

Simple Games

Spring 2026

- 1 Simple Games
- 2 Coalitional solution concepts
- 3 Problems on simple games
- 4 IsWeighted
- 5 Dimension

Simple Games

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- A simple game can be described by a pair (N, \mathcal{W}) :
 - N is a set of players,
 - $\mathcal{W} \subseteq \mathcal{P}(N)$ is a monotone set of **winning coalitions**, those coalitions X with $v(X) = 1$.
 - $\mathcal{L} = \mathcal{C}_N \setminus \mathcal{W}$ is the set of **losing coalitions** those coalitions X with $v(X) = 0$.

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- Members of $N = \{1, \dots, n\}$ are called **players** or **voters**.

Simple games: Representation

Due to monotonicity, any one of the following families of coalitions define a simple game on a set of players N :

- *winning coalitions* \mathcal{W} .
- *losing coalitions* \mathcal{L} .
- *minimal winning coalitions* \mathcal{W}^m

$$\mathcal{W}^m = \{X \in \mathcal{W}; \forall Z \in \mathcal{W}, Z \not\subseteq X\}$$
- *maximal losing coalitions* \mathcal{L}^M

$$\mathcal{L}^M = \{X \in \mathcal{L}; \forall Z \in \mathcal{L}, X \not\subseteq Z\}$$

This provides us with many representation forms for simple games.

Weighted voting games

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A simple game for which there exists a **quota** q and it is possible to assign to each $i \in N$ a **weight** w_i , so that

$$X \in \mathcal{W} \text{ iff } \sum_{i \in X} w_i \geq q.$$

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$$X \in \mathcal{W} \text{ iff } \sum_{i \in X} w_i \geq q.$$

- WVG can be represented by a tuple of integers $(q; w_1, \dots, w_n)$.
as **any weighted game admits such an integer realization**,
[Carreras and Freixas, Math. Soc.Sci., 1996]

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The game with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.

- Assume it is given by $(q; w_1, w_2, w_3, w_4)$.
- We have $w_1 + w_2 \geq q$ and $w_3 + w_4 \geq q$.
- Thus $\max\{w_1, w_2\} \geq q/2$ and $\max\{w_3, w_4\} \geq q/2$,
- So, $\max\{w_1, w_2\} + \max\{w_3, w_4\} \geq q$ which cannot be.

Input representations

- Simple Games
 - (N, \mathcal{W}) : extensive winning, (N, \mathcal{W}^m) : minimal winning
 - (N, \mathcal{L}) : extensive losing, (N, \mathcal{L}^M) maximal losing
 - (N, C) : monotone circuit winning
 - (N, F) : monotone formula winning,
- Weighted voting games: $(q; w_1, \dots, w_n)$

All numbers are integers

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The core of simple games

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- It is standard to assume that the grand coalition forms, even if the simple game is not superadditive.
- A player is a **veto player** if any coalition that does not contain it is losing. i.e., $v(C) = 0$, for any $C \subseteq N \setminus \{i\}$.
- Ex: Consider the unanimity game (N, v) where $v(C) = 0$, if $C \neq N$ and $v(N) = 1$.
The game indeed is a simple game and can be described in (minimal) winning form by $(N, \{N\})$.
In the unanimity game all players are veto players.

The core of simple games

Theorem

A simple game has non-empty core iff it has a veto player.

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A simple game has non-empty core iff it has a veto player.

- If Γ has a veto player i .
 - Consider the payoff $x_i = 1$ and $x_j = 0$, for $j \neq i$
 - For C with $i \in C$, $v(C) \leq 1$ and $x(C) = 1$.
 - For C with $i \notin C$, $v(C) = 0$ and $x(C) = 0$.
 - Thus, x is in the core.

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 - For C with $i \notin C$, $v(C) = 0$ and $x(C) = 0$.
 - Thus, x is in the core.
- If Γ has a non-empty core.
 - Consider a payoff x that is in the core.
 - $x(N) = v(N) = 1$, so there exists i with $x_i > 0$.
 - So, $x(N \setminus \{i\}) < 1$. But then, $v(N \setminus \{i\}) = 0$.
 - Then, by monotonicity, for C with $i \notin C$ $v(C) = 0$.
 - Thus, player i is a veto player.

