Mathematical Logic and Linguistics

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BGSMath Course
Class 7

Table of categorial connectives

	cont. mult.	disc. mult.	add.	qu.	norm. mod.	brack. mod.	exp.	limited contr. & weak.
primitive	$/$ $\backslash$ $\bullet$ I	$\uparrow$ $\downarrow$ $\odot$ J	$\&$ $\oplus$	$\bigwedge$ $\bigvee$	$\Box$ $\Diamond$	$[\,]^{-1}$ $\langle\rangle$	$!$ $?$	$\vert$ W
sem. inactive variants	•– –∘ ∘– –• ◐ ◑	ꜝ ꜜ ꜝ ꜜ ◓ ◒	$\sqcap$ $\sqcup$	$\forall$ $\exists$	■ ♦			
det. synth.	$\triangleleft^{-1}$ $\triangleright^{-1}$ $\triangleleft$ $\triangleright$	$\check{}$ $\hat{}$						diff.
nondet. synth.	$\div$ $\times$	$\triangleq$ $\Downarrow$ $\diamond$						$-$

Primitive types

1.	$\mathbf{Tp}^p_i$	$::=$	$\mathbf{Tp}^p_{i+j}/\mathbf{Tp}^{\overline{p}}_j$	$T(C/B)$	$=$	$T(B)\to T(C)$	over [5]
2.	$\mathbf{Tp}^p_j$	$::=$	$\mathbf{Tp}^{\overline{p}}_i\backslash\mathbf{Tp}^p_{i+j}$	$T(A\backslash C)$	$=$	$T(A)\to T(C)$	under [5]
3.	$\mathbf{Tp}^p_{i+j}$	$::=$	$\mathbf{Tp}^p_i\bullet\mathbf{Tp}^p_j$	$T(A\bullet B)$	$=$	$T(A)\&T(B)$	continuous product [5]
4.	$\mathbf{Tp}^p_0$	$::=$	I	$T(I)$	$=$	$\top$	continuous unit [4]
5, k.	$\mathbf{Tp}^p_{i+1}$	$::=$	$\mathbf{Tp}^p_{i+j}\uparrow_k\mathbf{Tp}^{\overline{p}}_j, 1\leq k\leq i{+}1$	$T(C\uparrow_k B)$	$=$	$T(B)\to T(C)$	extract [12]
6, k.	$\mathbf{Tp}^p_j$	$::=$	$\mathbf{Tp}^{\overline{p}}_{i+1}\downarrow_k\mathbf{Tp}^p_{i+j}, 1\leq k\leq i{+}1$	$T(A\downarrow_k C)$	$=$	$T(A)\to T(C)$	infix [12]
7, k.	$\mathbf{Tp}^p_{i+j}$	$::=$	$\mathbf{Tp}^p_{i+1}\odot_k\mathbf{Tp}^p_j, 1\leq k\leq i{+}1$	$T(A\odot_k B)$	$=$	$T(A)\&T(B)$	discontinuous product [12]
8.	$\mathbf{Tp}^p_1$	$::=$	J	$T(J)$	$=$	$\top$	discontinuous unit [12]
9.	$\mathbf{Tp}^p_i$	$::=$	$\mathbf{Tp}^p_i\&\mathbf{Tp}^p_i$	$T(A\&B)$	$=$	$T(A)\&T(B)$	additive conjunction [3, 8]
10.	$\mathbf{Tp}^p_i$	$::=$	$\mathbf{Tp}^p_i\oplus\mathbf{Tp}^p_i$	$T(A\oplus B)$	$=$	$T(A)+T(B)$	additive disjunction [3, 8]
11.	$\mathbf{Tp}^p_i$	$::=$	$\bigwedge V\mathbf{Tp}^p_i$	$T(\bigwedge vA)$	$=$	$F\to T(A)$	1st order univ. qu. [13]
12.	$\mathbf{Tp}^p_i$	$::=$	$\bigvee V\mathbf{Tp}^p_i$	$T(\bigvee vA)$	$=$	$F\&T(A)$	1st order exist. qu. [13]
13.	$\mathbf{Tp}^p_i$	$::=$	$\Box\mathbf{Tp}^p_i$	$T(\Box A)$	$=$	$\mathbf{L}T(A)$	universal modality [9]
14.	$\mathbf{Tp}^p_i$	$::=$	$\Diamond\mathbf{Tp}^p_i$	$T(\Diamond A)$	$=$	$\mathbf{M}T(A)$	existential modality [7]
15.	$\mathbf{Tp}^p_i$	$::=$	$[\,]^{-1}\mathbf{Tp}^p_i$	$T([\,]^{-1}A)$	$=$	$T(A)$	univ. bracket modality [10, 6]
16.	$\mathbf{Tp}^p_i$	$::=$	$\langle\rangle\mathbf{Tp}^p_i$	$T(\langle\rangle A)$	$=$	$T(A)$	exist. bracket modality [10, 6]
17.	$\mathbf{Tp}^p_0$	$::=$	$!\mathbf{Tp}^p_0$	$T(!A)$	$=$	$T(A)$	universal exponential [1]
18.	$\mathbf{Tp}^\circ_0$	$::=$	$?\mathbf{Tp}^\circ_0$	$T(?A)$	$=$	$T(A)^+$	existential exponential [13]
19.	$\mathbf{Tp}^p_{i+j}$	$::=$	$\mathbf{Tp}^p_{i+j}\vert\mathbf{Tp}^{\overline{p}}_j$	$T(B\vert A)$	$=$	$T(A)\to T(B)$	limited contraction [2]
20.	$\mathbf{Tp}^p_0$	$::=$	$W(w)$	$T(W(w))$	$=$	$\top$	limited weakening [11]

