Simple Games

Spring 2022

Simple Games

- Simple Games were introduced by (Taylor & Zwicker, 1999)

Simple Games

- Simple Games were introduced by (Taylor & Zwicker, 1999)
- A simpe game is a cooperative game (N, v) such that $v : \mathcal{C}_N \to \{0, 1\}$ and it is monotone.

Simple Games

- Simple Games were introduced by (Taylor & Zwicker, 1999)
- A simpe game is a cooperative game (N, v) such that $v : \mathcal{C}_N \to \{0, 1\}$ and it is monotone.
- A simple game can be described by a pair $(N, \mathcal{W})$:
  - N is a set of players,
  - $\mathcal{W} \subseteq \mathcal{P}(N)$ is a monotone set of winning coalitions, those coalitions X with $v(X) = 1$.
  - $\mathcal{L} = \mathcal{C}_N \backslash \mathcal{W}$ is the set of losing coalitions those coalitions X with $v(X) = 0$.

Simple Games

- Simple Games were introduced by (Taylor & Zwicker, 1999)
- A simpe game is a cooperative game (N, v) such that $v : \mathcal{C}_N \to \{0, 1\}$ and it is monotone.
- A simple game can be described by a pair $(N, \mathcal{W})$:
  - N is a set of players,
  - $\mathcal{W} \subseteq \mathcal{P}(N)$ is a monotone set of winning coalitions, those coalitions X with $v(X) = 1$.
  - $\mathcal{L} = \mathcal{C}_N \backslash \mathcal{W}$ is the set of losing coalitions those coalitions X with $v(X) = 0$.
- Members of $N = \{1, \ldots, n\}$ are called players or voters.

Simple games: Representation

Due to monotonicity, any one of the following families of coalitions define a simple game on a set of players N:

- *winning coalitions* $\mathcal{W}$.
- *losing coalitions* $\mathcal{L}$.
- *minimal winning coalitions* $\mathcal{W}^m$
  $\mathcal{W}^m = \{X \in \mathcal{W}; \forall Z \in \mathcal{W}, Z \not\subseteq X\}$
- *maximal losing coalitions* $\mathcal{L}^M$
  $\mathcal{L}^M = \{X \in \mathcal{L}; \forall Z \in \mathcal{L}, X \not\subseteq L\}$

This provides us with many representation forms for simple games.

Weighted voting games

Weighted voting games

- Weighted voting games (WVG)

Weighted voting games

- Weighted voting games (WVG)
  A simple game for which there exists a quota q and it is possible to assign to each $i \in N$ a weight w_i, so that
  $$X \in \mathcal{W} \text{ iff } \sum_{i \in X} w_i \geq q.$$

Weighted voting games

- Weighted voting games (WVG)
  A simple game for which there exists a quota q and it is possible to assign to each $i \in N$ a weight w_i, so that
  $$X \in \mathcal{W} \text{ iff } \sum_{i \in X} w_i \geq q.$$
- WVG can be represented by a tuple of integers
  $(q; w_1, \ldots, w_n)$.
  as any weighted game admits such an integer realization,
  [Carreras and Freixas, Math. Soc.Sci., 1996]

Vector weighted voting games

Vector weighted voting games

Vector weighted voting games

- Given two simple games $\Gamma_1 = (N, \mathcal{W}_1)$ and $\Gamma_2 = (N, \mathcal{W}_1)$ we can define their

Vector weighted voting games

- Given two simple games $\Gamma_1 = (N, \mathcal{W}_1)$ and $\Gamma_2 = (N, \mathcal{W}_1)$ we can define their
  - union

Vector weighted voting games

- Given two simple games $\Gamma_1 = (N, \mathcal{W}_1)$ and $\Gamma_2 = (N, \mathcal{W}_1)$ we can define their
  - union $\Gamma_1 \cup \Gamma_2 = (N, \mathcal{W}_1 \cup \mathcal{W}_2)$