Is the core empty?

- Determining if the core is empty or not can be done by checking for every player whether it is a veto player or not.
- For this it is enough to check whether $v(N \setminus \{i\}) = 0$.
- For reasonable v , polynomial time computable, this can be done in poly time

Shapley value and Banzhaf index

- Player i is **pivotal** for coalition C if $v(C) = 1$ and $v(C \setminus \{i\}) = 0$.
- The Shapley-Shubik index

$$\phi_i(\Gamma) = \frac{1}{n!} |\{\pi \mid i \text{ is pivotal for } S_\pi(i)\}|$$

$\phi_i(\Gamma)$ is the probability that the arrival of player i turns a losing coalition into a winning one.

- The Banzhaf index

$$\beta_i(\Gamma) = \frac{1}{2^{n-1}} |\{C \mid i \text{ is pivotal for } C\}|$$

$\beta_i(\Gamma)$ gives the probability of being pivotal over **random** coalitions.

When v is polytime computable, the associated problems belong to $\#P$.

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Problems on simple games

In general we state a property P , for simple games, and consider the associated decision problem which has the form:

Name: IsP

Input: A simple game/WVG/VWVG Γ

Question: Does Γ satisfy property P ?

Four properties

A simple game (N, \mathcal{W}) is

- **strong** if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$.
- **proper** if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.
- a **weighted voting game**.
- a **vector weighted voting game**.

IsStrong: Simple Games

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Proof

IsStrong: Simple Games losing forms

Γ is **strong** if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- Γ is **not strong** iff

$$\exists S \subseteq N : S \in \mathcal{L} \wedge N \setminus S \in \mathcal{L}$$

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- Γ is **not strong** iff

$$\exists S \subseteq N : S \in \mathcal{L} \wedge N \setminus S \in \mathcal{L}$$

which is equivalent to

$$\exists S \subseteq N : \exists L_1, L_2 \in \mathcal{L}^M : S \subseteq L_1 \wedge N \setminus S \subseteq L_2$$

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- This can be checked in polynomial time, given \mathcal{L}^M .

IsStrong: explicit forms

Γ is **strong** if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- Given (N, \mathcal{L}) , \mathcal{L}^M can be obtained in polynomial time.

IsStrong: explicit forms

Γ is **strong** if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- Given (N, \mathcal{L}) , \mathcal{L}^M can be obtained in polynomial time.
- Given (N, \mathcal{W}) , for $i \in N$ consider the family $\mathcal{W}_i = \{X \setminus \{i\} \mid X \in \mathcal{W}\}$ and $R = \cup_{i \in N} \mathcal{W}_i$.
- All the coalitions in $S = R \setminus \mathcal{W}$ are losing coalitions.
- Furthermore for a coalition $X \in \mathcal{L}^M$ and $i \notin X$, $X \cup \{i\} \in \mathcal{W}$.
- Therefore, $\mathcal{L}^M \subseteq S$ and $S^M = \mathcal{L}^M$.
- Then, we can compute \mathcal{L}^M from \mathcal{W} in polynomial time.

end proof

IsStrong: minimal winning forms

Γ is **strong** if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

Theorem

The ISSTRONG problem is coNP-complete when the input game is given in explicit minimal winning form.

Proof

IsStrong: minimal winning forms

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Proof

- The property can be expressed as

$$\forall S [(S \in \mathcal{W}) \text{ or } (S \notin \mathcal{W} \text{ and } N \setminus S \in \mathcal{W})]$$

- Observe that the property $S \in \mathcal{W}$ can be checked in polynomial time given S and \mathcal{W}^m .
- Thus the problem belongs to coNP.

IsStrong: minimal winning forms

- We provide a polynomial time reduction from the complement of the NP-complete **set splitting** problem.
- An instance of the **set splitting problem** is a collection C of subsets of a finite set N . The question is whether it is possible to partition N into two subsets P and $N \setminus P$ such that no subset in C is entirely contained in either P or $N \setminus P$.

