Congestion Games and Selfish Routing

Maria Serna

Fall 2016

Congestion games

Congestion games

A congestion game $(E, N, (d_e)_{e \in E}, (c_i)_{i \in N})$

- is defined on a finite set E of resources and
- has n players
- using a delay function d_e mapping $\mathbb{N}$ to the integers, for each resource e.
- The actions for each player are subsets of E.
- The cost functions are the following:

$$c_i(a_1, \ldots, a_n) = \sum_{e \in a_i} d_e(f_e(a_1, \ldots, a_n))$$

being $f_e(a_1, \ldots, a_n) = |\{i \mid e \in a_i\}|$.

Weighted congestion games

Weighted congestion games

A weighted congestion game $(E, N, (d_e)_{e \in E}, (c_i)_{i \in N}, (w_i)_{i \in N})$

- is defined on a finite set E of resources and
- has n players. Player i has an associated natural weight w_i.
- Using a delay function d_e mapping $\mathbb{N}$ to the integers, for each resource e.
- The actions for each player are subsets of E.
- The cost functions are the following:

$$c_i(a_1, \ldots, a_n) = \sum_{e \in a_i} d_e(f_e(a_1, \ldots, a_n))$$

being $f_e(a_1, \ldots, a_n) = \sum_{\{i | e \in a_i\}} w_i$.

Network weighted congestion games

Network weighted congestion games

A network weighted congestion game is defined on a directed graph $G = (V.E)$, $(N, G, (d_e)_{e\in E}, (c_i)_{i\in N}, (w_i)_{i\in N}, (s_i)_{i\in N}, (t_i)_{i\in N})$.

- The resources are the arcs in G.
- The game has n players. Player i has an associated natural weight w_i.
- Using a delay function d_e mapping $\mathbb{N}$ to the integers, for each arc $e \in E$.
- The action set for player i is the set of (s_i, t_i)-paths in G.
- The cost functions are the following:

$$c_i(a_1, \ldots, a_n) = \sum_{e\in a_i} d_e(f_e(a_1, \ldots, a_n))$$

being $f_e(a_1, \ldots, a_n) = \sum_{\{i|e\in a_i\}} w_i$.

Congestion games terminology

Congestion games terminology

- unweighted (vs. weighted)

Congestion games terminology

- unweighted (vs. weighted): $w_i = 1$.

Congestion games terminology

- unweighted (vs. weighted): $w_i = 1$.
- symmetric (vs. non-symmetric) strategies:

Congestion games terminology

- unweighted (vs. weighted): $w_i = 1$.
- symmetric (vs. non-symmetric) strategies: all the players have the same set of actions.

Congestion games terminology

- unweighted (vs. weighted): $w_i = 1$.
- symmetric (vs. non-symmetric) strategies: all the players have the same set of actions.
- symmetric congestion games:

Congestion games terminology

- unweighted (vs. weighted): $w_i = 1$.
- symmetric (vs. non-symmetric) strategies: all the players have the same set of actions.
- symmetric congestion games: unweighted with symmetric strategies.

Congestion games terminology

- unweighted (vs. weighted): $w_i = 1$.
- symmetric (vs. non-symmetric) strategies: all the players have the same set of actions.
- symmetric congestion games: unweighted with symmetric strategies.
- singleton congestion games:

Congestion games terminology

- unweighted (vs. weighted): $w_i = 1$.
- symmetric (vs. non-symmetric) strategies: all the players have the same set of actions.
- symmetric congestion games: unweighted with symmetric strategies.
- singleton congestion games: all possible actions have only one resource.

Congestion games terminology

- unweighted (vs. weighted): $w_i = 1$.
- symmetric (vs. non-symmetric) strategies: all the players have the same set of actions.
- symmetric congestion games: unweighted with symmetric strategies.
- singleton congestion games: all possible actions have only one resource.
- nonatomic network congestion games (vs. atomic)

Congestion games terminology

- unweighted (vs. weighted): $w_i = 1$.
- symmetric (vs. non-symmetric) strategies: all the players have the same set of actions.
- symmetric congestion games: unweighted with symmetric strategies.
- singleton congestion games: all possible actions have only one resource.
- nonatomic network congestion games (vs. atomic)
  In nonatomic congestion games the number of players is infinite and each player controls an infinitesimal weight of the total traffic.

