Simple Games

Spring 2022
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Simple Games were introduced by (Taylor & Zwicker, 1999)
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\[ v : C_N \rightarrow \{0, 1\} \]
and it is monotone.
Simple Games

- Simple Games were introduced by (Taylor & Zwicker, 1999)
- A **simple game** is a cooperative game \((N, \nu)\) such that \(\nu : \mathcal{C}_N \rightarrow \{0, 1\}\) and it is monotone.

A simple game can be described by a pair \((N, \mathcal{W})\):
- \(N\) is a set of players,
- \(\mathcal{W} \subseteq \mathcal{P}(N)\) is a monotone set of **winning coalitions**, those coalitions \(X\) with \(\nu(X) = 1\).
- \(\mathcal{L} = \mathcal{C}_N \setminus \mathcal{W}\) is the set of **losing coalitions** those coalitions \(X\) with \(\nu(X) = 0\).
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- Members of \(N = \{1, \ldots, n\}\) are called **players** or **voters**.
Simple games: Representation

Due to monotonicity, any one of the following families of coalitions define a simple game on a set of players $N$:

- **winning coalitions** $\mathcal{W}$.
- **losing coalitions** $\mathcal{L}$.
- **minimal winning coalitions** $\mathcal{W}^m$
  \[ \mathcal{W}^m = \{ X \in \mathcal{W} ; \forall Z \in \mathcal{W}, Z \not\subseteq X \} \]
- **maximal losing coalitions** $\mathcal{L}^M$
  \[ \mathcal{L}^M = \{ X \in \mathcal{L} ; \forall Z \in \mathcal{L}, X \not\subseteq L \} \]

This provides us with many representation forms for simple games.
Weighted voting games

A simple game for which there exists a quota $q$ and it is possible to assign to each $i \in \mathbb{N}$ a weight $w_i$, so that $X \in W$ iff $\sum_{i \in X} w_i \geq q$.

WVG can be represented by a tuple of integers $(q; w_1, \ldots, w_n)$.

as any weighted game admits such an integer realization, [Carreras and Freixas, Math. Soc. Sci., 1996]
Weighted voting games

- Weighted voting games (WVG)
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Vector weighted voting games

Given two simple games $\Gamma_1 = (N, W_1)$ and $\Gamma_2 = (N, W_2)$ we can define their union $\Gamma_1 \cup \Gamma_2 = (N, W_1 \cup W_2)$ and intersection $\Gamma_1 \cap \Gamma_2 = (N, W_1 \cap W_2)$. Both are simple games.

A simple game $\Gamma$ is a vector weighted voting game if there are WVGs $\Gamma_1, \ldots, \Gamma_k$, for some $k \geq 1$, so that $\Gamma = \Gamma_1 \cap \cdots \cap \Gamma_k$. 

AGT-MIRI Cooperative Game Theory

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- A simple game $\Gamma$ is a **vector weighted voting game** if there are WVGs $\Gamma_1, \ldots, \Gamma_k$, for some $k \geq 1$, so that $\Gamma = \Gamma_1 \cap \cdots \cap \Gamma_k$. 
Representability

There are simple games that are not WVGs. Assume it is given by \((q; w_1, w_2, w_3, w_4)\). We have

\[w_1 + w_2 \geq q\]

and

\[w_3 + w_4 \geq q\]

Thus,

\[\max\{w_1, w_2\} \geq q/2\]

and

\[\max\{w_3, w_4\} \geq q/2\]

So,

\[\max\{w_1, w_2\} + \max\{w_3, w_4\} \geq q\]

which cannot be.
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- Assume it is given by \((q; w_1, w_2, w_3, w_4)\).
- We have \( w_1 + w_2 \geq q \) and \( w_3 + w_4 \geq q \).
- Thus \( \max\{w_1, w_2\} \geq q/2 \) and \( \max\{w_3, w_4\} \geq q/2 \),
- So, \( \max\{w_1, w_2\} + \max\{w_3, w_4\} \geq q \) which cannot be.
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- Any simple game $\Gamma$ is a VWVGs.
Any simple game \( \Gamma \) is a VWVGs

Take a losing coalition \( C \) and consider the game in which players in \( C \) have weight 0 and players outside \( C \) 1, set the quote to 1.

