: (\phi, \psi)} \bullet R \qquad \frac{\Gamma_1 \Rightarrow N_1: \phi \qquad \Gamma_2 \Rightarrow N_2: \psi}{\Gamma_1, \Gamma_2 \Rightarrow \boxed{N_1 \bullet N_2}: (\phi, \psi)} \bullet R$$

$$\frac{}{\Lambda \Rightarrow \boxed{I}: 0} IR$$

Right synchronous discontinuous multiplicative rules

$$\frac{\Gamma_1 \Rightarrow \boxed{P_1} : \phi \qquad \Gamma_2 \Rightarrow \boxed{P_2} : \psi}{\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{P_1 \odot_k P_2} : (\phi, \psi)} \odot_k R \qquad \frac{\Gamma_1 \Rightarrow \boxed{P} : \phi \qquad \Gamma_2 \Rightarrow N : \psi}{\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{P \odot_k N} : (\phi, \psi)} \odot_k R$$

$$\frac{\Gamma_1 \Rightarrow N : \phi \qquad \Gamma_2 \Rightarrow \boxed{P} : \psi}{\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{N \odot_k P} : (\phi, \psi)} \odot_k R \qquad \frac{\Gamma_1 \Rightarrow N_1 : \phi \qquad \Gamma_2 \Rightarrow N_2 : \psi}{\Gamma_1 |_k \Gamma_2 \Rightarrow \boxed{N_1 \odot_k N_2} : (\phi, \psi)} \odot_k R$$

$$\frac{}{1 \Rightarrow \boxed{J} : 0} JR$$

Right synchronous additive rules

$$\frac{\Gamma \Rightarrow \boxed{P}: \phi}{\Gamma \Rightarrow \boxed{P \oplus B}: \iota_1 \phi} \oplus R_1 \qquad \frac{\Gamma \Rightarrow N: \phi}{\Gamma \Rightarrow \boxed{N \oplus B}: \iota_1 \phi} \oplus R_1$$

$$\frac{\Gamma \Rightarrow \boxed{P}: \psi}{\Gamma \Rightarrow \boxed{A \oplus P}: \iota_2 \psi} \oplus R_2 \qquad \frac{\Gamma \Rightarrow N: \psi}{\Gamma \Rightarrow \boxed{A \oplus N}: \iota_2 \psi} \oplus R_2$$

Definition of strongly focused DA, $\mathbf{DA_{Foc}}$

$\mathbf{DA_{Foc}}$ has all the same rules as $\mathbf{DA_{foc}}$ but not the Cut rules.

In addition, the sequents of $\mathbf{DA_{Foc}}$ are restricted:

A $\mathbf{DA_{Foc}}$ sequent cannot contain both a focus and a complex negative type.

Consequently, proofs in $\mathbf{DA_{Foc}}$ run in alternating phrases of positive rule application and negative rule application.

Structure of the reasoning of completeness

1. Lemma: Eta-expansion for $\mathbf{DA_{foc}}$. Proof by a simple induction on the structure of types. Corollary: Eta-expansion for **DA**.

2. Theorem: Embedding of **DA** with Cut into $\mathbf{DA_{foc}}$ with Cuts (and Eta). Proof by induction on the length of **DA** proofs.

3. Theorem: Cut-elimination for $\mathbf{DA_{foc}}$. Proof by double induction on the size of the Cut formula and the depth of the top-most Cut. Corollary: Cut-elimination for **DA**.

4. Theorem: embedding of $\mathbf{DA_{foc}}$ (without Cut) into $\mathbf{DA_{Foc}}$.

Embedding of **DA** into $\mathbf{DA_{foc}}$

Theorem 4.1 *For any configuration* Δ *and type* A*, we have that if* $\Delta \Rightarrow A$ *then* $\Delta \Longrightarrow_w A$.

Proof We proceed by induction on the length of **DA** proofs. For the non-axiomatic rules we apply the induction hypothesis (i.h.) for each premise of **DA** rules.

