1. Consider a cooperative game with three players $\{a, b, c\}$ and the following characteristic function

S	$\emptyset$	a	b	c	a,b	a,c	b,c	a,b,c
v	0	1	0	1	4	3	5	8

   Which imputations belong to the core?

   Hint. It might be useful to depict the imputations using barycentric coordinates. Points in the plane with three coordinates (x_1, x_2, x_3) points having $x_1 + x_2 + x_3 = 8$ with $x_1, x_3 \geq 1$ and $x_2 \geq 0$. Consider the equilateral triangle corresponding to the coordinates: left base $(1, 6, 1)$, right base $(1, 0, 7)$ and $(7, 0, 1)$.

2. Consider a cooperative game which is defined on an undirected graph $G = (V, E)$. The players are the vertices in the graph and for $S \subseteq V$, $v(S) = |\{u \in V \mid N(u) \cap S \neq \emptyset\}|$. As usual $N(u) = \{v \mid (u, v) \in E\}$.

   (a) Is the valuation function monotone? supperadditive? supermodular?

   (b) Is the core empty? Can this property be decided in polynomial time?

3. Consider an undirected graph $G = (V, E)$ and two vertices s and t. Assume that $n = |V|$ and $m = |E|$. Consider the cooperative game $\mathcal{C}(s, t) = (N, v)$ where $N = E$ and, for $S \subseteq N$, $v(S)$ is m minus the length of the shortest path from s to t in the graph (V, S), if such a path exists, and 0 otherwise. Is this game supperadditive? Does the game have an empty core?

4. Consider the following game which is defined by a parameter k. Each participant has a collection of old vinyl disks, not all of them in good state, that are willing to share. A company is able to obtain an accurate re-recording of an album provided that at least k copies are provided. In this setting the value of a coalition is the number of albums for which an accurate recording is possible. You can assume that there are n participants and m different albums.

   (a) Provide an expression of the valuation function.

   (b) Is the game convex?

   (c) Provide an accurate expression for the Shapley values. Can those values be computed in polynomial time?

5. In a simple game $(N, \mathcal{W})$. We say that player i is a *veto player* if $v(S) = 0$ for any $S \subseteq N \setminus \{i\}$.

   Prove that in a simple game in which there are veto players, the imputations in the core are exactly the set of payoff vectors for the grand coalition in which non veto players get 0.

6. Consider a the following decision process which runs on a leader-follower society with n members. The interaction among the society is represented by a bipartite directed graph $G = (L, F, E)$ having the following property: all the vertices in L have in-degree 0 and all

the vertices in F have out-degree 0. The decision process is defined by two parameters α, $0 \leq \alpha \leq 1$ and q, $0 \leq q \leq n$.

When the society has to reach a decision about some topic, each member takes an initial position. We model this situation as an initial decision vector $x \in \{0,1\}^n$. Then, each $i \in F$ looks at the values $p_{i1} = |\{(j,i) \in E \mid x_j = 1\}|$ and $p_{i0} = |\{(j,i) \in E \mid x_j = 0\}|$ and reconsiders its position according to the following algorithm.

- If $p_{i1} > \alpha(p_{i1} + p_{i0})$ and $p_{i0} < \alpha(p_{i1} + p_{i0})$, $x_i = 1$
- If $p_{i0} > \alpha(p_{i1} + p_{i0})$ and $p_{i1} < \alpha(p_{i1} + p_{i0})$, $x_i = 0$

Finally, the society reaches a "yea" (1) when $\sum_{i=1}^{n} x_i \geq q$, a "nay" (0) otherwise.

(a) Assuming that a coalition S is represented as the initial decision vector $x \in \{0,1\}^n$ defined as $x_i = 1$ iff $i \in S$, the decision system process defines a cooperative game assigning to a coalition S a value in $v(S) \in \{0,1\}$. Is this game simple?

(b) Provide a characterization of the games in the family with non-empty core.

(c) Can the Banzhaf value of player i be computed in polynomial time?

7. Consider a simple game $(N, \mathcal{W})$. We say that player i is a

- *passer* iff, $\forall S \subseteq N$, if $i \in S$, then $S \in \mathcal{W}$.
- *vetoer* iff, $\forall S \subseteq N$, if $i \notin S$, then $S \in \mathcal{L}$.
- *dictator* iff, $\forall S \subseteq N$, $S \in \mathcal{W}$ iff $i \in S$.

(a) Provide a simpler characterization of those properties.

(b) Under which of the forms of representation of simple games based on sets can those properties being decided in polynomial time?

8. Let $\Gamma = (N, W)$ be a simple game, its dual is the game $\Gamma^* = (N, W^*)$ where $W^* = S \subseteq N \mid N \setminus S \in W$. Show that if Γ is a WVG then Γ^* is also a WVG. Assume that a WVG is described by $\Gamma = (q; w_1, \ldots, w_n)$, provide a polynomial time algorithm to compute the weight and quota of its dual.