Any set that is not contained in \( C \) wins!
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- Take a losing coalition $C$ and consider the game in which players in $C$ have weight 0 and players outside $C$ 1, set the quote to 1.
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- The intersection of the above games describes $\Gamma$. 
Representability

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  - Take a losing coalition $C$ and consider the game in which players in $C$ have weight 0 and players outside $C$ 1, set the quote to 1.
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  - A winning coalition cannot be a subset of any losing coalition.
Representability

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  - Take a losing coalition \( C \) and consider the game in which players in \( C \) have weight 0 and players outside \( C \) 1, set the quote to 1.
  - Any set that is not contained in \( C \) wins!
  - The intersection of the above games describes \( \Gamma \).
    - A winning coalition cannot be a subset of any losing coalition.
- The dimension of a simple games is the minimum number of WVGs that allows its representation as VWVG
A representation as WVGs

The game $\Gamma$ with $N = \{1, 2, 3, 4\}$ where the minimal winning coalitions are the sets $\{1, 2\}$ and $\{3, 4\}$ is not a WVG.

The maximal losing coalitions are $\{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$.

This gives four WVG, according to the previous construction:

$\Gamma = [1; 0, 1, 0, 1] \cap [1; 0, 1, 1, 0] \cap [1; 1, 0, 0, 1] \cap [1; 1, 0, 1, 0]$. 
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A representation as WVGs

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- The maximal losing coalitions are $\{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$. 
- This gives four WVG, according to the previous construction

$$\Gamma = [1; 0, 1, 0, 1] \cap [1; 0, 1, 1, 0] \cap [1; 1, 0, 0, 1] \cap [1; 1, 0, 1, 0].$$
Input representations

- **Simple Games**
  - \((N, W)\): extensive winning, \((N, W^m)\): minimal winning
  - \((N, L)\): extensive losing, \((N, L^M)\): maximal losing
  - \((N, C)\): monotone circuit winning
  - \((N, F)\): monotone formula winning

- **Weighted voting games**: \((q; w_1, \ldots, w_n)\)

- **Vector weighted voting games**: \((q_1; w_1^1, \ldots, w_n^1), \ldots, (q_k; w_1^k, \ldots, w_n^k)\)

  All numbers are integers
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In general we state a property $P$, for simple games, and consider the associated decision problem which has the form:

- **Name:** IsP
- **Input:** A simple game/WVG/VWVG $\Gamma$
- **Question:** Does $\Gamma$ satisfy property $P$?
Four properties

A simple game \((N, \mathcal{W})\) is

- **strong** if \(S \not\in \mathcal{W}\) implies \(N \setminus S \in \mathcal{W}\).
- **proper** if \(S \in \mathcal{W}\) implies \(N \setminus S \not\in \mathcal{W}\).
- a **weighted voting game**.
- a **vector weighted voting game**.
IsStrong: Simple Games

Γ is strong if \( S \notin \mathcal{W} \) implies \( N \setminus S \in \mathcal{W} \)
IsStrong: Simple Games

Γ is strong if $S \not\in \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

**Theorem**

The **IsStrong** problem, when Γ is given in explicit winning or losing form or in maximal losing form can be solved in polynomial time.
IsStrong: Simple Games

Γ is strong if \( S \notin \mathcal{W} \) implies \( N \setminus S \in \mathcal{W} \)

Theorem

The IsStrong problem, when \( \Gamma \) is given in explicit winning or losing form or in maximal losing form can be solved in polynomial time.

- First observe that, given a family of subsets \( F \), we can check, for any set in \( F \), whether its complement is not in \( F \) in polynomial time.
- Therefore, the IsStrong problem, when the input is given in explicit losing form is polynomial time solvable.
IsStrong: Simple Games loosing forms

Γ is strong if $S \not\in \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

- A simple game is not strong iff

$$\exists S \subseteq N : S \in \mathcal{L} \land N \setminus S \in \mathcal{L}$$
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IsStrong: Simple Games loosing forms

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- A simple game is not strong iff