Congestion games terminology

- unweighted (vs. weighted): $w_i = 1$.
- symmetric (vs. non-symmetric) strategies: all the players have the same set of actions.
- symmetric congestion games: unweighted with symmetric strategies.
- singleton congestion games: all possible actions have only one resource.
- nonatomic network congestion games (vs. atomic)
  In nonatomic congestion games the number of players is infinite and each player controls an infinitesimal weight of the total traffic. Named also Selfish routing games.

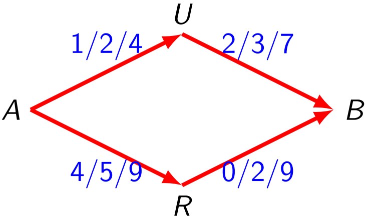
An example of a network congestion game

An example of a network congestion game

- There are three players.
- and a network (with a delay function on arcs)

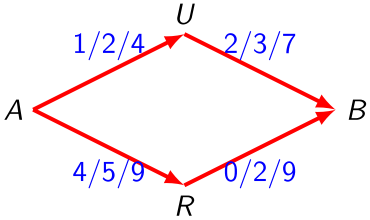
An example of a network congestion game

- There are three players.
- and a network (with a delay function on arcs)



An example of a network congestion game

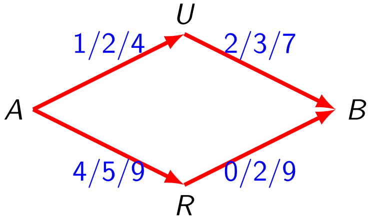
- There are three players.
- and a network (with a delay function on arcs)



- Player's objective: going from $s = A$ to $t = B$ as fast as possible.

An example of a network congestion game

- There are three players.
- and a network (with a delay function on arcs)



- Player's objective: going from $s = A$ to $t = B$ as fast as possible.
- Strategy profiles: paths from A to B.
- A NE?

An example of a weighted network congestion game

An example of a weighted network congestion game

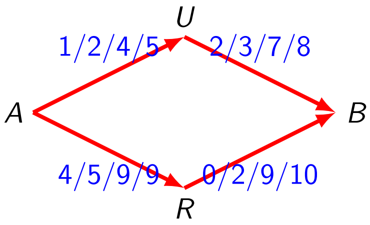
- There are three players with weights 1,1,2

An example of a weighted network congestion game

- There are three players with weights 1,1,2
- and a network (with a delay function on arcs)

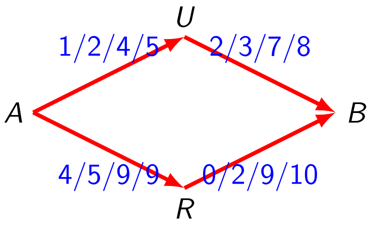
An example of a weighted network congestion game

- There are three players with weights 1,1,2
- and a network (with a delay function on arcs)



An example of a weighted network congestion game

- There are three players with weights 1,1,2
- and a network (with a delay function on arcs)



- Player's objective: send w_i units from $s = A$ to $t = B$ as fast as possible.
- Strategy profiles: paths from A to B.
- A NE?

Another family: Fair Cost Sharing Games

Another family: Fair Cost Sharing Games

A fair cost sharing game $(E, N, (c_e)_{e \in E})$

- is defined on a finite set E of resources and
- has n players
- a fixed cost c_e, for each resource e.
- The actions for each player are subsets of E.
- The cost functions are the following:

$$c_i(a_1, \dots, a_n) = \sum_{e \in a_i} \frac{c_e}{f_e(a_1, \dots, a_n)}$$

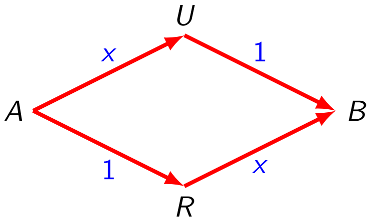
being $f_e(a_1, \dots, a_n) = |\{i \mid e \in a_i\}|$.

Selfish Routing: an Example

- Total traffic is $r = 1$.
- Network (with delay functions on arcs)

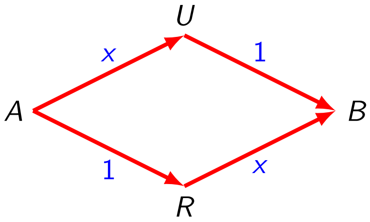
Selfish Routing: an Example

- Total traffic is $r = 1$.
- Network (with delay functions on arcs)



Selfish Routing: an Example

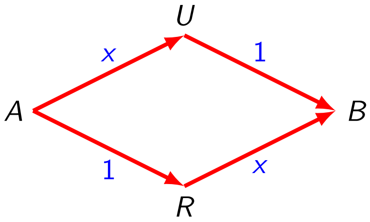
- Total traffic is $r = 1$.
- Network (with delay functions on arcs)



- Player's objective going from $s = A$ to $s = B$ with minimum delay.