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- We have to decide whether $P \subseteq N$ exists such that

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We associate to a set splitting instance (N, C) the simple game in explicit minimal winning form (N, C^m) .

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- Now assume that $P \subseteq N$ satisfies

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- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in C contains a set in C^m .

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- This means that P and $N \setminus P$ are losing coalitions in the game (N, C^m) .
- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$.
- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in C contains a set in C^m .
- Therefore, (N, C) has a set splitting iff (N, C^m) is not strong.

end proof

IsProper: winning forms

Γ is **proper** if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

Theorem

The ISPROPER problem, when the game is given in explicit winning or losing form or in minimal winning form, can be solved in polynomial time.

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Proof

- As before, given a family of subsets F , we can check, for any set in F , whether its complement is not in F in polynomial time. Taking into account the definitions, the ISPROPER problem is polynomial time solvable for the explicit forms

Is Proper: winning forms

- Γ is not proper iff

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- Which can be checked in polynomial time when \mathcal{W}^m is given.

end proof

IsProper: maximal losing form

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Theorem

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- A game is *not proper* iff

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- which is equivalent to

$$\exists S \subseteq N : \forall T_1, T_2 \in \mathcal{L}^M : S \not\subseteq T_1 \wedge N \setminus S \not\subseteq T_2$$

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- Therefore ISPROPER belongs to coNP.

IsProper: maximal losing form

To show that the problem is also coNP-hard we provide a reduction from the IsStrong problem for games given in extensive minimal winning form.

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To show that the problem is also coNP-hard we provide a reduction from the IsStrong problem for games given in extensive minimal winning form.

- If a family C of subsets of N is minimal then the family $\{N \setminus L : L \in C\}$ is maximal.
- Given a game $\Gamma = (N, \mathcal{W}^m)$, in minimal winning form, we construct the game $\Gamma' = (N, \{N \setminus L : L \in \mathcal{W}^m\})$ in maximal losing form.
- Which can be obtained in polynomial time.

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- Which can be obtained in polynomial time.
- Besides, Γ is strong iff Γ' is proper.

end proof

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Explicit forms

Lemma

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We can obtain \mathcal{W}^m and \mathcal{L}^M in polynomial time.

Once this is done we write, in polynomial time, the LP

$$\begin{array}{ll}
 \min q & \\
 \text{subject to} & w(S) \geq q \quad \text{if } S \in \mathcal{W}^m \\
 & w(S) < q \quad \text{if } S \in \mathcal{L}^M \\
 & 0 \leq w_i \quad \text{for all } 1 \leq i \leq n \\
 & 0 \leq q
 \end{array}$$

IsWeighted: Minimal and Maximal

Lemma

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- For $C \subseteq N$ we let $x_C \in \{0, 1\}^n$ denote the vector with the i 'th coordinate equal to 1 if and only if $i \in C$.

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Proof

- For $C \subseteq N$ we let $x_C \in \{0, 1\}^n$ denote the vector with the i 'th coordinate equal to 1 if and only if $i \in C$.
- In polynomial time we compute the boolean function Φ_{W^m} given by the DNF:

$$\Phi_{W^m}(x) = \bigvee_{S \in W^m} (\bigwedge_{i \in S} x_i)$$

IsWeighted: Minimal winning

By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \Leftrightarrow C \text{ is winning in the game given by } (N, W^m)$$

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IsWeighted: Minimal winning

By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \Leftrightarrow C \text{ is winning in the game given by } (N, W^m)$$

- It is well known that Φ_{W^m} is a threshold function iff the game given by (N, W^m) is weighted.
- Further Φ_{W^m} is monotonic (i.e. *positive*)
- But deciding whether a monotonic formula describes a threshold function can be solved in polynomial time.