- Identity axiom:

$$\frac{\overrightarrow{P} \Longrightarrow_w \boxed{P}}{\overrightarrow{P} \Longrightarrow_w P}\, foc \qquad \frac{\overrightarrow{\boxed{N}} \Longrightarrow_w N}{\overrightarrow{N} \Longrightarrow_w N}\, foc$$

- Cut rule: just apply *n-Cut*.
- Units

$$\frac{}{\Lambda \Rightarrow I}\, IR \qquad \rightsquigarrow \qquad \frac{\overline{\Lambda \Longrightarrow_w \boxed{I}}\, IR}{\Lambda \Longrightarrow_w I}\, foc$$

$$\frac{}{1 \Rightarrow J}\, JR \qquad \rightsquigarrow \qquad \frac{\overline{1 \Longrightarrow_w \boxed{J}}\, JR}{1 \Longrightarrow_w J}\, foc$$

Left unit rules apply as in the case of **DA.**

- Left discontinuous product: directly translates.
- Right discontinuous product. There are cases $P_1 \odot_k P_2$, $N_1 \odot_k N_2$, $N \odot_k P$ and $P \odot_k N$. We show one example:

$$\frac{\Delta \Rightarrow P \qquad \Gamma \Rightarrow N}{\Delta \,|_k\, \Gamma \Rightarrow P \odot_k N}\,\odot_k R \qquad \rightsquigarrow$$

$$\dfrac{\Gamma \Longrightarrow_w N \qquad \dfrac{\Delta \Longrightarrow_w P \qquad \dfrac{\overrightarrow{P} \Longrightarrow_w \boxed{P} \qquad \dfrac{\boxed{\overrightarrow{N}} \Longrightarrow_w N}{\overrightarrow{N} \Longrightarrow_w N}\,foc}{\overrightarrow{P} \,|_k\, \overrightarrow{N} \Rightarrow \boxed{P \odot_k N}}\,\odot_k R}{\Delta \,|_k\, \overrightarrow{N} \Longrightarrow_w \boxed{P \odot_k N}}\,n\text{-}Cut_2}{\dfrac{\Delta \,|_k\, \Gamma \Longrightarrow_w \boxed{P \odot_k N}}{\Delta \,|_k\, \Gamma \Longrightarrow_w P \odot_k N}\,foc}\,n\text{-}Cut_2$$

- Left discontinuous $\uparrow_k$ rule (the left rule for $\downarrow_k$ is entirely similar). Like in the case for the right discontinuous product $\odot_k$ rule, we only show one representative example:

$$\frac{\Gamma \Rightarrow P \qquad \Delta\langle \overrightarrow{N} \rangle \Rightarrow A}{\Delta\langle \overrightarrow{N{\uparrow_k}P} |_k \Gamma \rangle \Rightarrow A}\uparrow_k L \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma \Longrightarrow_w P \qquad \dfrac{\dfrac{\overrightarrow{P} \Longrightarrow_w \boxed{\mathrm{P}} \qquad \overrightarrow{\boxed{N}} \Longrightarrow_w N}{\overrightarrow{\boxed{N{\uparrow_k}P}} |_k \overrightarrow{P} \Longrightarrow_w N}\uparrow_k L \qquad \Delta\langle \overrightarrow{N} \rangle \Longrightarrow_w A}{\Delta\langle \overrightarrow{\boxed{N{\uparrow_k}P}} |_k \overrightarrow{P} \rangle \Longrightarrow_w A} n\text{-}Cut_1}{\Delta\langle \overrightarrow{\boxed{N{\uparrow_k}P}} |_k \Gamma \rangle \Longrightarrow_w A} n\text{-}Cut_2}{\Delta\langle \overrightarrow{N{\uparrow_k}P} |_k \Gamma \rangle \Longrightarrow_w A} foc$$

- Right discontinuous $\uparrow_k$ rule (the right discontinuous rule for $\downarrow_k$ is entirely similar):

$$\frac{\Delta \,|_k\, \overrightarrow{A} \Rightarrow B}{\Delta \Rightarrow B{\uparrow_k}A}\uparrow_k R \qquad \rightsquigarrow \qquad \frac{\Delta \,|_k\, \overrightarrow{A} \Longrightarrow_w B}{\Delta \Longrightarrow_w B{\uparrow_k}A}\uparrow_k R$$

- Product and implicative continuous rules. These follow the same pattern as the discontinuous case. We interchange the metalinguistic k-th intercalation $|_k$ with the metalinguistic concatenation ',', and we interchange $\odot_k$, $\uparrow_k$ and $\downarrow_k$ with $\bullet$, $/$, and $\backslash$ respectively.

