

Strategic games: Basic definitions and examples

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1 Game theory and CS

2 Strategic games

3 Congestion games

Basic References non-coop game theory

- Osborne. [An Introduction to Game Theory](#), Oxford University Press, 2004
- Nisan et al. Eds. [Algorithmic game theory](#), Cambridge University Press, 2007

Where to use game theory?

Game theory **studies** decisions made in an environment in which players interact.

game theory studies **choice of optimal behavior** when **personal costs and benefits** depend upon the **choices of all participants**.

What for?

Game theory looks for **states of equilibrium** sometimes called **solutions**

and analyzes interpretations/properties of such states

Game Theory for CS?

- Framework to analyze equilibrium states of protocols used by rational agents.
Price of anarchy/stability.
- Tool to design protocols for internet with guarantees.
Mechanism design.
- New concepts to analyze/justify behavior of on-line algorithms
Give guarantees of stability to dynamic network algorithms.
- Source of new computational problems to study.
Algorithmic game theory

Types of games

- Non-cooperative games
 - strategic games
 - extensive games
 - repeated games
 - Bayesian games
- Cooperative games
 - simple games
 - weighted games
 - ...

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Strategic game

A **strategic game** Γ (with ordinal preferences) consists of:

- A finite set $N = \{1, \dots, n\}$ of **players**.
- For each player $i \in N$, a nonempty set of **actions** A_i .
- Each player chooses his action **once**. Players choose actions **simultaneously**.
No player is informed, when he chooses his action, of the actions chosen by others.
- For each player $i \in N$, a **preference relation** (a complete, transitive, reflexive binary relation) $\preceq_i$ over the set $A = A_1 \times \dots \times A_n$.

It is frequent to specify the players' preferences by giving **utility functions** $u_i(a_1, \dots, a_n)$. Also called **pay-off functions**.

Example: Prisoner's Dilemma

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The story

- Two suspects in a major crime are held in separate cells.
- Evidence to convict each of them of a minor crime.
- No evidence to convict either of them of a major crime unless one of them finks.

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The penalties

- If **both stay quiet**, be convicted for a minor offense (**1 year**).
- If **only one finks**, he will be **freed** (and used as a witness) and the other will be convicted for a major offense (**4 years**).
- If **both fink**, each one will be convicted for a major offense with a reward for cooperation (**3 years each**).

Prisoner's Dilemma: Benefits?

Prisoner's Dilemma: Benefits?

The Prisoner's Dilemma **models a situation** in which

- there is a gain from **cooperation**,
- but each player has an incentive to **free ride**.

Prisoner's Dilemma: rules and preferences

Rules

- **Players** $N = \{\text{Suspect 1, Suspect 2}\}$.
- **Actions** $A_1 = A_2 = \{\text{Quiet, Fink}\}$.
- **Action profiles** $A = A_1 \times A_2 = \{(\text{Quiet, Quiet}), (\text{Quiet, Fink}), (\text{Fink, Quiet}), (\text{Fink, Fink})\}$

Preferences

- Player 1
 $(\text{Fink, Quiet}) \preceq_1 (\text{Quiet, Quiet}) \preceq_1 (\text{Fink, Fink}) \preceq_1 (\text{Quiet, Fink})$
- Player 2
 $(\text{Quiet, Fink}), \preceq_2 (\text{Quiet, Quiet}) \preceq_2 (\text{Fink, Fink}) \preceq_2 (\text{Fink, Quiet})$

Prisoner's Dilemma: rules and utilities

Rules

- **Players** $N = \{\text{Suspect 1, Suspect 2}\}$.
- **Actions** $A_1 = A_2 = \{\text{Quiet, Fink}\}$.
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 $\{(\text{Quiet, Quiet}), (\text{Quiet, Fink}), (\text{Fink, Quiet}), (\text{Fink, Fink})\}$

profile	u_1	u_2
(Fink, Quiet)	3	0
(Quiet, Quiet)	2	2
(Fink, Fink)	1	1
(Quiet, Fink)	0	3

Prisoner's Dilemma: rules and utilities

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Rationality: Players choose actions in order to maximize personal utility (**minimize cost**)

Prisoner's Dilemma: rules and costs

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- **Players** $N = \{\text{Suspect 1, Suspect 2}\}$.
- **Actions** $A_1 = A_2 = \{\text{Quiet, Fink}\}$.
- **Action profiles** $A = A_1 \times A_2$
 $\{(\text{Quiet, Quiet}), (\text{Quiet, Fink}), (\text{Fink, Quiet}), (\text{Fink, Fink})\}$

profile	c_1	c_2
(Fink, Quiet)	0	3
(Quiet, Quiet)	1	1
(Fink, Fink)	2	2
(Quiet, Fink)	3	0

