Cooperative Game Theory: More Solution concepts

Fall 2020

1 Other solution concepts

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Dummy player, symmetry, additivity, but not efficiency.

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- Can be computed by solving a polynomial number of exponentially large LPs.

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- The kernel of a superadditive game Γ, $\mathcal{K}(\Gamma)$ is the set of all imputations x such that, for any pair of players (i, j) either:
  - $S_{i,j}(x) = S_{j,i}(x)$, or
  - $S_{i,j}(x) > S_{j,i}(x)$ and $x_j = v(\{j\})$, or
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- The kernel always contains de nucleolus, thus it is non-empty.

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- There are games that have no stable sets [Lucas, 1968].