Concerning additives, conjunction Right translates directly and we consider then conjunction Left (disjunction is symmetric):

$$\frac{\Delta\langle\overrightarrow{P}\rangle \Rightarrow C}{\Delta\langle\overrightarrow{P\&M}\rangle \Rightarrow C}\&L \quad \leadsto \quad \dfrac{\dfrac{\dfrac{\dfrac{\overrightarrow{P} \Longrightarrow_w \boxed{\mathrm{P}}}{\overrightarrow{P} \Longrightarrow_w P}\,foc}{\overrightarrow{\boxed{\mathrm{P\&M}}} \Longrightarrow_w P}\,\&L_1}{\overrightarrow{P\&M} \Longrightarrow_w P}\,foc \qquad \Delta\langle\overrightarrow{P}\rangle \Longrightarrow_w C}{\Delta\langle\overrightarrow{P\&M}\rangle \Longrightarrow_w C}\,n\text{-}Cut_1$$

□

Cut-elimination for $\mathbf{DA_{foc}}$

We prove this by induction on the complexity (d, h) of top-most instances of *Cut*, where d is the size (number of connectives) appearing in the top-most Cut formula, and h is the depth of the top-most Cut.

Cut-elimination for $\mathbf{DA}_{\mathbf{foc}}$

We prove this by induction on the complexity (d, h) of top-most instances of *Cut*, where d is the size (number of connectives) appearing in the top-most Cut formula, and h is the depth of the top-most Cut.

There are four cases to consider: Cut with axiom in the minor premise, Cut with axiom in the major premise, principal Cuts, and permutation conversions.

Id cases

$$\frac{\overrightarrow{P} \Longrightarrow_w \boxed{P} \qquad \Delta\langle\overrightarrow{P}\rangle \Longrightarrow_w B \diamond \mathrm{foc}}{\Delta\langle\overrightarrow{P}\rangle \Longrightarrow_w B \diamond \mathrm{foc}}\ p\text{-}Cut_1 \qquad \rightsquigarrow \qquad \Delta\langle\overrightarrow{P}\rangle \Longrightarrow_w B \diamond \mathrm{foc}$$

$$\frac{\Delta \Longrightarrow_w N \diamond \mathrm{foc} \qquad \overrightarrow{\boxed{N}} \Longrightarrow_w N}{\Delta\langle\overrightarrow{N}\rangle \Longrightarrow_w B \diamond \mathrm{foc}}\ p\text{-}Cut_2 \qquad \rightsquigarrow \qquad \Delta\langle\overrightarrow{N}\rangle \Longrightarrow_w B \diamond \mathrm{foc}$$

foc cases

$$\dfrac{\dfrac{\Delta \Longrightarrow_w \boxed{P}}{\Delta \Longrightarrow_w P}\,\mathit{foc} \qquad \Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w A \diamond \mathrm{foc}}{\Gamma\langle \Delta \rangle \Longrightarrow_w A \diamond \mathrm{foc}}\,\mathit{n\text{-}Cut}_1 \quad \rightsquigarrow \quad \dfrac{\Delta \Longrightarrow_w \boxed{P} \qquad \Gamma\langle \overrightarrow{P} \rangle \Longrightarrow_w A \diamond \mathrm{foc}}{\Gamma\langle \Delta \rangle \Longrightarrow_w A \diamond \mathrm{foc}}\,\mathit{p\text{-}Cut}_1$$

$$\dfrac{\Delta \Longrightarrow_w N \qquad \dfrac{\Delta\langle \boxed{N} \rangle \Longrightarrow_w A}{\Gamma\langle \overrightarrow{N} \rangle \Longrightarrow_w A}\,\mathit{foc}}{\Gamma\langle \Delta \rangle \Longrightarrow_w A}\,\mathit{n\text{-}Cut}_2 \quad \rightsquigarrow \quad \dfrac{\Delta \Longrightarrow_w N \qquad \Gamma\langle \overrightarrow{\boxed{N}} \rangle \Longrightarrow_w A}{\Gamma\langle \Delta \rangle \Longrightarrow_w A}\,\mathit{p\text{-}Cut}_2$$