Sequent calculus

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The sort sA of a type A is the $i \in \mathcal{N}$ such that $A \in \mathbf{Tp}_i$. For example, $s(S{\uparrow_1}N){\uparrow_2}N = 2$.

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$$\begin{array}{rcl} \mathbf{Config} & ::= & \Lambda \mid \mathbf{TreeTerm}, \mathbf{Config} \mid [\mathbf{Config}] \\ \mathbf{TreeTerm} & ::= & 1 \mid \mathbf{Tp}_0 \mid \mathbf{Tp}_{i>0}\{\underbrace{\mathbf{Config} : \ldots : \mathbf{Config}}_{i\ \mathbf{Config}\text{'s}}\} \end{array}$$

For example, there is the configuration $\gamma = A\{C, 1 : B\{1\}, D\}, F, 1$ where $sA = 2$ and $sB = 1$ and $sC = sD = sF = 0$.

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The intuition is the following. Dotted nodes signify unbounded arity concatenations and a type labelling a mother node signifies a discontinuous type intercalated by its daughters. Leaf types are continuous, and a leaf 1 marks a point of discontinuity.

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$$\overrightarrow{A} = \begin{cases} A & \text{if } sA = 0 \\ A\{\underbrace{1 : \ldots : 1}_{sA\ 1\text{'s}}\} & \text{if } sA > 0 \end{cases}$$

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For example, $\overrightarrow{(S\uparrow_1 N)\uparrow_2 N} = (S\uparrow_1 N)\uparrow_2 N\{1 : 1\}$.

Where Γ is a configuration of sort i and $\Delta_1, \ldots, \Delta_i$ are configurations, the *fold* $\Gamma \otimes \langle \Delta_1 : \ldots : \Delta_i \rangle$ is the result of replacing the successive 1's in Γ by $\Delta_1, \ldots, \Delta_i$ respectively.

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For example, if γ_1, γ_2 and γ_3 are configurations, the fold $\gamma \otimes \langle \gamma_1 : \gamma_2 : \gamma_3 \rangle = A\{C, \gamma_1 : B\{\gamma_2\}, D\}, F, \gamma_3$.

Where Δ is a configuration of sort $i > 0$ and Γ is a configuration, the kth metalinguistic wrap, $1 \leq k \leq i$, $\Delta \,|_k\, \Gamma$ is given by

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i.e. $\Delta\,|_k\,\Gamma$ is the configuration resulting from replacing by Γ the kth separator in Δ.

For example, where γ' is a configuration,
$\gamma\,|_2\,\gamma' = A\{C, 1 : B\{\gamma'\}, D\}, F, 1$.

Rules of grammatical inference

Identity rules

$$\frac{}{\overrightarrow{A} \Rightarrow A}\ id \qquad \frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{A}\rangle \Rightarrow B}{\Delta\langle\Gamma\rangle \Rightarrow B}\ Cut$$

Multiplicatives

1. $$\frac{\Gamma \Rightarrow B \qquad \Delta\langle\overrightarrow{C}\rangle \Rightarrow D}{\Delta\langle\overrightarrow{C/B}, \Gamma\rangle \Rightarrow D}/L \qquad \frac{\Gamma, \overrightarrow{B} \Rightarrow C}{\Gamma \Rightarrow C/B}/R$$

2. $$\frac{\Gamma \Rightarrow A \qquad \Delta\langle\overrightarrow{C}\rangle \Rightarrow D}{\Delta\langle\Gamma, \overrightarrow{A\backslash C}\rangle \Rightarrow D}\backslash L \qquad \frac{\overrightarrow{A}, \Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C}\backslash R$$

3. $$\frac{\Delta\langle\overrightarrow{A}, \overrightarrow{B}\rangle \Rightarrow D}{\Delta\langle\overrightarrow{A\bullet B}\rangle \Rightarrow D}\bullet L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A\bullet B}\bullet R$$