Prisoner's Dilemma: rules and costs

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- **Actions** $A_1 = A_2 = \{\text{Quiet, Fink}\}$.
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(Quiet, Fink)	3	0

Rationality: Players choose actions in order to minimize personal cost

Prisoner's Dilemma: bi-matrix representation

We can represent the game in a compact way on a **bi-matrix**.

utility	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

cost	Quiet	Fink
Quiet	1,1	3,0
Fink	0,3	2,2

Example: Matching Pennies

The story

- Two people choose, simultaneously, whether to show the head or tail of a coin.
- If they show same side, person 2 pays person 1 1eur, otherwise person 1 pays person 2 1eur.
- Payoff are equal to **the amounts of money involved**.

Example: Matching Pennies

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Head	1,-1	-1,1
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This is an example of a zero-sum game

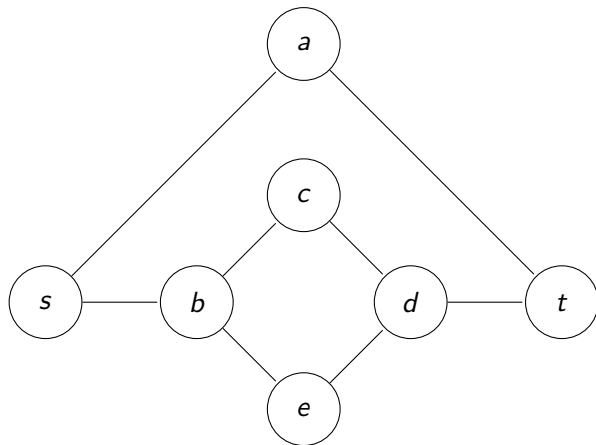
Example: Sending from s to t

The story

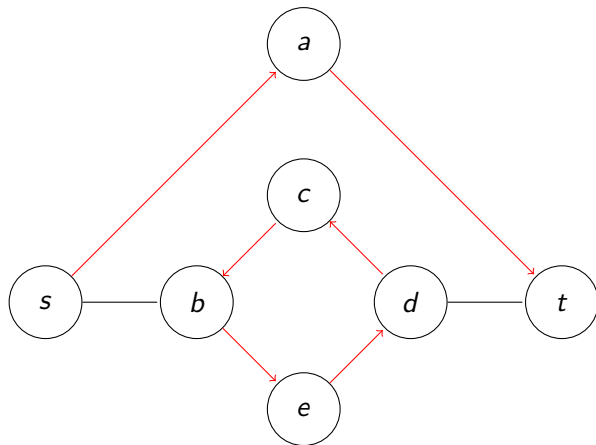
- We have a graph $G = (V, E)$ and two vertices $s, t \in V$.
- There is one player for each vertex $v \in V$, $v \neq t$.
- The set of actions for player u is $N_G(u)$.
- A strategy profile is a set of vertices $(v_1, \dots, v_{n-1})$.
- Pay-offs are defined as follows:
player u gets 1 if the shortest path joining s to t in the digraph induced by $v_1, \dots, v_{n-1}$ contains (u, v_u) , otherwise gets 0.

Players are selfish but the system can get a different reward/cost.
For example the cost of the shortest path.

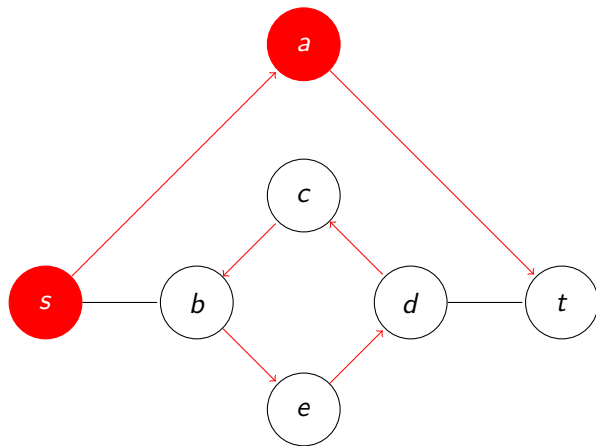
Sending from s to t : example



Sending from s to t : strategy profile (1)