Principal cases

Principal cut of $\uparrow_k$:

$$\dfrac{\dfrac{\Delta|_k \overrightarrow{P_1} \Longrightarrow_w P_2 \diamond \text{foc}}{\Delta \Longrightarrow_w P_2 \uparrow_k P_1 \diamond \text{foc}}\uparrow_k R \qquad \dfrac{\Gamma_1 \Longrightarrow_w \boxed{P_1} \qquad \Gamma_2\langle \overrightarrow{P_2} \rangle \Longrightarrow_w A}{\Gamma_2\langle \overrightarrow{\boxed{P_2 \uparrow_k P_1}} |_k \Gamma_1 \rangle \Longrightarrow_w A}\uparrow_k L}{\Gamma_2\langle \Delta|_k \Gamma_1 \rangle \Longrightarrow_w A \diamond \text{foc}} \textit{p-Cut}_2$$

$$\leadsto$$

$$\dfrac{\Gamma_1 \Longrightarrow_w \boxed{P_1} \qquad \dfrac{\Delta|_k \overrightarrow{P_1} \Longrightarrow P_2 \diamond \text{foc} \qquad \Gamma_2\langle \overrightarrow{P_2} \rangle \Longrightarrow_w A}{\Gamma_2\langle \Delta|_k \overrightarrow{P_1} \rangle \Longrightarrow_w A \diamond \text{foc}} \textit{n-Cut}_1}{\Gamma_2\langle \Delta|_k \Gamma_1 \rangle \Longrightarrow_w A \diamond \text{foc}} \textit{p-Cut}_1$$

The case of $\downarrow_k$ is entirely similar to the $\uparrow_k$ case.

Principal Cut of discontinuous product:

$$\cfrac{\cfrac{\Delta_1 \Longrightarrow_w \boxed{P} \qquad \Delta_2 \Longrightarrow_w N}{\Delta_1 |_k \Delta_2 \Longrightarrow_w \boxed{P \odot_k N}}\odot_k R \qquad \cfrac{\Gamma\langle \overrightarrow{P} |_k \overrightarrow{N} \rangle \Longrightarrow_w A \diamond \text{foc}}{\Gamma\langle \overrightarrow{P \odot_k N} \rangle \Longrightarrow_w A \diamond \text{foc}}\odot_k L}{\Gamma\langle \Delta_1 |_k \Delta_2 \rangle \Longrightarrow_w A \diamond \text{foc}}\textit{p-Cut}_1$$

$$\leadsto$$

$$\cfrac{\Delta_2 \Longrightarrow_w N \qquad \cfrac{\Delta_1 \Longrightarrow \boxed{P} \qquad \Gamma\langle \overrightarrow{P} |_k \overrightarrow{N} \rangle \Longrightarrow_w A \diamond \text{foc}}{\Gamma\langle \Delta_1 |_k \overrightarrow{N} \rangle \Longrightarrow_w A \diamond \text{foc}}\textit{p-Cut}_1}{\Gamma\langle \Delta_1 |_k \Delta_2 \rangle \Longrightarrow_w A \diamond \text{foc}}\textit{n-Cut}_2$$

Principal Cut of additive conjunction:

$$\dfrac{\dfrac{\Delta \Longrightarrow_w Q \diamond \text{foc} \qquad \Delta \Longrightarrow_w A \diamond \text{foc}}{\Delta \Longrightarrow_w Q\&A \diamond \text{foc}}\,\&R \qquad \dfrac{\Gamma\langle \overrightarrow{\boxed{Q}} \rangle \Longrightarrow_w B}{\Gamma\langle \overrightarrow{\boxed{Q\&A}} \Longrightarrow_w B}\,\&L}{\Gamma\langle \Delta \rangle \Longrightarrow_w B \diamond \text{foc}}\,p\text{-}Cut_2$$

$$\rightsquigarrow$$

$$\dfrac{\Delta \Longrightarrow Q \diamond \text{foc} \qquad \Gamma\langle \overrightarrow{\boxed{Q}} \rangle \Longrightarrow_w B}{\Gamma\langle \Delta \rangle \Longrightarrow_w B \diamond \text{foc}}\,p\text{-}Cut_2$$

