Mathematical Logic and Linguistics

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Cut elimination or Cut admissibility

- Consider the Cut rule:

$$\frac{\Gamma \Rightarrow A \qquad \Delta(A) \Rightarrow B}{\Delta(\Gamma) \Rightarrow B}\ Cut$$

- We want to provide an algorithm that given a proof $\mathcal{D}$, we transform it into a new one $\mathcal{D}^*$ such that it is Cut-free, i.e. all the Cut rule instances have been "removed".
- This algorithm should preserve the *derivational semantics* of $\mathcal{D}$, i.e..

$$[[\mathcal{D}]] \quad = \quad [[\mathcal{D}^*]]$$

Cut elimination or Cut admissibility

Cut elimination or Cut admissibility

- Define the *length* $|A|$ of a formula A as the number of connectives it contains.
- Define the *length* $|\Delta|$ of a configuration Δ as the sum of the lengths of its formula-occurrences.
- Given an instance of the Cut rule, we define its *Cut complexity* as:

$$\left| \frac{\Delta \Rightarrow A \qquad \Gamma(A) \Rightarrow B}{\Delta(\Gamma) \Rightarrow B} \, Cut \right| \quad \triangleq \quad |\Delta| + |\Gamma| + |A| + |B|$$

Strategy of the proof

Strategy of the proof

- Suppose we have a proof $\mathcal{D}$ whose last rule is Cut, but its premises do not use the Cut rule, i.e. they are Cut-free:

$$\mathcal{D} \quad = \quad \dfrac{\dfrac{\vdots\, \mathcal{D}_1}{\Delta \Rightarrow A} \qquad \dfrac{\vdots\, \mathcal{D}_2}{\Gamma(A) \Rightarrow B}}{\Delta(\Gamma) \Rightarrow B}\ Cut$$

  We want to transform $\mathcal{D}$ into a new proof with strictly decreased Cut-complexities, or a new proof which is already Cut-free.

Reduction steps

Reduction steps

As previously mentioned, assume we have a proof with only one Cut and whose last rule is this Cut. We have the following reductions:

- The so-called principal cases.
- The so-called permutation conversions.

Principal cases

Principal cases

$$\dfrac{\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow B/A}\,/R \qquad \dfrac{\Theta \Rightarrow A \qquad \Delta(B) \Rightarrow C}{\Delta(B/A, \Theta) \Rightarrow C}\,/L}{\Delta(\Gamma, \Theta) \Rightarrow C}\,Cut_1 \qquad \rightsquigarrow$$

$$\dfrac{\Theta \Rightarrow A \qquad \dfrac{\Gamma, A \Rightarrow B \qquad \Delta(B) \Rightarrow C}{\Delta(\Gamma, A) \Rightarrow C}\,Cut_3}{\Delta(\Gamma, \Theta) \Rightarrow C}\,Cut_2$$

- Observe that we have a new proof with exactly two Cuts, but whose Cut complexities are strictly smaller. For:

$$\begin{aligned} |Cut_2| &= |\Delta()| + |\Gamma| + |\Theta| + |A| + |C| \\ &< |\Delta()| + |\Gamma| + |\Theta| + |C| + |A| + |B| + 1 \\ &= |\Delta()| + |\Gamma| + |\Theta| + |C| + |B/A| \\ &= |Cut_1| \end{aligned}$$

Principal cases continued

Principal cases continued

Similarly we have:

$$|Cut_3| \quad < \quad |Cut_1|$$

There are two other principal cases:

- $\backslash$ case (Exercise).
- $\bullet$ case.

Principal cases continued

Principal cases continued

Product $\bullet$ case:

$$\dfrac{\dfrac{\Delta \Rightarrow A \qquad \Gamma \Rightarrow B}{\Delta, \Gamma \Rightarrow A \bullet B}\bullet L \qquad \dfrac{\Theta(A, B) \Rightarrow C}{\Theta(A \bullet B) \Rightarrow C}\bullet R}{\Theta(\Delta, \Gamma) \Rightarrow C}\, Cut_1$$

$$\rightsquigarrow$$

$$\dfrac{\Gamma \Rightarrow B \qquad \dfrac{\Delta \Rightarrow A \qquad \Theta(A, B) \Rightarrow C}{\Theta(\Delta, B) \Rightarrow C}\, Cut_2}{\Theta(\Delta, \Gamma) \Rightarrow C}\, Cut_3$$

Since $|A, B| < |A \bullet B|$, therefore:

$$\begin{array}{lcl} |Cut_2| & < & |Cut_1| \\ |Cut_3| & < & |Cut_1| \end{array}$$

Principal cases continued

Principal cases continued

Case involving the continuous unit:

$$\dfrac{\dfrac{}{\Lambda \Rightarrow I}\,IR \qquad \dfrac{\Delta(\Lambda) \Rightarrow A}{\Delta(I) \Rightarrow A}\,IL}{\Delta(\Lambda) \Rightarrow A}\,Cut$$

$$\rightsquigarrow$$

$$\Delta(\Lambda) \Rightarrow A$$

Permutation conversions

Permutation conversions

$$\dfrac{\Delta \Rightarrow A \qquad \dfrac{\Theta \Rightarrow B \qquad \Gamma(C; A) \Rightarrow D}{\Gamma(C/B, \Theta; A) \Rightarrow D}\,/L}{\Gamma(C/B, \Theta; \Delta) \Rightarrow D}\,Cut_1$$

$$\rightsquigarrow$$

$$\dfrac{\Theta \Rightarrow B \qquad \dfrac{\Delta \Rightarrow A \qquad \Gamma(C; D) \Rightarrow D}{\Gamma(C; \Delta) \Rightarrow D}\,Cut_2}{\Gamma(C/B, \Theta; \Delta) \Rightarrow D}\,/L$$

Observe that

$$\begin{array}{rcl} |Cut_2| & = & |\Gamma| + |\Delta| + |C| + |D| \\ & < & |\Gamma| + |\Delta| + |C| + |B| + 1 + |D|, \text{ since } |C/B| = |C| + |B| + 1 \\ & = & |Cut_1| \end{array}$$

Other cases are similar, and are left as exercises.

Identity case

Identity case

$$\dfrac{\dfrac{}{A \Rightarrow A}\,Id \qquad \Delta(A) \Rightarrow B}{\Delta(A) \Rightarrow B}\,Cut$$

$$\rightsquigarrow$$

$$\dfrac{}{\Delta(A) \Rightarrow A}$$

The proof of Cut elimination or Cut admissibility

- The number of Cuts in a **L**-proof is finite.
- Apply iteratively the previous reductions to top-most Cuts.
- Each iteration properly reduces the Cut complexity.
- Cut complexity cannot be negative.
- Therefore we are done.

Applications of Cut admissibility

Applications of Cut admissibility

We have only to consider Cut-free proofs. Let us see some of its nice corollaries:

- The *subformula property*.
- In a Cut-free proof, since the length of the premises are strictly smaller than the lengths of their premises, it turns out that the proof-search space is finite.
- Therefore, proof-derivability in **L** is decidable.
- The so-called *finite reading property* holds.