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which is equivalent to

$$\exists S \subseteq N : \exists L_1, L_2 \in \mathcal{L}^M : S \subseteq L_1 \land N \setminus S \subseteq L_2$$
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- which is equivalent to there are two maximal losing coalitions \( L_1 \) and \( L_2 \) such that \( L_1 \cup L_2 = N \).
IsStrong: Simple Games loosing forms

\( \Gamma \) is \textit{strong} if \( S \notin \mathcal{W} \) implies \( N \setminus S \in \mathcal{W} \)

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  which is equivalent to there are two maximal losing coalitions \( L_1 \) and \( L_2 \) such that \( L_1 \cup L_2 = N \).
- This can be checked in polynomial time, given \( \mathcal{L}^M \).
\( \Gamma \) is strong if \( S \notin \mathcal{W} \) implies \( N \setminus S \in \mathcal{W} \)

- Given \((N, \mathcal{W})\), for \( i \in N \) consider the family \( \mathcal{W}_i = \{ X \setminus \{i\} \mid X \in \mathcal{W} \} \) and \( R = \bigcup_{i \in N} \mathcal{W}_i \).
- All the coalitions in \( R \setminus \mathcal{W} \) are losing coalitions.
- Furthermore for a coalition \( X \in \mathcal{L}^M \) and \( i \notin X \), \( X \cup \{i\} \in \mathcal{W} \).
- Therefore, \( \mathcal{L}^M \subseteq R \setminus \mathcal{W} \) and \( (R \setminus \mathcal{W})^M = \mathcal{L}^M \).
- Then, we compute \( \mathcal{L}^M \) from \( \mathcal{W} \) in polynomial time and then use the algorithm for the maximal losing form.
IsStrong: minimal winning forms

Γ is strong if $S \notin \mathcal{W}$ implies $N \setminus S \in \mathcal{W}$

Theorem

The \textbf{IsStrong} problem is coNP-complete when the input game is given in explicit minimal winning form.
IsStrong: minimal winning forms

Γ is strong if \( S \notin \mathcal{W} \) implies \( N \setminus S \in \mathcal{W} \)

**Theorem**

The *IsStrong* problem is coNP-complete when the input game is given in explicit minimal winning form.

- The property can be expressed as

  \[
  \forall S \left[ (S \in \mathcal{W}) \text{ or } (S \notin \mathcal{W} \text{ and } N \setminus S \in \mathcal{W}) \right]
  \]

- Observe that the property \( S \in \mathcal{W} \) can be checked in polynomial time given \( S \) and \( \mathcal{W}^m \).

- Thus the problem belongs to coNP.
IsStrong: minimal winning forms

- We provide a polynomial time reduction from the complement of the NP-complete set splitting problem.
- An instance of the set splitting problem is a collection $C$ of subsets of a finite set $N$. The question is whether it is possible to partition $N$ into two subsets $P$ and $N \setminus P$ such that no subset in $C$ is entirely contained in either $P$ or $N \setminus P$. 
We provide a polynomial time reduction from the complement of the NP-complete set splitting problem.

An instance of the set splitting problem is a collection $C$ of subsets of a finite set $N$. The question is whether it is possible to partition $N$ into two subsets $P$ and $N \setminus P$ such that no subset in $C$ is entirely contained in either $P$ or $N \setminus P$.

We have to decide whether $P \subseteq N$ exists such that

$$\forall S \in C : S \not\subseteq P \land S \not\subseteq N \setminus P$$
IsStrong: minimal winning forms

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- We have to decide whether $P \subseteq N$ exists such that

$$\forall S \in C : S \not\subseteq P \land S \not\subseteq N \setminus P$$

We associate to a set splitting instance $(N, C)$ the simple game in explicit minimal winning form $(N, C^m)$.
IsStrong: minimal winning form

- $C^m$ can be computed in polynomial time, given $C$. Why?
IsStrong: minimal winning form

- $C^m$ can be computed in polynomial time, given $C$. Why?
- Now assume that $P \subseteq N$ satisfies
  \[
  \forall S \in C : S \not\subseteq P \land S \not\subseteq N \setminus P
  \]
IsStrong: minimal winning form

- $C^m$ can be computed in polynomial time, given $C$. Why?
- Now assume that $P \subseteq N$ satisfies
  \[
  \forall S \in C : S \not\subseteq P \land S \not\subseteq N \setminus P
  \]
  This means that $P$ and $N \setminus P$ are losing coalitions in the game $(N, C^m)$. 

