Some definitions:
A valuation function \( v : C_n \rightarrow \mathbb{R} \) is said to be
- **monotone** when \( v(C) \leq v(D) \), for \( C \subseteq D \subseteq N \).
- **superadditive** when \( v(C \cup D) \geq v(C) + v(D) \), for every pair of disjoint coalitions \( C, D \subseteq N \).
- **supermodular** when \( v(C \cup D) + v(C \cap D) \geq v(C) + v(D) \), for \( C, D \subseteq N \).

A coalitional game \((N, v)\) is **convex** iff \( v \) is supermodular.

6. Consider a cooperative game with three players \{a, b, c\} and the following characteristic function

\[
\begin{array}{ccccccc}
S & \emptyset & a & b & c & a,b & a,c & b,c & a,b,c \\
v & 0 & 1 & 0 & 1 & 4 & 3 & 5 & 8 \\
\end{array}
\]

Which imputations belong to the core?

Hint. It might be useful to depict the imputations using barycentric coordinates. Points in the plane with three coordinates \((x_1, x_2, x_3)\) having \( x_1 + x_2 + x_3 = 8 \) with \( x_1, x_3 \geq 1 \) and \( x_2 \geq 0 \). Consider the equilateral triangle corresponding to the coordinates: left base \((1, 6, 1)\), right base \((1, 0, 7)\) and \((7, 0, 1)\).

7. Consider a cooperative game which is defined on an undirected graph \( G = (V, E) \). The players are the vertices in the graph and for \( S \subseteq V \), \( v(S) = |\{u \in V \mid N(u) \cap S \neq \emptyset\}| \). As usual \( N(u) = \{v \mid (u, v) \in E\} \).

   (a) Is the valuation function monotone? supperadditive? supermodular?
   (b) Is the core empty? Can this property be decided in polynomial time?

8. Consider an undirected graph \( G = (V, E) \) and two vertices \( s \) and \( t \). Assume that \( n = |V| \) and \( m = |E| \). Consider the cooperative game \( C(s, t) = (N, v) \) where \( N = E \) and, for \( S \subseteq N \), \( v(S) \) is \( m \) minus the length of the shortest path from \( s \) to \( t \) in the graph \( (V, S) \), if such a path exists, and 0 otherwise. Is this game supperadditive? Does the game have an empty core?

9. Consider the following game which is defined by a parameter \( k \). Each participant has a collection of old vinyl disks, not all of them in good state, that are willing to share. A company is able to obtain an accurate re-recording of an album provided that at least \( k \) copies are provided. In this setting the value of a coalition is the number of albums for which an accurate recording is possible. You can assume that there are \( n \) participants and \( m \) different albums.

   (a) Provide an expression of the valuation function.
   (b) Is the game convex?
   (c) Provide an accurate expression for the Shapley values. Can those values be computed in polynomial time?