Some definitions:
A valuation function $v : \mathcal{C}_n \to \mathbb{R}$ is said to be

- *monotone* when $v(C) \leq v(D)$, for $C \subseteq D \subseteq N$.
- *superadditive* when $v(C \cup D) \geq v(C) + v(D)$, for every pair of disjoint coalitions $C, D \subseteq N$.
- *supermodular* when $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$, for $C, D \subseteq N$.

A coalitional game (N, v) is *convex* iff v is supermodular.

5 Cooperative games

5.1. Consider a cooperative game which is defined on an undirected graph $G = (V, E)$. The players are the vertices in the graph and for $S \subseteq V$, $v(S) = |\{u \in V \mid N(u) \cap S \neq \emptyset\}|$. As usual $N(u) = \{v \mid (u, v) \in E\}$.

(a) Is the valuation function monotone? supperadditive? supermodular?

(b) Is the core empty? Can this property be decided in polynomial time?

5.2. Consider an undirected graph $G = (V, E)$ and two vertices s and t. Assume that $n = |V|$ and $m = |E|$. Consider the cooperative game $\mathcal{C}(s, t) = (N, v)$ where $N = E$ and, for $S \subseteq N$, $v(S)$ is m minus the length of the shortest path from s to t in the graph (V, S), if such a path exists, and 0 otherwise. Is this game supperadditive? Does the game have an empty core?

5.3. Consider the following game which is defined by a parameter k. Each participant has a collection of old vinyl disks, not all of them in good state, that are willing to share. A company is able to obtain an accurate re-recording of an album provided that at least k copies are provided. In this setting the value of a coalition is the number of albums for which an accurate recording is possible. You can assume that there are n participants and m different albums.

(a) Provide an expression of the valuation function.

(b) Is the game convex?

(c) Provide an accurate expression for the Shapley values. Can those values be computed in polynomial time?

5.4. The *unanimity game on N with respect to coalition $F \subseteq N$* is the game $\Gamma_F = (N, v_F)$ where

$$v_F(S) = \begin{cases} 1 & \text{if } F \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

(a) Can the core be empty? If so, analyze the computational complexity of deciding if the core is non-empty. When the core is nonempty, an outcome in the core can be computed in polynomial time?

(b) Can the Shapley values be computed in polynomial time?

5.5. **The diameter game.**

Consider a cooperative game which is defined on an undirected connected graph $G = (V, E)$. The players are the edges in the graph. For $X \subseteq E$, let $G_X = (V, X)$ be the graph formed by V and the edges in X. The valuation function is the following

$$v(X) = \begin{cases} 2|X| - diam(G_X) & \text{if } G_X \text{ is connected} \\ \frac{|X|}{2} & \text{otherwise,} \end{cases}$$

where $diam(H)$ is the diameter of the graph H.

(a) Is the valuation function monotone? supperadditive? supermodular?

(b) Are there connected graphs such that the core of the associated diameter game is non-empty?

5.6. Consider a simple game $(N, \mathcal{W})$. We say that player i is a

- *passer* iff, $\forall S \subseteq N$, if $i \in S$, then $S \in \mathcal{W}$.
- *vetoer* iff, $\forall S \subseteq N$, if $i \notin S$, then $S \in \mathcal{L}$.
- *dictator* iff, $\forall S \subseteq N$, $S \in \mathcal{W}$ iff $i \in S$.

(a) Provide a simpler characterization of those properties.

(b) Under which of the forms of representation of simple games based on sets can those properties being decided in polynomial time?

5.7. **Games on social networks** One of the criticism to simple games is the fact of assuming that any coalition can be formed. In the context in which the players participate in a social networks, a natural restriction on a coalition to take effect is that its members should at least be able to establish some level of communication among themselves.

For simplicity you can assume that a simple game $\Gamma = (N, \mathcal{W})$ is defined and that the social network is an undirected graph $H = (N, E)$.

On top of that we can came out with different combinations for defining winning coalitions in an associated *social game*, Γ_s on N. Consider the following options:

(a) A coalition X is winning in Γ_s iff X wins in Γ and $H[X]$ has no isolated vertices.

(b) A coalition X is winning in Γ_s iff X wins in Γ and $H[X]$ is connected.

(c) A coalition X is winning in Γ_s iff there is $Y \subseteq X$, so that Y wins in Γ and $H[Y]$ is connected.

Under which of the options (a), (b) or (c) is Γ_s a simple game?

5.8. Assume that a WVG is described by $\Gamma = (q; w_1, \ldots, w_n)$. Analyze the computational complexity of the following problems.

- Compute the smallest number of players that can form a wining coalition in Γ.
- Compute the biggest number of players that can form a losing coalition in Γ.

5.9. Assume that a WVG is described by $\Gamma = (q; w_1, \ldots, w_n)$. Analyze the computational complexity of the following problems.

- Given $\Gamma = (q; w_1, \ldots, w_n)$ and $i \in \{1 \ldots n\}$, is i a dummy player in Γ?
- Given $\Gamma = (q; w_1, \ldots, w_n)$ and $i, j \in \{1 \ldots n\}$, are i and j symmetric players in Γ?
- Given $\Gamma = (q; w_1, \ldots, w_n)$ and $i \in \{1 \ldots n\}$, compute the Banzhaf value of i in Γ?

5.10. Consider a the following decision process which runs on a leader-follower society with n members. The interaction among the society is represented by a bipartite directed graph $G = (L, F, E)$ having the following property: all the vertices in L have in-degree 0 and all the vertices in F have out-degree 0. The decision process is defined by two parameters α, $0 \leq \alpha \leq 1$ and q, $0 \leq q \leq n$.

