An introduction to Computational Social Choice

Fall 2020

Social Choice Theory

- Mathematical theory for aggregating individual preferences into collective decisions
- Originated in ancient Greece. Formal foundations:
  - 18th Century (Condorcet and Borda)
  - 19th Century: Charles Dodgson (a.k.a. Lewis Carroll)
  - 20th Century: Nobel prizes to Arrow and Sen
- Objective: Methods to select a collective outcome based on (possibly different) individual preferences.

Social Choice Theory

- Set of voters $N = \{1, \ldots, n\}$
- Set of alternatives $A = \{1, \ldots, m\}$
- Voter i has a preference ranking over alternatives $\succ_i$
- Preference ranking $\succ$ is the collection of all voters' rankings

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Ply	1	2	3
	a	c	b
	b	a	c
	c	b	a

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- voting rule $=$ social choice function

Voting rules: Plurality

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- Each voter awards one point to her top alternative
- Alternative with the most point wins

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N	1	2	3	4	5
	a	a	a	b	b
	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

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Winner
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- Most frequently used voting rule
- Many political elections use plurality

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Problems?

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	a	a	a	b	b	4
	b	b	b	c	c	3
	c	c	c	d	d	2
	d	d	d	e	e	1
	e	e	e	a	a	0

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Total
a: 12
b: 17
c: 12
d: 7
e: 2

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	a	a	a	b	b	4
	b	b	b	c	c	3
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- A modified Borda Count is used in the Eurovision Song Context, points to the top 10 songs with 12, 10, 8,9,.. ,1 points

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	b	b	b	c	c
	c	c	c	d	d
	d	d	d	e	e
	e	e	e	a	a

$k = 3$ Total
a: 3
b: 5
c: 5
d: 2
e: 0

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- Approval voting was used for papal conclaves between 1294 and 1621.
- Used to select potential consensus candidates for an election.

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- The family include many rules
  - Plurality $s = (1, 0, \ldots, 0)$
  - Borda $s = (m - 1, m - 2, \ldots, 0)$
  - k-aproval $s = (1, \ldots, 1, 0, \ldots, 0)$
  - Veto $s = (0, \ldots, 0, 1)$
  - ...

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a, b

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2nd round Winner
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- Similar to the French presidential election system
  - Problem: vote division
  - Happened in the 2002 French presidential election

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	Loser
R1	e

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	e	e	e	a	a	e	a	c

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R1	e
R2	d
R3	c
R4	a

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- Social choice: The top alternative in σ^*

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- Maximin
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Which rule to use?

- We just introduced infinitely many rules
- How do we know which is the "right" rule to use? Axioms, Characterization theorems, Impossibility Theorems
- Impossibility versus Computational hardness

1. Social Choice

2. Some properties of voting rules

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  The majority preference prefers x to y
- A Condorcet winner is an alternative that beats every other alternative in pairwise election
- A Condorcet paradox happens when the majority preference has a cycle.

Condorcet Paradox: Example

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N	1	2	3	Majority Pref
	a	c	b	$a \succ b$
	b	a	c	$b \succ c$
	c	b	a	$c \succ a$

Condorcet Paradox: Example

N	1	2	3	Majority Pref
	a	c	b	$a \succ b$
	b	a	c	$b \succ c$
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Also known as Dodgson's Paradox (Alice in Wonderland by Charles L. Dodgson alias Lewis Carroll)

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  - Kemeny, Copeland, Maximin ARE Condorcet consistent.
  - What is the complexity of Existence of Condorcet winner, obtaining the Condorcet winner . . .

Strategy-proofness

- A voting rule is strategy-proof if there exists no profile where some voter can obtain a preferred outcome by changing her preferences.
- Which voting rules are strategy-proof?
- Do they have good properties?
- When they are not, can the manipulation be computed easily?

Problems

E-manipulation: Given a set C of candidates, a set V of nonmanipulative voters, a set S of manipulative voters, with $S \cap V = \emptyset$, and a candidate $c \in C$. Is there a way to set the preference lists of the voters in S such that, under election system E, c is the (a) winner?

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E-Bribery: Given a set C of candidates, a set V of voters, a candidate $c \in C$, and a nonnegative integer k. Is there a way to set the preference lists of at most k voters such that, under election system E, c is the (a) winner?

Problems

E-Control under additive candidates: Given a set C of candidates, a pool D of potential additional candidates, a candidate $c \in C$, and a set of voters V with preferences over $C \cup D$. Is there a set $D' \subseteq D$, such that setting the set of candidates to $C \cup D'$, under election system E, c is the (a) winner?

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E-Destructive control under additive candidates: Given a set C of candidates, a pool D of potential additional candidates, a candidate $c \in C$, and a set of voters V with preferences over $C \cup D$. Is there a set $D' \subseteq D$, such that setting the set of candidates to $C \cup D'$, under election system E, c is not a winner?