Principal Cut of additive conjunction, another case:

$$\dfrac{\dfrac{\Delta \Longrightarrow_w M \diamond \text{foc} \qquad \Delta \Longrightarrow_w A \diamond \text{foc}}{\Delta \Longrightarrow_w M\&A \diamond \text{foc}}\,\&R \qquad \dfrac{\Gamma\langle \overrightarrow{M} \rangle \Longrightarrow_w B}{\Gamma\langle \overrightarrow{\boxed{M\&A}} \rangle \Longrightarrow_w B}\,\&L}{\Gamma\langle \Delta \rangle \Longrightarrow_w B \diamond \text{foc}}\,p\text{-}Cut_2$$

$$\rightsquigarrow$$

$$\dfrac{\Delta \Longrightarrow M \diamond \text{foc} \qquad \Gamma\langle \overrightarrow{M} \rangle \Longrightarrow_w B}{\Gamma\langle \Delta \rangle \Longrightarrow_w B \diamond \text{foc}}\,n\text{-}Cut_1$$

Commutation conversions

Left commutative *p-Cut* conversions:

$$\dfrac{\dfrac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w N}{\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow_w N}\,foc \qquad \Gamma\langle\overrightarrow{\boxed{N}}\rangle \Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{Q}\rangle\rangle \Longrightarrow_w C}\,p\text{-}Cut_2$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w N \qquad \Gamma\langle\overrightarrow{\boxed{N}}\rangle \Longrightarrow_w C}{\Gamma\langle\Delta\langle\overrightarrow{\boxed{Q}}\rangle\rangle \Longrightarrow_w C}\,p\text{-}Cut_2}{\Gamma\langle\Delta\langle\overrightarrow{Q}\rangle\rangle \Longrightarrow_w C}\,foc$$

$$
\dfrac{\dfrac{\Delta\langle \overrightarrow{A}|_k \overrightarrow{B}\rangle \Longrightarrow_w \boxed{P}}{\Delta\langle \overrightarrow{A \odot_k B}\rangle \Longrightarrow_w \boxed{P}}\,\odot_k L \qquad \Gamma\langle \overrightarrow{P}\rangle \Longrightarrow_w C \diamond \text{foc}}{\Gamma\langle \Delta\langle \overrightarrow{A \odot_k B}\rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\, p\text{-}Cut_1
$$

$$\rightsquigarrow$$

$$
\dfrac{\dfrac{\Delta\langle \overrightarrow{A}|_k \overrightarrow{B}\rangle \Longrightarrow_w \boxed{P} \qquad \Gamma\langle \overrightarrow{P}\rangle \Longrightarrow_w C \diamond \text{foc}}{\Gamma\langle \Delta\langle \overrightarrow{A}|_k \overrightarrow{B}\rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\, p\text{-}Cut_1}{\Gamma\langle \Delta\langle \overrightarrow{A \odot_k B}\rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\,\odot_k L
$$

$$\dfrac{\dfrac{\Delta\langle \overrightarrow{A}|_k \overrightarrow{B}\rangle \Longrightarrow_w N \diamond \text{foc}}{\Delta\langle \overrightarrow{A \odot_k B}\rangle \Longrightarrow_w N \diamond \text{foc}}\ \odot_k L \qquad \Gamma\langle \boxed{\overrightarrow{N}} \rangle \Longrightarrow_w C}{\Gamma\langle \Delta\langle \overrightarrow{A \odot_k B}\rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\ p\text{-}Cut_2$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Delta\langle \overrightarrow{A}|_k \overrightarrow{B}\rangle \Longrightarrow_w N \diamond \text{foc} \qquad \Gamma\langle \boxed{\overrightarrow{N}} \rangle \Longrightarrow_w C}{\Gamma\langle \Delta\langle \overrightarrow{A}|_k \overrightarrow{B}\rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\ p\text{-}Cut_2}{\Gamma\langle \Delta\langle \overrightarrow{A \odot_k B}\rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\ \odot_k L$$

$$\dfrac{\dfrac{\Gamma_1 \Longrightarrow_w \boxed{P_1} \qquad \Gamma_2\langle \overrightarrow{\boxed{N_1}} \rangle \Longrightarrow_w N}{\Gamma_2\langle \overrightarrow{\boxed{N_1 \uparrow_k P_1}} |_k \Gamma_1 \rangle \Longrightarrow_w N}\,{\uparrow_k L} \qquad \Theta\langle \overrightarrow{\boxed{N}} \rangle \Longrightarrow_w C}{\Theta\langle \Gamma_2\langle \overrightarrow{\boxed{N_1 \uparrow_k P_1}} |_k \Gamma_1 \rangle\rangle \Longrightarrow_w C}\, p\text{-}Cut_2$$