Vector weighted voting games

- Given two simple games $\Gamma_1 = (N, \mathcal{W}_1)$ and $\Gamma_2 = (N, \mathcal{W}_1)$ we can define their
  - union $\Gamma_1 \cup \Gamma_2 = (N, \mathcal{W}_1 \cup \mathcal{W}_2)$
  - intersection

Vector weighted voting games

- Given two simple games $\Gamma_1 = (N, \mathcal{W}_1)$ and $\Gamma_2 = (N, \mathcal{W}_1)$ we can define their
  - union $\Gamma_1 \cup \Gamma_2 = (N, \mathcal{W}_1 \cup \mathcal{W}_2)$
  - intersection $\Gamma_1 \cap \Gamma_2 = (N, \mathcal{W}_1 \cap \mathcal{W}_2)$

  Both are simple games

Vector weighted voting games

- Given two simple games $\Gamma_1 = (N, \mathcal{W}_1)$ and $\Gamma_2 = (N, \mathcal{W}_1)$ we can define their
  - union $\Gamma_1 \cup \Gamma_2 = (N, \mathcal{W}_1 \cup \mathcal{W}_2)$
  - intersection $\Gamma_1 \cap \Gamma_2 = (N, \mathcal{W}_1 \cap \mathcal{W}_2)$

  Both are simple games
- A simple game Γ is a vector weighted voting game if there are WVGs $\Gamma_1, \ldots, \Gamma_k$, for some $k \geq 1$, so that $\Gamma = \Gamma_1 \cap \cdots \cap \Gamma_k$.

Representability

Representability

- There are simple games that are not WVGs

Representability

- There are simple games that are not WVGs
  The game with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.

Representability

- There are simple games that are not WVGs
  The game with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.
  - Assume it is given by $(q; w_1, w_2, w_3, w_4)$.

Representability

- There are simple games that are not WVGs
  The game with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.
  - Assume it is given by $(q; w_1, w_2, w_3, w_4)$.
  - We have $w_1 + w_2 \geq q$ and $w_3 + w_4 \geq q$.
  - Thus $\max\{w_1, w_2\} \geq q/2$ and $\max\{w_3, w_4\} \geq q/2$,
  - So, $\max\{w_1, w_2\} + \max\{w_3, w_4\} \geq q$ which cannot be.

Representability

Representability

- Any simple game Γ is a VWVGs

Representability

- Any simple game Γ is a VWVGs
  - Take a losing coalition C and consider the game in which players in C have weight 0 and players outside C 1, set the quote to 1.
    Any set that is not contained in C wins!

Representability

- Any simple game Γ is a VWVGs
  - Take a losing coalition C and consider the game in which players in C have weight 0 and players outside C 1, set the quote to 1.
    Any set that is not contained in C wins!
  - The intersection of the above games describes Γ.

Representability

- Any simple game Γ is a VWVGs
  - Take a losing coalition C and consider the game in which players in C have weight 0 and players outside C 1, set the quote to 1.
    Any set that is not contained in C wins!
  - The intersection of the above games describes Γ.
    A winning coalition cannot be a subset of any losing coalition.

Representability

- Any simple game Γ is a VWVGs
  - Take a losing coalition C and consider the game in which players in C have weight 0 and players outside C 1, set the quote to 1.
    Any set that is not contained in C wins!
  - The intersection of the above games describes Γ.
    A winning coalition cannot be a subset of any losing coalition.
- The dimension of a simple games is the minimum number of WVGs that allows its representation as VWVG

A representation as WVGs

A representation as WVGs

The game Γ with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.

A representation as WVGs

The game Γ with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.

- The maximal losing coalitions are $\{\{1, 3\}, \{1, 4\}, \{2, 3\}\{2, 4\}\}$

A representation as WVGs

The game Γ with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.