Selfish Routing: an Example

- Total traffic is $r = 1$.
- Network (with delay functions on arcs)



- Player's objective going from $s = A$ to $s = B$ with minimum delay.
- Strategy profiles: flows from A to B with total flow $r = 1$

Selfish routing: strategy profiles

Selfish routing: strategy profiles

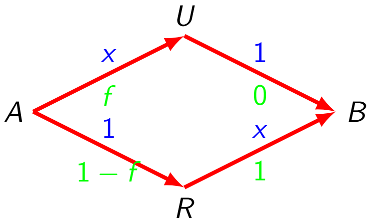
- Traffic as Flows:

Selfish routing: strategy profiles

- Traffic as Flows:
  - A flow vector f giving the routing of traffic.
  - f_P = amount of traffic routed on $(s_i - t_i)$ path P.

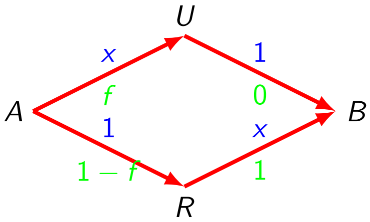
Selfish routing: strategy profiles

- Traffic as Flows:
  - A flow vector f giving the routing of traffic.
  - f_P = amount of traffic routed on $(s_i - t_i)$ path P.



Selfish routing: strategy profiles

- Traffic as Flows:
  - A flow vector f giving the routing of traffic.
  - f_P = amount of traffic routed on $(s_i - t_i)$ path P.



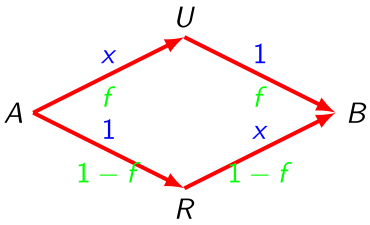
- Notation: for a path P and a feasible flow f, $C^P(f)$ denotes the cost corresponding to the traffic routed through P by f.

Selfish routing: equilibria

- A flow is a Nash equilibrium (or is a Nash flow) if all flow is routed on min-latency paths (given current edge congestion)

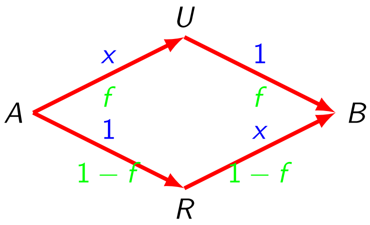
Selfish routing: equilibria

- A flow is a Nash equilibrium (or is a Nash flow) if all flow is routed on min-latency paths (given current edge congestion)



Selfish routing: equilibria

- A flow is a Nash equilibrium (or is a Nash flow) if all flow is routed on min-latency paths (given current edge congestion)



- This flow is not a Nash flow unless $f = 1/2$.

Selfish routing: equilibria

Selfish routing: equilibria

Theorem

A feasible flow x is an equilibrium flow iff for any feasible flow y

$$\sum_{e\in E} d_e(x[e])x[e] \leqslant \sum_{e\in E} d_e(y[e])x[e].$$

Selfish routing: equilibria

Theorem

A feasible flow x is an equilibrium flow iff for any feasible flow y

$$\sum_{e \in E} d_e(x[e])x[e] \leqslant \sum_{e \in E} d_e(y[e])x[e].$$

- Called Variational Inequality (Smith 79 and Dafermos 80)

Selfish routing: equilibria

Theorem

A feasible flow x is an equilibrium flow iff for any feasible flow y

$$\sum_{e \in E} d_e(x[e])x[e] \leqslant \sum_{e \in E} d_e(y[e])x[e].$$

- Called Variational Inequality (Smith 79 and Dafermos 80)
- As a consequence all Nash flows have the same cost per edge.

Selfish routing: equilibria

Theorem

A feasible flow x is an equilibrium flow iff for any feasible flow y

$$\sum_{e \in E} d_e(x[e])x[e] \leqslant \sum_{e \in E} d_e(y[e])x[e].$$

- Called Variational Inequality (Smith 79 and Dafermos 80)
- As a consequence all Nash flows have the same cost per edge.
- Do PNE exist?

