

Some definitions:

A valuation function $v : \mathcal{C}_n \rightarrow \mathbb{R}$ is said to be

- *monotone* when $v(C) \leq v(D)$, for $C \subseteq D \subseteq N$.
- *superadditive* when $v(C \cup D) \geq v(C) + v(D)$, for every pair of disjoint coalitions $C, D \subseteq N$.
- *supermodular* when $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$, for $C, D \subseteq N$.

A coalitional game (N, v) is *convex* iff v is supermodular.

5 Cooperative games

- 5.1. Consider a cooperative game which is defined on an undirected graph $G = (V, E)$. The players are the vertices in the graph and for $S \subseteq V$, $v(S) = |\{u \in V \mid N(u) \cap S \neq \emptyset\}|$. As usual $N(u) = \{v \mid (u, v) \in E\}$.
- (a) Is the valuation function monotone? superadditive? supermodular?
 - (b) Is the core empty? Can this property be decided in polynomial time?
- 5.2. Consider an undirected graph $G = (V, E)$ and two vertices s and t . Assume that $n = |V|$ and $m = |E|$. Consider the cooperative game $\mathcal{C}(s, t) = (N, v)$ where $N = E$ and, for $S \subseteq N$, $v(S)$ is m minus the length of the shortest path from s to t in the graph (V, S) , if such a path exists, and 0 otherwise. Is this game superadditive? Does the game have an empty core?
- 5.3. Consider the following game which is defined by a parameter k . Each participant has a collection of old vinyl disks, not all of them in good state, that are willing to share. A company is able to obtain an accurate re-recording of an album provided that at least k copies are provided. In this setting the value of a coalition is the number of albums for which an accurate recording is possible. You can assume that there are n participants and m different albums.
- (a) Provide an expression of the valuation function.
 - (b) Is the game convex?
 - (c) Provide an accurate expression for the Shapley values. Can those values be computed in polynomial time?

5.4. The *unanimity game* on N with respect to coalition $F \subseteq N$ is the game $\Gamma_F = (N, v_F)$ where

$$v_F(S) = \begin{cases} 1 & \text{if } F \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Can the core be empty? If so, analyze the computational complexity of deciding if the core is non-empty. When the core is nonempty, an outcome in the core can be computed in polynomial time?
- (b) Can the Shapley values be computed in polynomial time?

5.5. **The diameter game.**

Consider a cooperative game which is defined on an undirected connected graph $G = (V, E)$. The players are the edges in the graph. For $X \subseteq E$, let $G_X = (V, X)$ be the graph formed by V and the edges in X . The valuation function is the following

$$v(X) = \begin{cases} 2|X| - \text{diam}(G_X) & \text{if } G_X \text{ is connected} \\ \frac{|X|}{2} & \text{otherwise,} \end{cases}$$

where $\text{diam}(H)$ is the diameter of the graph H .

- (a) Is the valuation function monotone? superadditive? supermodular?
- (b) Are there connected graphs such that the core of the associated diameter game is non-empty?

5.6. Consider a simple game (N, \mathcal{W}) . We say that player i is a

- *passer* iff, $\forall S \subseteq N$, if $i \in S$, then $S \in \mathcal{W}$.
- *vetoer* iff, $\forall S \subseteq N$, if $i \notin S$, then $S \in \mathcal{L}$.
- *dictator* iff, $\forall S \subseteq N$, $S \in \mathcal{W}$ iff $i \in S$.

- (a) Provide a simpler characterization of those properties.
- (b) Under which of the forms of representation of simple games based on sets can those properties being decided in polynomial time?

5.7. **Games on social networks** One of the criticism to simple games is the fact of assuming that any coalition can be formed. In the context in which the players participate in a social networks, a natural restriction on a coalition to take effect is that its members should at least be able to establish some level of communication among themselves.

For simplicity you can assume that a simple game $\Gamma = (N, \mathcal{W})$ is defined and that the social network is an undirected graph $H = (N, E)$.

On top of that we can come out with different combinations for defining winning coalitions in an associated *social game*, Γ_s on N . Consider the following options:

- (a) A coalition X is winning in Γ_s iff X wins in Γ and $H[X]$ has no isolated vertices.
- (b) A coalition X is winning in Γ_s iff X wins in Γ and $H[X]$ is connected.
- (c) A coalition X is winning in Γ_s iff there is $Y \subseteq X$, so that Y wins in Γ and $H[Y]$ is connected.

Under which of the options (a), (b) or (c) is Γ_s a simple game?

5.8. Assume that a WVG is described by $\Gamma = (q; w_1, \dots, w_n)$. Analyze the computational complexity of the following problems.

- Compute the smallest number of players that can form a winning coalition in Γ .
- Compute the biggest number of players that can form a losing coalition in Γ .

5.9. Assume that a WVG is described by $\Gamma = (q; w_1, \dots, w_n)$. Analyze the computational complexity of the following problems.

- Given $\Gamma = (q; w_1, \dots, w_n)$ and $i \in \{1 \dots n\}$, is i a dummy player in Γ ?
- Given $\Gamma = (q; w_1, \dots, w_n)$ and $i, j \in \{1 \dots n\}$, are i and j symmetric players in Γ ?
- Given $\Gamma = (q; w_1, \dots, w_n)$ and $i \in \{1 \dots n\}$, compute the Banzhaf value of i in Γ ?

5.10. Consider the following decision process which runs on a leader-follower society with n members. The interaction among the society is represented by a bipartite directed graph $G = (L, F, E)$ having the following property: all the vertices in L have in-degree 0 and all the vertices in F have out-degree 0. The decision process is defined by two parameters α , $0 \leq \alpha \leq 1$ and q , $0 \leq q \leq n$.