- The maximal losing coalitions are $\{\{1, 3\}, \{1, 4\}, \{2, 3\}\{2, 4\}\}$
- This gives four WVG, according to the previous construction

$$\Gamma = [1; 0, 1, 0, 1] \cap [1; 0, 1, 1, 0] \cap [1; 1, 0, 0, 1] \cap [1; 1, 0, 1, 0].$$

Input representations

- Simple Games
  $(N, \mathcal{W})$: extensive wining, $(N, \mathcal{W}^m)$: minimal wining
  $(N, \mathcal{L})$: extensive losing, $(N, \mathcal{L}^M)$ maximal losing
  (N, C): monotone circuit winning
  (N, F): monotone formula winning,
- Weighted voting games: $(q; w_1, \ldots, w_n)$
- Vector weighted voting games:
  $(q_1; w_1^1, \ldots, w_n^1), \ldots, (q_k; w_1^k, \ldots, w_n^k)$

All numbers are integers

Problems on simple games

In general we state a property $\mathtt{P}$, for simple games, and consider the associated decision problem which has the form:

> *Name:* $\textsc{IsP}$
> *Input: A simple game/WVG/VWVG* Γ
> *Question: Does* Γ *satisfy property* P*?*

Four properties

A simple game $(N, \mathcal{W})$ is

- strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$.
- proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.
- a weighted voting game.
- a vector weighted voting game.

IsStrong: Simple Games

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

IsStrong: Simple Games

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

Theorem

The IsStrong *problem, when Γ is given in explicit winning or losing form or in maximal losing form can be solved in polynomial time.*

IsStrong: Simple Games

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

Theorem

The IsStrong *problem, when Γ is given in explicit winning or losing form or in maximal losing form can be solved in polynomial time.*

- First observe that, given a family of subsets F, we can check, for any set in F, whether its complement is not in F in polynomial time.
- Therefore, the IsStrong problem, when the input is given in explicit losing form is polynomial time solvable.

IsStrong: Simple Games loosing forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- A simple game is not strong iff

$$\exists S \subseteq N : S \in \mathcal{L} \wedge N \setminus S \in \mathcal{L}$$

IsStrong: Simple Games loosing forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- A simple game is not strong iff

$$\exists S \subseteq N : S \in \mathcal{L} \ \wedge \ N \setminus S \in \mathcal{L}$$

which is equivalent to

$$\exists S \subseteq N : \exists L_1, L_2 \in \mathcal{L}^M : S \subseteq L_1 \ \wedge \ N \setminus S \subseteq L_2$$

IsStrong: Simple Games loosing forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- A simple game is not strong iff

$$\exists S \subseteq N : S \in \mathcal{L} \ \wedge \ N \setminus S \in \mathcal{L}$$

  which is equivalent to

$$\exists S \subseteq N : \exists L_1, L_2 \in \mathcal{L}^M : S \subseteq L_1 \ \wedge \ N \setminus S \subseteq L_2$$

- which is equivalent to there are two maximal losing coalitions L_1 and L_2 such that $L_1 \cup L_2 = N$.

IsStrong: Simple Games loosing forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- A simple game is not strong iff

$$\exists S \subseteq N : S \in \mathcal{L} \ \wedge \ N \setminus S \in \mathcal{L}$$

which is equivalent to

$$\exists S \subseteq N : \exists L_1, L_2 \in \mathcal{L}^M : S \subseteq L_1 \ \wedge \ N \setminus S \subseteq L_2$$

- which is equivalent to there are two maximal losing coalitions L_1 and L_2 such that $L_1 \cup L_2 = N$.
- This can be checked in polynomial time, given $\mathcal{L}^M$.

$\square$

IsStrong: explicit winning forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- Given $(N, \mathcal{W})$, for $i \in N$ consider the family $\mathcal{W}_i = \{X \setminus \{i\} \mid X \in \mathcal{W}\}$ and $R = \cup_{i \in N} \mathcal{W}_i$.
- All the coalitions in $R \setminus W$ are losing coalitions.
- Furthermore for a coalition $X \in \mathcal{L}^M$ and $i \notin X$, $X \cup \{i\} \in \mathcal{W}$.
- Thereofore, $\mathcal{L}^M \subseteq R \setminus W$ and $(R \setminus W)^M = \mathcal{L}^M$.
- Then, we compute $\mathcal{L}^M$ from $\mathcal{W}$ in polynomial time and then use the algoritm for the maximal losing form.