When the society has to reach a decision about some topic, each member takes an initial position. We model this situation as an initial decision vector $x \in \{0, 1\}^n$. Then, each $i \in F$ looks at the values $p_{i1} = |\{(j, i) \in E \mid x_j = 1\}|$ and $p_{i0} = |\{(j, i) \in E \mid x_j = 0\}|$ and reconsiders its position according to the following algorithm.

- If $p_{i1} > \alpha(p_{i1} + p_{i0})$ and $p_{i0} < \alpha(p_{i1} + p_{i0})$, $x_i = 1$
- If $p_{i0} > \alpha(p_{i1} + p_{i0})$ and $p_{i1} < \alpha(p_{i1} + p_{i0})$, $x_i = 0$

Finally, the society reaches a "yea" (1) when $\sum_{i=1}^{n} x_i \geq q$, a "nay" (0) otherwise.

(a) Assuming that a coalition S is represented as the initial decision vector $x \in \{0, 1\}^n$ defined as $x_i = 1$ iff $i \in S$, the decision system process defines a cooperative game assigning to a coalition S a value in $v(S) \in \{0, 1\}$. Is this game simple?

(b) Provide a characterization of the games in the family with non-empty core.

(c) Can the Banzhaf value of player i be computed in polynomial time?

5.11. For a given undirected graph $G = (V, E)$, the associated *vertex cover game* has $N = V$ and in it a coalition wins iff and only if X is a vertex cover in G.

(a) Show that vertex cover games are simple games.

(b) Are there games in which the core is non-empty?

(c) Analyze the computational complexity of the IsProper and IsStrong problem on vertex cover games.

(d) Analyze the computational complexity of computing a winning coalition with smallest possible size.

5.12. For a given undirected graph $G = (V, E)$, the associated *domination game* has $N = V$ and in it a coalition wins if and only if X is a dominating set in G.

(a) Show that domination games are simple games.

(b) Are there domination games in which the core is non-empty?

(c) Analyze the computational complexity of the IsProper and IsStrong problem on domination games. (Hint. It might be useful to study the properties of the complement of a maximal independent set in G.)

(d) Analyze the computational complexity of computing a winning coalition with smallest possible size.

5.13. For a given undirected graph $G = (V, E)$, the associated *isolation game* has $N = V$ and in it a coalition X loses if and only if there is a vertex $u \in V$ such that $X \cap N[u] = \emptyset$.

(a) Show that isolation games are simple games.

(b) Are there isolation games in which the core is non-empty?

(c) Analyze the computational complexity of the IsProper and IsStrong problem on isolation games.

(d) Analyze the computational complexity of computing a winning coalition with smallest possible size.

6 Social Choice

6.1. A voting system is said to be *partition consistent* if whenever some alternative wins in all the subelections that results in partitioning the voters into two disjoints groups, this alternative also wins the election as a whole. Here we assume that the voters keep their preferences in all the subelections.

Show that Plurality is partition consistent but that Copeland is not.

6.2. In a weighthed election each player i has been assigned a weight w_i. This weight is used as a multiplying factor to the number of points that player i assigns to an alternative. In this way any voting rule assigning points can be extended to the weighted case.

The E-Coalitional-Weighted-Manipulation problem E-CWM is defined as follows. The input is a weighthed election (A, N, w) where player i has weight w_i, a set $M \subset N$, a preference P_i, for each player $i \notin M$, and an alternative $d \in A$. The question is whether it is possible to find preferences for the palyers in M so that in the joint preference profile c is a winner.

Given a Partition instance $(k_1, \ldots, k_n)$ with $\sum_{i=1}^{n} k_i = 2K$, we construct the following instance of the Copeland-CWM problem.

- $A = \{a, b, c, d\}$ and d is the distinguished alternative that we want to make a winner.
- The weights and preference of the 4 voters not in M are

Weight	Preference
2K+2	$d > a > b > c$
2K+2	$c > d > b > a$
K+1	$a > b > c > d$
K+1	$b > a > c > d$

- M has n players, player $m_i \in M$ has weight k_i.

Using this construction show that Copeland-CWM is NP-complete.

6.3. Consider the CONSTRUCTIVE COALITIONAL MANIPULATION problem (CCM). We are given a set of alternatives A, the preferences on A of a set of voters S (the nonmanipulators' votes), another set of voters T whose preferences are still open (the manipulators' votes), and a preferred candidate $p \in A$. We are asked whether there is a way to cast the votes in T so that p wins the election.

Let us consider another voting system: *Cup (sequential binary comparisons)*. The cup is defined by a balanced binary tree T with one leaf per candidate, and an assignment of candidates to leaves (each leaf gets one candidate). Each non-leaf node is assigned the winner of the pairwise election of the node's children; the candidate assigned to the root wins.

The cup voting protocol assumes that T and the assignment of candidates to leaves is known by the voters before they vote.

In the binary tree representing the cup, we can consider each node to be a subelection. When considering the CCM problem, we may say that the voters in T only order the candidates in that subelection since the place of the other candidates in the order is irrelevant for the subelection.

We say that a candidate can obtain a particular result in the subelection if it does so for some coalitional vote on T. This defines the set of potential winners for each subelection.

(a) Show that a candidate can win a subelection at node u in T if and only if it can win in one of its children subelections, and it can defeat one of the potential winners of the sibling child in a pairwise election.

(b) Show that, for the cup voting protocol (given the tree and the assignment of candidates to leaves), CCM can be solved in polynomial time.