IsStrong: minimal winning form

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- Now assume that $P \subseteq N$ satisfies

$$\forall S \in C : S \not\subseteq P \land S \not\subseteq N \setminus P$$

- This means that $P$ and $N \setminus P$ are losing coalitions in the game $(N, C^m)$.
- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$. 
IsStrong: minimal winning form

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- Now assume that $P \subseteq N$ satisfies
  \[
  \forall S \in C : S \not\subseteq P \land S \not\subseteq N \setminus P
  \]
  This means that $P$ and $N \setminus P$ are losing coalitions in the game $(N, C^m)$.
- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$.
- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in $C$ contains a set in $C^m$. 
IsStrong: minimal winning form

- $C^m$ can be computed in polynomial time, given $C$. Why?
- Now assume that $P \subseteq N$ satisfies

$$\forall S \in C : S \not\subseteq P \land S \not\subseteq N \setminus P$$

- This means that $P$ and $N \setminus P$ are losing coalitions in the game $(N, C^m)$.
- So, $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C^m$.
- This implies $S \not\subseteq P$ and $S \not\subseteq N \setminus P$, for any $S \in C$ since any set in $C$ contains a set in $C^m$.
- Therefore, $(N, C)$ has a set splitting iff $(N, C^m)$ is not strong.
IsProper: winning forms

Γ is proper if $S \in W$ implies $N \setminus S \notin W$.

**Theorem**

*The IsProper problem, when the game is given in explicit winning or losing form or in minimal winning form, can be solved in polynomial time.*
\[ \Gamma \text{ is proper if } S \in \mathcal{W} \text{ implies } N \setminus S \notin \mathcal{W}. \]

**Theorem**

The \textsc{IsProper} problem, when the game is given in explicit winning or losing form or in minimal winning form, can be solved in polynomial time.

- As before, given a family of subsets \( F \), we can check, for any set in \( F \), whether its complement is not in \( F \) in polynomial time.
- Taking into account the definitions, the \textsc{IsProper} problem is polynomial time solvable for the explicit forms.
IsProper: winning forms

- $\Gamma$ is not proper iff

$$\exists S \subseteq N : S \in \mathcal{W} \land N \setminus S \in \mathcal{W}$$
IsProper: winning forms

- $\Gamma$ is not proper iff

$$\exists S \subseteq N : S \in \mathcal{W} \land N \setminus S \in \mathcal{W}$$

- which is equivalent to

$$\exists S \subseteq N : \exists W_1, W_2 \in \mathcal{W}^m : W_1 \subseteq S \land W_2 \subseteq N \setminus S.$$
IsProper: winning forms

- $\Gamma$ is not proper iff
  \[ \exists S \subseteq N : S \in \mathcal{W} \land N \setminus S \in \mathcal{W} \]

- which is equivalent to
  \[ \exists S \subseteq N : \exists W_1, W_2 \in \mathcal{W}^m : W_1 \subseteq S \land W_2 \subseteq N \setminus S. \]

- equivalent to there are two minimal winning coalitions $W_1$ and $W_2$ such that $W_1 \cap W_2 = \emptyset$. 

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IsProper: winning forms

- Γ is not proper iff

\[ \exists S \subseteq N : S \in W \land N \setminus S \in W \]

- which is equivalent to

\[ \exists S \subseteq N : \exists W_1, W_2 \in W^m : W_1 \subseteq S \land W_2 \subseteq N \setminus S. \]

- equivalent to there are two minimal winning coalitions \( W_1 \) and \( W_2 \) such that \( W_1 \cap W_2 = \emptyset \).

- Which can be checked in polynomial time when \( W^m \) is given.
IsProper: maximal losing form

Γ is proper if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$.

Theorem

The IsProper problem is coNP-complete when the input game is given in extensive maximal losing form.
**IsProper: maximal losing form**

Γ is proper if $S \in W$ implies $N \setminus S \notin W$.

**Theorem**

The IsProper problem is coNP-complete when the input game is given in extensive maximal losing form.