4. $$\frac{\Delta\langle\Lambda\rangle \Rightarrow A}{\Delta\langle\overrightarrow{I}\rangle \Rightarrow A}IL \qquad \frac{}{\Lambda \Rightarrow I}IR$$

$$5,k. \quad \frac{\Gamma \Rightarrow B \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{C{\uparrow_k}B} \,|_k\, \Gamma \rangle \Rightarrow D} \uparrow_k L \qquad \frac{\Gamma \,|_k\, \overrightarrow{B} \Rightarrow C}{\Gamma \Rightarrow C{\uparrow_k}B} \uparrow_k R$$

$$6,k. \quad \frac{\Gamma \Rightarrow A \qquad \Delta\langle \overrightarrow{C} \rangle \Rightarrow D}{\Delta\langle \Gamma \,|_k\, \overrightarrow{A{\downarrow_k}C} \rangle \Rightarrow D} \downarrow_k L \qquad \frac{\overrightarrow{A} \,|_k\, \Gamma \Rightarrow C}{\Gamma \Rightarrow A{\downarrow_k}C} \downarrow_k R$$

$$7,k. \quad \frac{\Delta\langle \overrightarrow{A} \,|_k\, \overrightarrow{B} \rangle \Rightarrow D}{\Delta\langle \overrightarrow{A{\odot_k}B} \rangle \Rightarrow D} \odot_k L \qquad \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1 \,|_k\, \Gamma_2 \Rightarrow A{\odot_k}B} \odot_k R$$

$$8. \quad \frac{\Delta\langle 1 \rangle \Rightarrow A}{\Delta\langle \overrightarrow{J} \rangle \Rightarrow A} JL \qquad \frac{}{1 \Rightarrow J} JR$$

Additives

9. $$\frac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\&B}\rangle \Rightarrow C}\,\&L_1 \qquad \frac{\Gamma\langle\overrightarrow{B}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\&B}\rangle \Rightarrow C}\,\&L_2$$

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A\&B}\,\&R$$

10. $$\frac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow C \qquad \Gamma\langle\overrightarrow{B}\rangle \Rightarrow C}{\Gamma\langle\overrightarrow{A\oplus B}\rangle \Rightarrow C}\,\oplus L$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A\oplus B}\,\oplus R_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A\oplus B}\,\oplus R_2$$

Quantifiers

11. $$\frac{\Gamma\langle\overrightarrow{A[t/v]}\rangle \Rightarrow B}{\Gamma\langle\overrightarrow{\bigwedge vA}\rangle \Rightarrow B}\bigwedge L \qquad \frac{\Gamma \Rightarrow A[a/v]}{\Gamma \Rightarrow \bigwedge vA}\bigwedge R^{\dagger}$$

12. $$\frac{\Gamma\langle\overrightarrow{A[a/v]}\rangle \Rightarrow B}{\Gamma\langle\overrightarrow{\bigvee vA}\rangle \Rightarrow B}\bigvee L^{\dagger} \qquad \frac{\Gamma \Rightarrow A[t/v]}{\Gamma \Rightarrow \bigvee vA}\bigvee R$$

† indicates that there is no *a* in the conclusion

Normal modalities

13. $$\frac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow B}{\Gamma\langle\overrightarrow{\Box A}\rangle \Rightarrow B}\,\Box L \qquad \frac{\boxtimes\Gamma \Rightarrow A}{\boxtimes\Gamma \Rightarrow \Box A}\,\Box R$$

14. $$\frac{\boxtimes\Gamma\langle\overrightarrow{A}\rangle \Rightarrow \oplus B}{\boxtimes\Gamma\langle\overrightarrow{\Diamond A}\rangle \Rightarrow \oplus B}\,\Diamond L \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Diamond A}\,\Diamond R$$

$\boxtimes$/$\oplus$ marks a structure all the types of which have principal connective a box/diamond

Bracket modalities

15. $$\frac{\Gamma\langle\overrightarrow{A}\rangle \Rightarrow B}{\Gamma\langle\overrightarrow{[[\,]^{-1}A]}\rangle \Rightarrow B}[\,]^{-1}L \qquad \frac{[\Gamma] \Rightarrow A}{\Gamma \Rightarrow [\,]^{-1}A}[\,]^{-1}R$$

16. $$\frac{\Gamma\langle[\overrightarrow{A}]\rangle \Rightarrow B}{\Gamma\langle\overrightarrow{\langle\rangle A}\rangle \Rightarrow B}\langle\rangle L \qquad \frac{\Gamma \Rightarrow A}{[\Gamma] \Rightarrow \langle\rangle A}\langle\rangle R$$

Exponentials

17. $$\frac{\Gamma\langle A\rangle \Rightarrow B}{\Gamma\langle !A\rangle \Rightarrow B}\ !L \qquad \frac{!A_1,\ldots,!A_n \Rightarrow A}{!A_1,\ldots,!A_n \Rightarrow !A}\ !R$$

$$\frac{\Delta\langle !A,\Gamma\rangle \Rightarrow B}{\Delta\langle \Gamma,!A\rangle \Rightarrow B}\ !P \qquad \frac{\Delta\langle \Gamma,!A\rangle \Rightarrow B}{\Delta\langle !A,\Gamma\rangle \Rightarrow B}\ !P$$

$$\frac{\Delta\langle !A_0,\ldots,!A_n,[!A_0,\ldots,!A_n,\Gamma]\rangle \Rightarrow B}{\Delta\langle !A_0,\ldots,!A_n,\Gamma\rangle \Rightarrow B}\ !C$$

18. $$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow ?A}\ ?R$$

$$\frac{\Gamma \Rightarrow A \qquad \Delta \Rightarrow ?A}{\Gamma,\Delta \Rightarrow ?A}\ ?M$$

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