IsStrong: minimal winning forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

Theorem

The IsStrong *problem is coNP-complete when the input game is given in explicit minimal winning form.*

IsStrong: minimal winning forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

Theorem

The IsStrong *problem is coNP-complete when the input game is given in explicit minimal winning form.*

- The property can be expressed as

$$\forall S \; [(S \in \mathcal{W}) \text{ or } (S \notin \mathcal{W} \text{ and } N \setminus S \in \mathcal{W})]$$

- Observe that the property $S \in \mathcal{W}$ can be checked in polynomial time given S and $\mathcal{W}^m$.
- Thus the problem belongs to coNP.

IsStrong: minimal winning forms

- We provide a polynomial time reduction from the complement of the NP-complete set splitting problem.
- An instance of the set splitting problem is a collection C of subsets of a finite set N. The question is whether it is possible to partition N into two subsets P and $N \setminus P$ such that no subset in C is entirely contained in either P or $N \setminus P$.

IsStrong: minimal winning forms

- We provide a polynomial time reduction from the complement of the NP-complete set splitting problem.
- An instance of the set splitting problem is a collection C of subsets of a finite set N. The question is whether it is possible to partition N into two subsets P and $N \setminus P$ such that no subset in C is entirely contained in either P or $N \setminus P$.
- We have to decide whether $P \subseteq N$ exists such that

$$\forall S \in C : S \not\subseteq P \wedge S \not\subseteq N \setminus P$$

IsStrong: minimal winning forms

- We provide a polynomial time reduction from the complement of the NP-complete set splitting problem.
- An instance of the set splitting problem is a collection C of subsets of a finite set N. The question is whether it is possible to partition N into two subsets P and $N \setminus P$ such that no subset in C is entirely contained in either P or $N \setminus P$.
- We have to decide whether $P \subseteq N$ exists such that

$$\forall S \in C : S \not\subseteq P \wedge S \not\subseteq N \setminus P$$

We associate to a set splitting instance (N, C) the simple game in explicit minimal winning form (N, C^m).

IsStrong: minimal winning form

- C^m can be computed in polynomial time, given C. Why?

IsStrong: minimal winning form

- C^m can be computed in polynomial time, given C. Why?
- Now assume that $P \subseteq N$ satisfies

$$\forall S \in C : S \not\subseteq P \wedge S \not\subseteq N \setminus P$$

IsStrong: minimal winning form

- C^m can be computed in polynomial time, given C. Why?
- Now assume that $P \subseteq N$ satisfies

$$\forall S \in C : S \not\subseteq P \wedge S \not\subseteq N \setminus P$$

- This means that P and $N \setminus P$ are losing coalitions in the game (N, C^m).

IsStrong: minimal winning form

- C^m can be computed in polynomial time, given C. Why?
- Now assume that $P \subseteq N$ satisfies

$$\forall S \in C : S \not\subseteq P \wedge S \not\subseteq N \setminus P$$

- This means that P and $N \setminus P$ are losing coalitions in the game (N, C^m).
- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$.

IsStrong: minimal winning form

- C^m can be computed in polynomial time, given C. Why?
- Now assume that $P \subseteq N$ satisfies

$$\forall S \in C : S \not\subseteq P \wedge S \not\subseteq N \setminus P$$

- This means that P and $N \setminus P$ are losing coalitions in the game (N, C^m).
- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$.
- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in C contains a set in C^m.

IsStrong: minimal winning form

- C^m can be computed in polynomial time, given C. Why?
- Now assume that $P \subseteq N$ satisfies

$$\forall S \in C : S \not\subseteq P \wedge S \not\subseteq N \setminus P$$

- This means that P and $N \setminus P$ are losing coalitions in the game (N, C^m).
- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$.
- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in C contains a set in C^m.
- Therefore, (N, C) has a set splitting iff (N, C^m) is not strong.