IsWeighted: Maximal losing

- we can prove a similar result given (N, L^M) .
- The **dual** of game $\Gamma = (N, \mathcal{W})$ is the game $\Gamma^d = (N, \mathcal{W}^d)$ where $S \in \mathcal{W}^d$ iff $N \setminus S \notin \mathcal{W}$.
- Observe that Γ is weighted iff Γ^d is weighted.
- We can compute a monotone CNF formula describing the losing coalitions of Γ . Negating this formula we get a DNF on negated variables. Replacing \bar{x}_i by y_i we get a DNF describing \mathcal{W}^d .
- As the formula can be computed in polynomial time the result follows.

end proof

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Both are simple games

- A simple game Γ is a **vector weighted voting game** if there are WVGs $\Gamma_1, \dots, \Gamma_k$, for some $k \geq 1$, so that $\Gamma = \Gamma_1 \cap \dots \cap \Gamma_k$.

Vector weighted voting games

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 - union** $\Gamma_1 \cup \Gamma_2 = (N, \mathcal{W}_1 \cup \mathcal{W}_2)$
 - intersection** $\Gamma_1 \cap \Gamma_2 = (N, \mathcal{W}_1 \cap \mathcal{W}_2)$

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- A simple game Γ is a **vector weighted voting game** if there are WVGs $\Gamma_1, \dots, \Gamma_k$, for some $k \geq 1$, so that $\Gamma = \Gamma_1 \cap \dots \cap \Gamma_k$.
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- Any simple game is a vector weighted voting game.
- The **dimension** of a simple games is the minimum number of WWGs that allows its representation as VWVG

Dimension

Theorem

For a simple game $\Gamma = (N, \mathcal{L}^M)$, $\dim(\Gamma) \leq |\mathcal{L}^M|$.

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- The intersection of the above games describes Γ , as a minimal winning coalition cannot be a subset of any losing coalition.



Hardness

Theorem

Let d_1 and d_2 two fixed integers with $1 \leq d_2 < d_1$. Then the problem of deciding whether the intersection of d_1 WVGs can be represented as the intersection of d_2 WVGs is NP-hard.

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The reduction is from the NP-complete problem

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Input: $n + 1$ integer values, x_1, \dots, x_n and b

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- In G_ℓ , $1 \leq \ell \leq d$
 - $w_\ell(p_i) = 2x_i$, $1 \leq i \leq k$.
 - For $1 \leq j \leq d$ and $j \neq \ell$, $w_\ell(q_j) = w_\ell(q'_j) = 0$
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- $\Gamma(I)$ is the intersection of $\Gamma_1, \dots, \Gamma_d$

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- If $x(S) \leq b - 1$, for $1 \leq \ell \leq d$, $w_\ell(S) \leq 2x(S) + 2 \leq 2b + 1$, and S loses in Γ_ℓ .
- So, $\Gamma(I)$ is equivalent to the WVG in which $w(p_i) = x_i$, $1 \leq i \leq k$, $w(q_j) = w(q'_j) = 0$, $1 \leq j \leq d$, and $q = b + 1$

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- Let $\alpha \neq \beta$ such that $h(\alpha) = h(\beta) = \gamma$.

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- Let $\alpha \neq \beta$ such that $h(\alpha) = h(\beta) = \gamma$, consider H_γ
 - Let W be the total weight of players p_i , $1 \leq i \leq k$, and q_j, q'_j , $j \notin \{\alpha, \beta\}$
 - Let $w_\alpha, w_\beta, w'_\alpha, w'_\beta$ the weight of players $q_\alpha, q_\beta, q'_\alpha, q'_\beta$.
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- The coalition with players p_i with $i \in J$ and $\{q'_1, \dots, q'_d\}$ wins in $\Gamma(I)$, so $W + w'_\alpha + w'_\beta \geq q^*$.
- So, $2W + w_\alpha + w_\beta + w'_\alpha + w'_\beta$ must be $> q^*$ and $\leq q^*$, we get a contradiction

End Proof

Hardness

Theorem

Computing the dimension of a VWVG is NP-hard

Theorem

The ISWEIGHTED problem is NP hard for VWVGs

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