When the society has to reach a decision about some topic, each member takes an initial position. We model this situation as an initial decision vector $x \in \{0, 1\}^n$. Then, each $i \in F$ looks at the values $p_{i1} = |\{(j, i) \in E \mid x_j = 1\}|$ and $p_{i0} = |\{(j, i) \in E \mid x_j = 0\}|$ and reconsiders its position according to the following algorithm.

- If $p_{i1} > \alpha(p_{i1} + p_{i0})$ and $p_{i0} < \alpha(p_{i1} + p_{i0})$, $x_i = 1$
- If $p_{i0} > \alpha(p_{i1} + p_{i0})$ and $p_{i1} < \alpha(p_{i1} + p_{i0})$, $x_i = 0$

Finally, the society reaches a "yea" (1) when $\sum_{i=1}^n x_i \geq q$, a "nay" (0) otherwise.

- (a) Assuming that a coalition S is represented as the initial decision vector $x \in \{0, 1\}^n$ defined as $x_i = 1$ iff $i \in S$, the decision system process defines a cooperative game assigning to a coalition S a value in $v(S) \in \{0, 1\}$. Is this game simple?
- (b) Provide a characterization of the games in the family with non-empty core.
- (c) Can the Banzhaf value of player i be computed in polynomial time?

- 5.11. For a given undirected graph $G = (V, E)$, the associated *vertex cover game* has $N = V$ and in it a coalition wins iff and only if X is a vertex cover in G .
- (a) Show that vertex cover games are simple games.
 - (b) Are there games in which the core is non-empty?
 - (c) Analyze the computational complexity of the ISPROPER and ISSTRONG problem on vertex cover games.
 - (d) Analyze the computational complexity of computing a winning coalition with smallest possible size.
- 5.12. For a given undirected graph $G = (V, E)$, the associated *domination game* has $N = V$ and in it a coalition wins if and only if X is a dominating set in G .
- (a) Show that domination games are simple games.
 - (b) Are there domination games in which the core is non-empty?
 - (c) Analyze the computational complexity of the ISPROPER and ISSTRONG problem on domination games. (Hint. It might be useful to study the properties of the complement of a maximal independent set in G .)
 - (d) Analyze the computational complexity of computing a winning coalition with smallest possible size.
- 5.13. For a given undirected graph $G = (V, E)$, the associated *isolation game* has $N = V$ and in it a coalition X loses if and only if there is a vertex $u \in V$ such that $X \cap N[u] = \emptyset$.
- (a) Show that isolation games are simple games.
 - (b) Are there isolation games in which the core is non-empty?
 - (c) Analyze the computational complexity of the ISPROPER and ISSTRONG problem on isolation games.
 - (d) Analyze the computational complexity of computing a winning coalition with smallest possible size.

6 Social Choice

- 6.1. A voting system is said to be *partition consistent* if whenever some alternative wins in all the subelections that results in partitioning the voters into two disjoint groups, this alternative also wins the election as a whole. Here we assume that the voters keep their preferences in all the subelections.

Show that Plurality is partition consistent but that Copeland is not.

- 6.2. In a weighted election each player i has been assigned a weight w_i . This weight is used as a multiplying factor to the number of points that player i assigns to an alternative. In this way any voting rule assigning points can be extended to the weighted case.

The E-COALITIONAL-WEIGHTED-MANIPULATION problem E-CWM is defined as follows. The input is a weighted election (A, N, w) where player i has weight w_i , a set $M \subset N$, a preference P_i , for each player $i \notin M$, and an alternative $d \in A$. The question is whether it is possible to find preferences for the players in M so that in the joint preference profile c is a winner.

Given a PARTITION instance (k_1, \dots, k_n) with $\sum_{i=1}^n k_i = 2K$, we construct the following instance of the Copeland-CWM problem.

- $A = \{a, b, c, d\}$ and d is the distinguished alternative that we want to make a winner.
- The weights and preference of the 4 voters not in M are

Weight	Preference
2K+2	$d > a > b > c$
2K+2	$c > d > b > a$
K+1	$a > b > c > d$
K+1	$b > a > c > d$

- M has n players, player $m_i \in M$ has weight k_i .

Using this construction show that Copeland-CWM is NP-complete.

- 6.3. Consider the CONSTRUCTIVE COALITIONAL MANIPULATION problem (CCM). We are given a set of alternatives A , the preferences on A of a set of voters S (the nonmanipulators' votes), another set of voters T whose preferences are still open (the manipulators' votes), and a preferred candidate $p \in A$. We are asked whether there is a way to cast the votes in T so that p wins the election.

Let us consider another voting system: *Cup (sequential binary comparisons)*. The cup is defined by a balanced binary tree T with one leaf per candidate, and an assignment of candidates to leaves (each leaf gets one candidate). Each non-leaf node is assigned the winner of the pairwise election of the node's children; the candidate assigned to the root wins.

The cup voting protocol assumes that T and the assignment of candidates to leaves is known by the voters before they vote.

In the binary tree representing the cup, we can consider each node to be a subelection. When considering the CCM problem, we may say that the voters in T only order the candidates in that subelection since the place of the other candidates in the order is irrelevant for the subelection.

We say that a candidate can obtain a particular result in the subelection if it does so for some coalitional vote on T . This defines the set of potential winners for each subelection.

- (a) Show that a candidate can win a subelection at node u in T if and only if it can win in one of its children subelections, and it can defeat one of the potential winners of the sibling child in a pairwise election.
- (b) Show that, for the cup voting protocol (given the tree and the assignment of candidates to leaves), CCM can be solved in polynomial time.