IsProper: winning forms

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

Theorem

The IsProper *problem, when the game is given in explicit winning or losing form or in minimal winning form, can be solved in polynomial time.*

IsProper: winning forms

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

Theorem

The IsProper *problem, when the game is given in explicit winning or losing form or in minimal winning form, can be solved in polynomial time.*

- As before, given a family of subsets F, we can check, for any set in F, whether its complement is not in F in polynomial time.
  Taking into account the definitions, the IsProper problem is polynomial time solvable for the explicit forms

IsProper: winning forms

- Γ is not proper iff

$$\exists S \subseteq N : S \in \mathcal{W} \ \wedge \ N \setminus S \in \mathcal{W}$$

IsProper: winning forms

- Γ is not proper iff

$$\exists S \subseteq N : S \in \mathcal{W} \ \wedge \ N \setminus S \in \mathcal{W}$$

- which is equivalent to

$$\exists S \subseteq N : \exists W_1, W_2 \in \mathcal{W}^m : W_1 \subseteq S \ \wedge \ W_2 \subseteq N \setminus S.$$

IsProper: winning forms

- Γ is not proper iff

$$\exists S \subseteq N : S \in \mathcal{W} \;\wedge\; N \setminus S \in \mathcal{W}$$

- which is equivalent to

$$\exists S \subseteq N : \exists W_1, W_2 \in \mathcal{W}^m : W_1 \subseteq S \;\wedge\; W_2 \subseteq N \setminus S.$$

- equivalent to there are two minimal winning coalitions W_1 and W_2 such that $W_1 \cap W_2 = \emptyset$.

IsProper: winning forms

- Γ is not proper iff

$$\exists S \subseteq N : S \in \mathcal{W} \;\wedge\; N \setminus S \in \mathcal{W}$$

- which is equivalent to

$$\exists S \subseteq N : \exists W_1, W_2 \in \mathcal{W}^m : W_1 \subseteq S \;\wedge\; W_2 \subseteq N \setminus S.$$

- equivalent to there are two minimal winning coalitions W_1 and W_2 such that $W_1 \cap W_2 = \emptyset$.
- Which can be checked in polynomial time when $\mathcal{W}^m$ is given.

IsProper: maximal losing form

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

Theorem

The IsProper *problem is coNP-complete when the input game is given in extensive maximal losing form.*

IsProper: maximal losing form

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

Theorem

The IsProper *problem is coNP-complete when the input game is given in extensive maximal losing form.*

- A game is *not proper* iff

$$\exists S \subseteq N : S \notin \mathcal{L} \ \wedge \ N \setminus S \notin \mathcal{L}$$

- which is equivalet to

$$\exists S \subseteq N : \forall T_1, T_2 \in \mathcal{L}^M : S \not\subseteq T_1 \ \wedge \ N \setminus S \not\subseteq T_2$$

IsProper: maximal losing form

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

Theorem

The IsProper *problem is coNP-complete when the input game is given in extensive maximal losing form.*

- A game is *not proper* iff

$$\exists S \subseteq N : S \notin \mathcal{L} \ \wedge \ N \setminus S \notin \mathcal{L}$$

- which is equivalet to

$$\exists S \subseteq N : \forall T_1, T_2 \in \mathcal{L}^M : S \not\subseteq T_1 \ \wedge \ N \setminus S \not\subseteq T_2$$

- Therefore IsProper belongs to coNP.

IsProper: maximal losing form

To show that the problem is also coNP-hard we provide a reduction from the IsStrong problem for games given in extensive minimal winning form.

IsProper: maximal losing form

To show that the problem is also coNP-hard we provide a reduction from the IsStrong problem for games given in extensive minimal winning form.

- If a family C of subsets of N is minimal then the family $\{N \setminus L : L \in C\}$ is maximal.
- Given a game $\Gamma = (N, \mathcal{W}^m)$, in minimal winning form, we construct the game $\Gamma' = (N, \{N \setminus L : L \in \mathcal{W}^m\})$ in maximal losing form.
- Which can be obtained in polynomial time.

IsProper: maximal losing form

To show that the problem is also coNP-hard we provide a reduction from the IsStrong problem for games given in extensive minimal winning form.