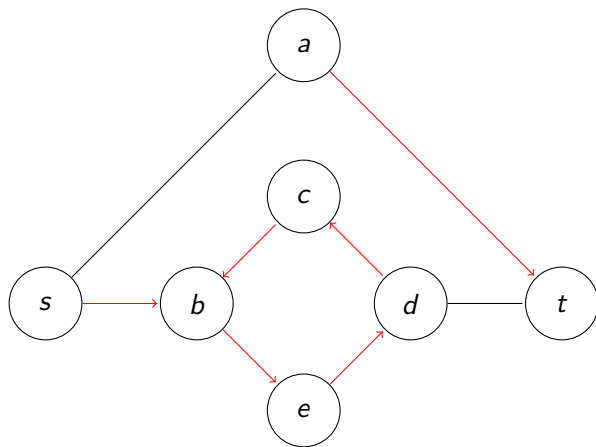


Sending from s to t : pay-offs (1)

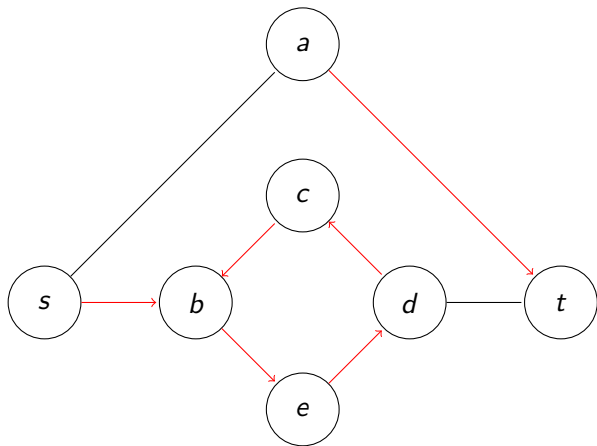


Red nodes get pay-off 1, other nodes get pay-off 0.

Sending from s to t : strategy profile (2)



Sending from s to t : strategy profile (2)



All nodes get pay-off 0.

Strategies: Notation

A **strategy of player** $i \in N$ in a strategic game Γ is an action $a_i \in A_i$.

A **strategy profile** $s = (s_1, \dots, s_n)$ consists of a strategy for each player.

For each $s = (s_1, \dots, s_n)$ and $s'_i \in A_i$ we denote by

$$(s_{-i}, s'_i) = (s_1, \dots, s_{i-1}, s'_i, s_{i+1}, \dots, s_n)$$

$$s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$$

is not an strategy profile but can be seen as an strategy for the other players.

Best response

Let Γ be an strategic game defined through pay-off functions
The set of **best responses** for player i to s_{-i} is

$$BR(s_{-i}) = \{a_i \in A_i \mid u_i(s_{-i}, a_i) = \max_{a'_i \in A_i} u_i(s_{-i}, a'_i)\}$$

Those are the actions that give maximum pay-off provided the other players do not change their strategies.

Solution concepts

- Pure Nash equilibrium
- (Mixed) Nash equilibrium
- Dominant strategies
- Strong Nash equilibrium
- Correlated equilibrium

Dominant strategies

A **dominant strategy** for player i is an strategy s_i^* if regardless of what other players do the outcome is better for player i .

Formally, for every strategy profile $s = (s_1, \dots, s_n)$,
 $u_i(s) \leq u_i(s_{-i}, s_i^*)$.

Pure Nash equilibrium

A **pure Nash equilibrium** is an strategy profile $s^* = (s_1^*, \dots, s_n^*)$ such that
 no player i can do better choosing an action different from s_i^* ,
 given that every other player j adheres to s_j^* :

*for every player i and for every action $a_i \in A_i$ it holds
 $u_i(s_{-i}^*, s_i^*) \geq u_i(s_{-i}^*, a_i)$.*

*Equivalently, for every player i and for every action $a_i \in A_i$
 it holds $s_i^* \in BR(s_{-i}^*)$.*

Pure Nash Equilibrium

- Is a strategy profile in which **all players are happy**.
- Identified with a fixed point of an iterative process of computing a **best response**.
- However, **the game is played only once!**
- GT deals with the existence and analysis of equilibria assuming rational behavior.
players try to maximize their benefit
- GT does not provide algorithmic tools for computing such equilibrium if one exists.

More games

utility	Quiet	Fink
Quiet	2,2	0,3
Fink	3,0	1,1

utility	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

utility	Bach	Stravinsky
Bach	2,1	0,0
Stravinsky	0,0	1,2

utility	swerve	don't sw
swerve	3,3	2,4
don't sw	4,2	1,1

Dominant strategies? Nash equilibria?