$$\rightsquigarrow$$

$$\dfrac{\Gamma_1 \Longrightarrow_w \boxed{P_1} \qquad \dfrac{\Gamma_1\langle \overrightarrow{\boxed{N_1}} \rangle \Longrightarrow_w N \qquad \Theta\langle \overrightarrow{\boxed{N}} \rangle \Longrightarrow_w C}{\Theta\langle \Gamma_2\langle \overrightarrow{\boxed{N_1}} \rangle\rangle \Longrightarrow_w C}\, p\text{-}Cut_2}{\Theta\langle \Gamma_2\langle \overrightarrow{\boxed{N_1 \uparrow_k P_1}} |_k \Gamma_1 \rangle\rangle \Longrightarrow_w C}\,{\uparrow_k L}$$

Additive case:

$$\dfrac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle \Longrightarrow_w \boxed{P} \qquad \Gamma\langle\overrightarrow{B}\rangle \Longrightarrow_w \boxed{P}}{\Gamma\langle\overrightarrow{A\oplus B}\rangle \Longrightarrow_w \boxed{P}}\oplus L \qquad \Delta\langle\overrightarrow{P}\rangle \Longrightarrow_w C \diamond \text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\textit{p-Cut}_1$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma\langle\overrightarrow{A}\rangle \Longrightarrow_w \boxed{P} \qquad \Delta\langle\overrightarrow{P}\rangle \Longrightarrow_w C \diamond \text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{A}\rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\textit{p-Cut}_1 \qquad \dfrac{\Gamma\langle\overrightarrow{B}\rangle \Longrightarrow_w \boxed{P} \qquad \Delta\langle\overrightarrow{P}\rangle \Longrightarrow_w C \diamond \text{foc}}{\Delta\langle\Gamma\langle\overrightarrow{B}\rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\textit{p-Cut}_1}{\Delta\langle\Gamma\langle\overrightarrow{A\oplus B}\rangle\rangle \Longrightarrow_w C \diamond \text{foc}}\oplus L$$

Etc.

□

Embedding of $\mathbf{DA_{foc}}$ (without Cut) into $\mathbf{DA_{Foc}}$

The following theorem entails the embedding of $\mathbf{DA_{foc}}$ (without Cut) into $\mathbf{DA_{Foc}}$ since a $\mathbf{DA_{Foc}}$ sequent cannot contain both a focus and a complex negative type, and if there is no focus and no complex negative type the sequent is already of the form required for $\mathbf{DA_{Foc}}$.

Theorem. For any configuration Δ and type A, we have that if $\Delta \Longrightarrow_w A$ with one focalised formula and no asynchronous formula occurrence, then $\Delta \Longrightarrow A$ with the same formula focalised. If $\Delta \Longrightarrow_w A$ with no focalised formula and with at least one asynchronous formula, then $\Delta \Longrightarrow A$.

Proof.

We proceed by induction on the size of (number of connectives in) of $\mathbf{DA_{foc}}$ sequents. We consider Cut-free $\mathbf{DA_{foc}}$ proofs which match the sequents of this theorem. If the last rule is logical (i.e., it is not an instance of the *foc* rule) the i.h. applies directly and we get $\mathbf{DA_{Foc}}$ proofs of the same end-sequent. Now, let us suppose that the last rule is not logical, i.e. it is an instance of the *foc* rule. Let us suppose that the end sequent $\Delta \Longrightarrow_w A$ is a synchronous sequent. Suppose for example that the focalised formula is in the succedent:

$$\frac{\Delta \Longrightarrow_w \boxed{P}}{\Delta \Longrightarrow_w P}\ foc$$

The sequent $\Delta \Longrightarrow_w \boxed{P}$ arises from a synchronous rule to which we can apply i.h.