- If a family C of subsets of N is minimal then the family $\{N \setminus L : L \in C\}$ is maximal.
- Given a game $\Gamma = (N, \mathcal{W}^m)$, in minimal winning form, we construct the game $\Gamma' = (N, \{N \setminus L : L \in \mathcal{W}^m\})$ in maximal losing form.
- Which can be obtained in polynomial time.
- Besides, Γ is strong iff Γ' is proper.

Weighted voting games

Weighted voting games

Some of the proofs are based on reductions from the NP-complete problem PARTITION:

Weighted voting games

Some of the proofs are based on reductions from the NP-complete problem PARTITION:

Name: PARTITION

Input: n integer values, $x_1, \ldots, x_n$

Question: Is there $S \subseteq \{1, \ldots, n\}$ *for which*

$$\sum_{i \in S} x_i = \sum_{i \notin S} x_i.$$

Weighted voting games

Some of the proofs are based on reductions from the NP-complete problem PARTITION:

Name: PARTITION
Input: n integer values, $x_1, \ldots, x_n$
Question: Is there $S \subseteq \{1, \ldots, n\}$ *for which*

$$\sum_{i \in S} x_i = \sum_{i \notin S} x_i.$$

Observe that, for any instance of the PARTITION problem in which the sum of the n input numbers is odd, the answer must be NO.

Weighted voting games

Theorem

The IsStrong *and the* IsProper *problems, when the input is described by an integer realization of a weighted game* $(q; w)$*, are coNP-complete.*

Weighted voting games

Theorem

The IsStrong *and the* IsProper *problems, when the input is described by an integer realization of a weighted game* $(q; w)$*, are coNP-complete.*

- From the definitions of strong, proper it is straightforward to show that both problems belong to coNP.

Weighted voting games

Theorem

The IsStrong *and the* IsProper *problems, when the input is described by an integer realization of a weighted game* $(q; w)$, *are coNP-complete.*

- From the definitions of strong, proper it is straightforward to show that both problems belong to coNP.
- Observe that the weighted game with integer representation $(2; 1, 1, 1)$ is both proper and strong.

Hardness

We transform an instance $x = (x_1, \ldots, x_n)$ of PARTITION into a realization of a weighted game according to the following schema

$$f(x) = \begin{cases} (q(x); x) & \text{when } x_1 + \cdots + x_n \text{ is even,} \\ (2; 1, 1, 1) & \text{otherwise.} \end{cases}$$

Hardness

We transform an instance $x = (x_1, \dots, x_n)$ of PARTITION into a realization of a weighted game according to the following schema

$$f(x) = \begin{cases} (q(x); x) & \text{when } x_1 + \cdots + x_n \text{ is even,} \\ (2; 1, 1, 1) & \text{otherwise.} \end{cases}$$

- Function f can be computed in polynomial time provided q does.

Hardness

We transform an instance $x = (x_1, \dots, x_n)$ of PARTITION into a realization of a weighted game according to the following schema

$$f(x) = \begin{cases} (q(x); x) & \text{when } x_1 + \cdots + x_n \text{ is even,} \\ (2; 1, 1, 1) & \text{otherwise.} \end{cases}$$

- Function f can be computed in polynomial time provided q does.
- Independently of q, when $x_1 + \cdots + x_n$ is *odd*, x is a NO input for partition, but $f(x)$ is a YES instance of ISSTRONG or ISPROPER.

IsStrong

Assume that $x_1 + \cdots + x_n$ is *even*.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
Set $q(x) = s + 1$.

IsStrong

Assume that $x_1 + \cdots + x_n$ is *even*.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
Set $q(x) = s + 1$.

- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both S and $N \setminus S$ are losing coalitions and $f(x)$ is not strong.

IsStrong

Assume that $x_1 + \cdots + x_n$ is *even*.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
Set $q(x) = s + 1$.

- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both S and $N \setminus S$ are losing coalitions and $f(x)$ is not strong.
- If S and $N \setminus S$ are losing coalitions in $f(x)$.
  If $\sum_{i \in S} x_i < s$ then $\sum_{i \notin S} x_i \geq s + 1$, $N \setminus S$ should be winning.
  Thus $\sum_{i \in S} x_i = \sum_{i \notin S} x_i = s$, and there exists a partition of x.

IsProper

Assume that $x_1 + \cdots + x_n$ is *even*.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
Set $q(x) = s$.

IsProper

Assume that $x_1 + \cdots + x_n$ is *even*.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
Set $q(x) = s$.

- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both S and $N \setminus S$ are winning coalitions and $f(x)$ is not proper.

IsProper

Assume that $x_1 + \cdots + x_n$ is *even*.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
Set $q(x) = s$.

- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both S and $N \setminus S$ are winning coalitions and $f(x)$ is not proper.
- When $f(x)$ is not proper

$$\exists S \subseteq N : \sum_{i \in S} x_i \geq s \wedge \sum_{i \notin S} x_i \geq s,$$

and thus $\sum_{i \in S} x_i = s$.

Explicit forms

Lemma

The IsWeighted *problem can be solved in polynomial time when the input game is given in explicit winning or losing form.*

Explicit forms

Lemma

The IsWeighted *problem can be solved in polynomial time when the input game is given in explicit winning or losing form.*

We can obtain $\mathcal{W}^m$ and $\mathcal{L}^M$ in polynomial time.
Once this is done we write, in polynomial time, the LP

$$\begin{array}{lll} \texttt{min } q & & \\ \texttt{subject to} & w(S) \geq q & \texttt{if } S \in W^m \\ & w(S) < q & \texttt{if } S \in L^M \\ & 0 \leq w_i & \texttt{for all } 1 \leq i \leq n \\ & 0 \leq q & \end{array}$$

IsWeighted: Minimal and Maximal

Lemma

The IsWeighted *problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.*

IsWeighted: Minimal and Maximal

Lemma

The IsWeighted *problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.*

- For $C \subseteq N$ we let $x_C \in \{0,1\}^n$ denote the vector with the i'th coordinate equal to 1 if and only if $i \in C$.

IsWeighted: Minimal and Maximal

Lemma

The ISWEIGHTED *problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.*

- For $C \subseteq N$ we let $x_C \in \{0,1\}^n$ denote the vector with the i'th coordinate equal to 1 if and only if $i \in C$.
- In polynomial time we compute the boolean function Φ_{W^m} given by the DNF:

$$\Phi_{W^m}(x) = \bigvee_{S \in W^m} (\wedge_{i \in S} x_i)$$

IsWheigthed: Minimal winning

By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \Leftrightarrow C \text{ is winning in the game given by } (N, W^m)$$

IsWheigthed: Minimal winning

By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \Leftrightarrow C \text{ is winning in the game given by } (N, W^m)$$

- It is well known that Φ_{W^m} is a threshold function iff the game given by (N, W^m) is weighted.

IsWheigthed: Minimal winning

By construction we have the following:

$$\Phi_{W^m}(x_C) = 1 \Leftrightarrow C \text{ is winning in the game given by } (N, W^m)$$

- It is well known that Φ_{W^m} is a threshold function iff the game given by (N, W^m) is weighted.
- Further Φ_{W^m} is monotonic (i.e. *positive*)
- But deciding whether a monotonic formula describes a threshold function can be solved in polynomial time.

IsWheigthed: Maximal loosing

- we can prove a similar result given (N, L^M).
- The dual of game $\Gamma = (N, \mathcal{W})$ is the game $\Gamma^d = (N, \mathcal{W}^d)$ where
  $S \in \mathcal{W}^d$ iff $N \setminus S \notin \mathcal{W}$.
- Observe that Γ is weighted iff Γ^d is weighted.
- We can compute a monotone CNF formula describing the loosing coalitions of Γ. Negating this formula we get a DNF on negated variables. Replacing $\overline{x}_i$ by y_i we get a DNF describing $\mathcal{W}^d$.
- As the formula can be computed in polynomial time the result follows.