Selfish routing: equilibria existence

- As for the atomic case we can consider a potential, for a given flow x

$$\Psi(x) = \sum_{e \in E} \int_0^{x[e]} d_e(u)du$$

Selfish routing: equilibria existence

- As for the atomic case we can consider a potential, for a given flow x

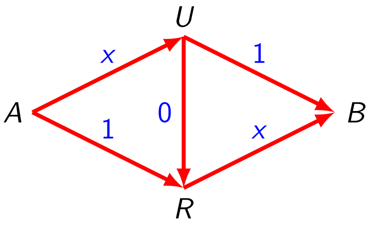
$$\Psi(x) = \sum_{e \in E} \int_0^{x[e]} d_e(u)du$$

Theorem

A feasible flow x is an equilibrium flow iff x is a minimum of Ψ over the set of feasible flows.

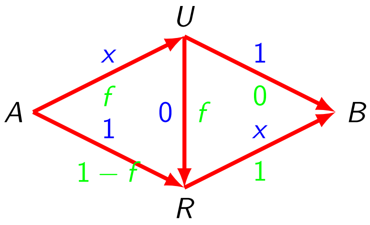
Another example

Another example



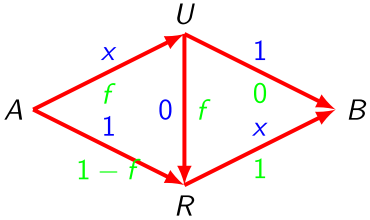
Another example

- A feasible flow:



Another example

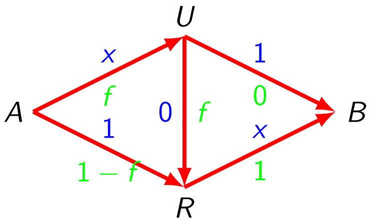
- A feasible flow:



- Is this a NE?

Another example

- A feasible flow:



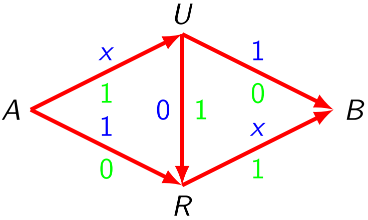
- Is this a NE? No unless $f = 1$

Selfish routing: equilibria

- A Nash flow

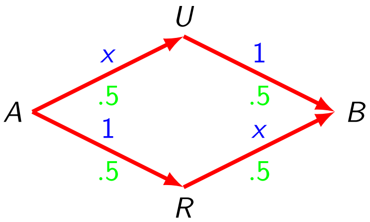
Selfish routing: equilibria

- A Nash flow

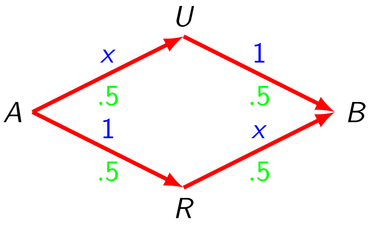


Braess' paradox: Nash flows

Braess' paradox: Nash flows

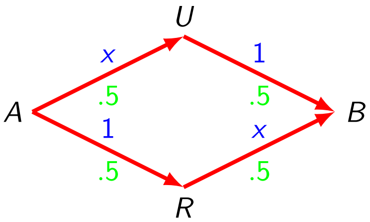


Braess' paradox: Nash flows

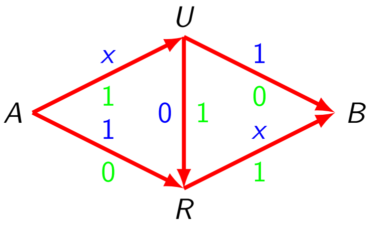


Delay is 1.5. We add a fast connection

Braess' paradox: Nash flows

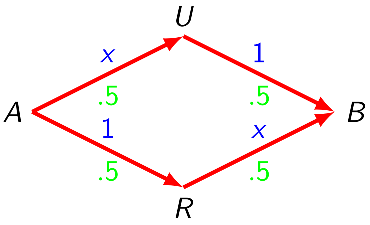


Delay is 1.5. We add a fast

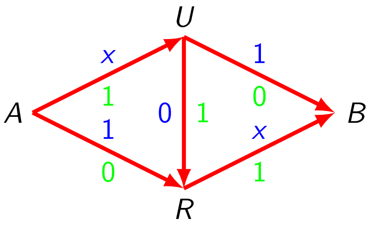


connection

Braess' paradox: Nash flows



Delay is 1.5. We add a fast



connection

and delay is 2!