Let us suppose now that the end-sequent contains at least one asynchronous formula. We see three cases which are illustrative:

(1) a. $\Delta\langle\overrightarrow{A{\odot_k}B}\rangle \Longrightarrow_w \boxed{P}$

b. $\Delta\langle\overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w B{\uparrow_k}A$

c. $\Delta\langle\overrightarrow{\boxed{Q}}\rangle \Longrightarrow_w A\&B$

In case (1a), we have by Eta expansion that $\overrightarrow{A\odot_k B}\Longrightarrow_w \boxed{\overrightarrow{A\odot_k B}}$. We apply to this sequent the invertible $\odot_k$ left rule, whence $\overrightarrow{A}|_k\overrightarrow{B}\Longrightarrow_w \boxed{\overrightarrow{A\odot_k B}}$. In this case we have the following proof in $\mathbf{DA_{foc}}$:

$$\dfrac{\dfrac{\overrightarrow{A}\ |_k\ \overrightarrow{B}\Longrightarrow_w \boxed{\overrightarrow{A\odot_k B}} \qquad \Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w \boxed{P}}{\Delta\langle\overrightarrow{A}\ |_k\ \overrightarrow{B}\rangle\Longrightarrow_w \boxed{P}}\ p\text{-}Cut_1}{\Delta\langle\overrightarrow{A}\ |_k\ \overrightarrow{B}\rangle\Longrightarrow_w P}\ foc$$

To the above $\mathbf{DA_{foc}}$ proof we apply Cut-elimination and we get the Cut-free $\mathbf{DA_{foc}}$ end-sequent $\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle \Longrightarrow_w P$. We have $|\Delta\langle\overrightarrow{A}|_k\overrightarrow{B}\rangle\Longrightarrow_w P| < |\Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow_w P|$. We can apply then i.h. and we derive the provable $\mathbf{DA_{Foc}}$ sequent $\Delta\langle\overrightarrow{A}\ |_k\ \overrightarrow{B}\rangle\Longrightarrow P$ to which we can apply the left $\odot_k$ rule. We have obtained $\Delta\langle\overrightarrow{A\odot_k B}\rangle\Longrightarrow P$.

In the same way, we have in case (1b) that $\overrightarrow{\boxed{B{\uparrow_k}A}}|_k\overrightarrow{A}\Longrightarrow_w B$. Thus we have the following proof in $\mathbf{DA_{foc}}$:

$$\dfrac{\dfrac{\Delta\langle\overrightarrow{\boxed{Q}}\rangle\Longrightarrow_w B{\uparrow_k}A \qquad \overrightarrow{\boxed{B{\uparrow_k}A}}|_k\overrightarrow{A}\Longrightarrow_w B}{\Delta\langle\overrightarrow{\boxed{Q}}\rangle|_k\overrightarrow{A}\Longrightarrow_w B}\ p\text{-}Cut_2}{\Delta\langle\overrightarrow{Q}\rangle|_k\overrightarrow{A}\Longrightarrow_w B}\ foc$$

As before, we apply Cut-elimination to the above proof. We get the Cut-free $\mathbf{DA_{foc}}$ end-sequent $\Delta\langle\overrightarrow{Q}\rangle|_k\overrightarrow{A}\Longrightarrow_w B$. It has size less than $|\Delta\langle\overrightarrow{Q}\rangle\Longrightarrow_w B{\uparrow_k}A|$. We can apply i.h. and we get the $\mathbf{DA_{Foc}}$ provable sequent $\Delta\langle\overrightarrow{Q}\rangle|_k\overrightarrow{A}\Longrightarrow B$ to which we apply the $\uparrow_k$ right rule.

In case (1c):

$$\frac{\Delta\langle\boxed{\overrightarrow{\mathrm{Q}}}\rangle \Longrightarrow_w A\&B}{\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow_w A\&B}\ foc$$

by applying the *foc* rule and the invertibility of $\&R$ we get the provable $\mathbf{DA_{foc}}$ sequents $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow_w A$ and $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow_w B$. These sequents have smaller size than $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow_w A\&B$. The aforementioned sequents have a Cut-free proof in $\mathbf{DA_{foc}}$. We apply i.h. and we get $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow A$ and $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow B$. We apply the $\&$ right rule in $\mathbf{DA_{Foc}}$, and we get $\Delta\langle\overrightarrow{Q}\rangle \Longrightarrow A\&B$.

□