- A game is not proper iff
  \[
  \exists S \subseteq N : S \notin L \land N \setminus S \notin L
  \]
- which is equivalent to
  \[
  \exists S \subseteq N : \forall T_1, T_2 \in L^M : S \not\subseteq T_1 \land N \setminus S \not\subseteq T_2
  \]
IsProper: maximal losing form

Γ is proper if \( S \in \mathcal{W} \) implies \( N \setminus S \notin \mathcal{W} \).

**Theorem**

The IsProper problem is coNP-complete when the input game is given in extensive maximal losing form.

- A game is not proper iff
  \[
  \exists S \subseteq N : S \notin \mathcal{L} \land N \setminus S \notin \mathcal{L}
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- which is equivalent to
  \[
  \exists S \subseteq N : \forall T_1, T_2 \in \mathcal{L}^M : S \not\subseteq T_1 \land N \setminus S \not\subseteq T_2
  \]
- Therefore IsProper belongs to coNP.
IsProper: maximal losing form

To show that the problem is also coNP-hard we provide a reduction from the IsStrong problem for games given in extensive minimal winning form.
To show that the problem is also coNP-hard we provide a reduction from the IsStrong problem for games given in extensive minimal winning form.

- If a family $C$ of subsets of $N$ is minimal then the family $\{ N \setminus L : L \in C \}$ is maximal.
- Given a game $\Gamma = (N, \mathcal{W}^m)$, in minimal winning form, we construct the game $\Gamma' = (N, \{ N \setminus L : L \in \mathcal{W}^m \})$ in maximal losing form.
- Which can be obtained in polynomial time.
To show that the problem is also coNP-hard we provide a reduction from the IsStrong problem for games given in extensive minimal winning form.

- If a family $C$ of subsets of $N$ is minimal then the family \( \{ N \setminus L : L \in C \} \) is maximal.
- Given a game $\Gamma = (N, \mathcal{W}^m)$, in minimal winning form, we construct the game $\Gamma' = (N, \{ N \setminus L : L \in \mathcal{W}^m \})$ in maximal losing form.
- Which can be obtained in polynomial time.
- Besides, $\Gamma$ is strong iff $\Gamma'$ is proper.
Weighted voting games

Some of the proofs are based on reductions from the NP-complete problem \textit{Partition}:

**Name:** \textit{Partition}

**Input:** \(n\) integer values, \(x_1, \ldots, x_n\)

**Question:** Is there \(S \subseteq \{1, \ldots, n\}\) for which \(x_i \in S \implies x_i = \sum_{j \in S} x_j\)?

Observe that, for any instance of the \textit{Partition} problem in which the sum of the \(n\) input numbers is odd, the answer must be no.
Weighted voting games

Some of the proofs are based on reductions from the NP-complete problem Partition:
Weighted voting games

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\]
Weighted voting games

Some of the proofs are based on reductions from the NP-complete problem PARTITION:

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Observe that, for any instance of the PARTITION problem in which the sum of the \(n\) input numbers is odd, the answer must be NO.
Theorem

The \textsc{IsStrong} and the \textsc{IsProper} problems, when the input is described by an integer realization of a weighted game \((q; w)\), are coNP-complete.
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- From the definitions of strong, proper it is straightforward to show that both problems belong to coNP.
Weighted voting games

Theorem

The **IsStrong** and the **IsProper** problems, when the input is described by an integer realization of a weighted game \((q; w)\), are coNP-complete.

- From the definitions of strong, proper it is straightforward to show that both problems belong to coNP.
- Observe that the weighted game with integer representation \((2; 1, 1, 1)\) is both proper and strong.
We transform an instance $x = (x_1, \ldots, x_n)$ of \textsc{Partition} into a realization of a weighted game according to the following schema

$$f(x) = \begin{cases} (q(x); x) & \text{when } x_1 + \cdots + x_n \text{ is even}, \\ (2; 1, 1, 1) & \text{otherwise.} \end{cases}$$
Hardness

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- Function $f$ can be computed in polynomial time provided $q$ does.
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\end{cases}$$

- Function $f$ can be computed in polynomial time provided $q$ does.
- Independently of $q$, when $x_1 + \cdots + x_n$ is odd, $x$ is a NO input for partition, but $f(x)$ is a YES instance of ISSTRONG or ISPROPER.
Assume that $x_1 + \cdots + x_n$ is even.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
Set $q(x) = s + 1$. 
IsStrong