Examples of Nash equilibrium

- Prisoner's Dilemma, (Fink, Fink).
- Bach or Stravinsky, (Bach, Bach), (Stravinsky, Stravinsky).
- Matching Pennies, none.
- Chicken, (swerve, don't sw), (don't sw, swerve).

Example: Sending from s to t

The story

- We have a graph $G = (V, E)$ and two vertices $s, t \in V$.
- There is one player for each vertex $v \in V$, $v \neq t$.
- The set of actions for player u is $N_G(u)$.
- A strategy profile is a set of vertices $(v_1, \dots, v_{n-1})$.
- Pay-offs are defined as follows:
player u gets 1 if the shortest path joining s to t in the digraph induced by $v_1, \dots, v_{n-1}$ contains (u, v_u) , otherwise gets 0.

Exercise: Dominant strategies? Nash equilibria?

Pure Nash equilibrium

- First notion of equilibrium for non-cooperative games.
- There are strategic games with no pure Nash equilibrium.
- There are games with more than one pure Nash equilibrium.
- How to compute a Nash equilibrium if there is one?

Mixed strategies

Until now players were selecting as strategy an **action**.

A **mixed strategy** for player i is a distribution (lottery) σ_i on the set of actions A_i .

The utility function for player i is the **expected utility** under the joint distribution $\sigma = (\sigma_1, \dots, \sigma_n)$ assuming independence.

$$U_i(\sigma) = \sum_{(a_1, \dots, a_n) \in A} \sigma_1(a_1) \cdots \sigma_n(a_n) u_i(a_1, \dots, a_n)$$

Mixed Nash equilibrium

A **mixed Nash equilibrium** is a profile $\sigma^* = (\sigma_1^*, \dots, \sigma_n^*)$ such that no player i can get better utility choosing a distribution different from σ_i^* , given that every other player j adheres to σ_j^* .

Theorem (Nash)

Every strategic game has a mixed Nash equilibrium.

From a computational point of view, mixed strategies present an additional representation problem.

In CS we can store only rational numbers. It is known

- For two player game there are always a mixed Nash equilibrium with rational probabilities.
- There are three player games without rational mixed Nash equilibrium.

[Schoenebeck and Vadhan: eccc 51, 2005]

NE in the Matching pennies game

utility	Head	Tail
Head	1,-1	-1,1
Tail	-1,1	1,-1

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- We know that the game has no PNE

NE in the Matching pennies game

utility	Head	Tail
Head	1,-1	-1,1
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- We know that the game has no PNE
- Is $((.2, .8), (.4, .6))$ a NE?
- Is $((.5, .5), (.5, .5))$ a NE?

Checking for a Nash equilibrium

Given a distribution σ_i on A_i define the **support** of σ_i to be the set

$$\text{supp}(\sigma_i) = \{a_i \mid \sigma_i(a_i) \neq 0\}$$

Theorem

A mixed strategy profile σ is a Nash equilibrium iff, for any player i and any action $a_i \in \text{supp}(\sigma_i)$, a_i is a best response to σ_{-i}

Basic problems

Is (pure) Nash (ISN/ISP_N)

Given a strategic game Γ and a mixed (pure) strategy profile s , decide whether s is a Nash equilibrium of Γ .

Exists pure Nash? (EP_N)

Given a strategic game Γ , decide whether Γ has a Pure Nash equilibrium.

Compute (pure) Nash (CN,CP_N)

Given a strategic game Γ , compute a (pure) Nash equilibrium (if it exists).

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Congestion games

Congestion games

A congestion game

- is defined on a finite set E of resources and
- has n players
- using a delay function d mapping $E \times \mathbb{N}$ to the integers.
- The actions for each player are subsets of E .
- The pay-off functions are the following:

$$u_i(a_1, \dots, a_n) = - \left(\sum_{e \in a_i} d(e, f(a_1, \dots, a_n, e)) \right)$$

being $f(a_1, \dots, a_n, e) = |\{i \mid e \in a_i\}|$.

Network congestion games

Network congestion games

A network congestion game

- is defined on a directed graph $G = (V, E)$ resources are the edges
- has n players
- using a delay function d mapping $E \times \mathbb{N}$ to the integers.
- The actions for each player are paths from s_i to t_i , for some $s_i, t_i \in V(G)$.
- The pay-off functions are the following:

$$u_i(a_1, \dots, a_n) = - \left(\sum_{e \in a_i} d(e, f(a_1, \dots, a_n, e)) \right)$$

being $f(a_1, \dots, a_n, e) = |\{i \mid e \in a_i\}|$.