Assume that $x_1 + \cdots + x_n$ is even.
Let $s = (x_1 + \cdots + x_n)/2$ and $N = \{1, \ldots, n\}$.
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- If there is $S \subset N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both $S$ and $N \setminus S$ are losing coalitions and $f(x)$ is not strong.
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- If $S$ and $N \setminus S$ are losing coalitions in $f(x)$.
  If $\sum_{i \in S} x_i < s$ then $\sum_{i \notin S} x_i \geq s + 1$, $N \setminus S$ should be winning. Thus $\sum_{i \in S} x_i = \sum_{i \notin S} x_i = s$, and there exists a partition of $x$. 
Assume that $x_1 + \cdots + x_n$ is even.

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- If there is $S \subseteq N$ such that $\sum_{i \in S} x_i = s$, then $\sum_{i \notin S} x_i = s$, thus both $S$ and $N \setminus S$ are winning coalitions and $f(x)$ is not proper.

- When $f(x)$ is not proper

\[
\exists S \subseteq N : \sum_{i \in S} x_i \geq s \land \sum_{i \notin S} x_i \geq s,
\]

and thus $\sum_{i \in S} x_i = s$. 
1 Simple Games

2 Problems on simple games

3 IsWeighted
Explicit forms

Lemma

The IsWeighted problem can be solved in polynomial time when the input game is given in explicit winning or losing form.
Lemma

The \textsc{IsWeighted} problem can be solved in polynomial time when the input game is given in explicit winning or losing form.

We can obtain $W^m$ and $L^M$ in polynomial time. Once this is done we write, in polynomial time, the LP

$$\begin{align*}
\min q \\
\text{subject to} & \quad w(S) \geq q \quad \text{if } S \in W^m \\
& \quad w(S) < q \quad \text{if } S \in L^M \\
& \quad 0 \leq w_i \quad \text{for all } 1 \leq i \leq n \\
& \quad 0 \leq q
\end{align*}$$
The **IsWeighted** problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.
IsWeighted: Minimal and Maximal

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The IsWeighted problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.

For $C \subseteq N$ we let $x_C \in \{0, 1\}^n$ denote the vector with the $i$'th coordinate equal to 1 if and only if $i \in C$. 
IsWeighted: Minimal and Maximal

Lemma

The IsWeighted problem can be solved in polynomial time when the input game is given in explicit minimal winning or maximal losing form.

For $C \subseteq N$ we let $x_C \in \{0, 1\}^n$ denote the vector with the $i$'th coordinate equal to 1 if and only if $i \in C$.

In polynomial time we compute the boolean function $\Phi_{W^m}$ given by the DNF:

$$\Phi_{W^m}(x) = \bigvee_{S \in W^m} (\bigwedge_{i \in S} x_i)$$
IsWeighted: Minimal winning

By construction we have the following:

\[ \Phi_{W^m}(x_C) = 1 \iff C \text{ is winning in the game given by } (N, W^m) \]
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It is well known that \( \Phi_{W^m} \) is a threshold function iff the game given by \( (N, W^m) \) is weighted.
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- It is well known that \( \Phi_{W^m} \) is a threshold function iff the game given by \( (N, W^m) \) is weighted.
- Further \( \Phi_{W^m} \) is monotonic (i.e. positive)
- But deciding whether a monotonic formula describes a threshold function can be solved in polynomial time.
IsWeighted: Maximal loosing

- we can prove a similar result given \((N, L^M)\).
- The dual of game \(\Gamma = (N, \mathcal{W})\) is the game \(\Gamma^d = (N, \mathcal{W}^d)\) where
  \(S \in \mathcal{W}^d\) iff \(N \setminus S \notin \mathcal{W}\).
- Observe that \(\Gamma\) is weighted iff \(\Gamma^d\) is weighted.
- We can compute a monotone CNF formula describing the loosing coalitions of \(\Gamma\). Negating this formula we get a DNF on negated variables. Replacing \(\overline{x_i}\) by \(y_i\) we get a DNF describing \(\mathcal{W}^d\).
- As the formula can be computed in polynomial time the result follows.