Combinatorial Optimization Games

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Induced subgraph games

A game is described by an undirected, weighted graph $G = (N, E)$ with $|N| = n$ and $|E| = m$ and an integer edge weight function $w$. The weight of edge $(i, j) \in E$ is denoted by $w_{i,j}$.

In the game $\Gamma(G, w) = (N, v)$ the set of players is $N$, and the value $v$ of a coalition $C \subseteq N$ is $v(C) = \sum_{i, j \in C, i < j} \sum_{(i, j) \in E} w_{i,j}$.

Usually self-loops are allowed when we want that the value of a singleton is different from 0.

Observe that $v(\emptyset) = 0$ and $v(N) = w(E)$. 
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## Induced subgraph games

Induced subgraph games model aspects of social networks. The value of each coalition (team, club) is determined by the relationships among its members: a player assigns a positive utility to being in a coalition with his friends and a negative utility to being in a coalition with his enemies.

The representation is succinct as long as the number of bits required to encode edge weights is polynomial in $|N|$: using an adjacency matrix to represent the graph requires only $n^2$ entries. Weights can be exponential in $n$ and still have polynomial size.
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- Weights can be exponential in \(n\) and still have polynomial size.
Completeness?

Consider the game $\Gamma = (N, v)$, where $n = \{1, 2, 3\}$ and $v(C) = \begin{cases} 0 & \text{if } |C| \leq 1 \\ 1 & \text{if } |C| = 2 \\ 6 & \text{if } |C| = 3 \end{cases}$.

Assume that $\Gamma(G, w)$ realizes $\Gamma$. By the first condition all self-loops must have weight 0. By the second condition any pair of different vertices must be connected by an edge with weight 1. So $G$ must be a triangle. But then $v(\{1, 2, 3\}) = 3 \neq 6$. 

Is this a complete representation?

All simple games can be represented as induced subgraph games?

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**AGT-MIRI**

Cooperative Game Theory
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AGT-MIRI Cooperative Game Theory
Properties of valuations

- **monotone** if \( v(C) \leq v(D) \) for \( C \subseteq D \subseteq N \).
- **superadditive** if \( v(C \cup D) \geq v(C) + v(D) \), for every pair of disjoint coalitions \( C, D \subseteq N \).
- **supermodular** \( v(C \cup D) + v(C \cap D) \geq v(C) + v(D) \).
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- A game $(N, v)$ is **convex** iff $v$ is supermodular.
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- A game \((N, \nu)\) is convex iff \( \nu \) is supermodular.
- Since we allow for negative edge weights, induced subgraph games are not necessarily monotone.
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- A game $(N, \nu)$ is **convex** iff $\nu$ is supermodular.

Since we allow for negative edge weights, induced subgraph games are not necessarily monotone.

However, when all edge weights are non-negative, induced subgraph games are **convex**.
Can the core be empty?

The core of $\Gamma(N,v)$ is the set of all imputations $x$ such that $v(S) \leq x(S)$, for each coalition $S \subseteq N$. 

AGT-MIRI
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Theorem

If $\Gamma = (N, v)$ is a convex game, then $\Gamma$ has a non-empty core.

Fix an arbitrary permutation $\pi$, and let $x_i$ be the marginal contribution of $i$ with respect to $\pi$.

Let us show that $(x_1, \ldots, x_n)$ is in the core of $\Gamma$.

For $C \subseteq N$, we can assume that $C = \{i_1, \ldots, i_s\}$ where $\pi(i_1) < \cdots < \pi(i_s)$.

So, $v(C) = v(\{i_1\}) - v(\emptyset) + v(\{i_1, i_2\}) - v(\{i_1\}) + \cdots + v(C) - v(C \setminus \{i_s\})$.

By supermodularity we have, $v(\{i_1, \ldots, i_j-1, i_j\}) - v(\{i_1, \ldots, i_j-1\}) \leq v(\{1, \ldots, i_j\}) - v(\{1, \ldots, i_j-1\})$.

Therefore $v(C) \leq x(C)$ and $v(N) = x(N)$.

Observe that we have shown that the vector formed by the Shapley value is in the core of a convex game.
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  - Therefore \( v(C) \leq x(C) \) and \( v(N) = x(N) \).
- Observe that we have shown that the vector formed by the Shapley value is in the core of a convex game.
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- For \( C \subseteq N \), let \( \delta_i(C) = v(C \cup \{i\}) - v(C) \).
- The Shapley value of player \( i \) in a game \( \Gamma = (N, v) \) with \( n \) players is

\[
\Phi_i(\Gamma) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \delta_i(S_\pi(i))
\]
Properties of the Shapley value:

- Efficiency: $\Phi_1 + \ldots + \Phi_n = v(N)$
- Dummy: if $i$ is a dummy, $\Phi_i = 0$
- Symmetry: if $i$ and $j$ are symmetric, $\Phi_i = \Phi_j$
- Additivity: $\Phi_i(G_1 + G_2) = \Phi_i((G_1) + \Phi_i(G_2)$

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$$\Phi(i) = \frac{1}{2} \sum_{(i,j) \in E} w_{i,j}.$$
Shapley value: Computation

Let $\{e_1, \ldots, e_m\}$ be the set of edges in $G$. We can decompose the graph $G$ into $m$ graphs $G_1, \ldots, G_m$, where for $1 \leq j \leq m$ the graph $G_j = (V, \{e_j\})$.

Considering the same weight as in the original graph, let $\Gamma_j = \Gamma(G_j, w)$. According to the definitions:

$$\Gamma = \Gamma(G, w) = \Gamma_1 + \cdots + \Gamma_m.$$

By the additivity axiom, for each player $i \in N$ we have

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We have to compute $\Phi_i(\Gamma_j)$. When $i$ is not incident to $e_j$, $i$ is a dummy in $\Gamma_j$ and $\Phi_i(\Gamma_j) = 0$.

When $e_j = (i, \ell)$ for some $\ell \in \mathbb{N}$, players $i$ and $\ell$ are symmetric in $\Gamma_j$.

Since the value of the grand coalition in $\Gamma_j$ equals $w(i, \ell)$, by efficiency and symmetry we get $\Phi_i(\Gamma_j) = w(i, \ell) / 2$. 
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The Shapley value of player $i$ in $\Gamma(G, w)$ is

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Corollary

The Shapley values of induced subgraph games can be computed in polynomial time.
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Can the core be empty?

Theorem

Consider a game $\Gamma(G, w)$, the following are equivalent

- The vector of Shapley values is in the core
- $(G, w)$ has no negative cut
- The core is non-empty
Can the core be empty?

The Shapley value is in the core iff $G$ has no negative cut.

Let $e(S, x) = v(S) - x(S)$ be the excess of coalition $S$ at the imputation $x$. Thus, $x$ is in the core iff $e(x, S) \leq 0 \forall S \subseteq N$.

For the Shapley values, $e(S, \Phi)$ is $-1/2$ times the weight of the edges going from $S$ to $N \setminus S$. Hence the Shapley value is in the core if and only if there is no negative cut $(S, N \setminus S)$. 
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The Shapley value is in the core iff $G$ has no negative cut.

- Let $e(S, x) = v(S) - x(S)$ be the excess of coalition $S$ at the imputation $x$.
- Thus, $x$ is in the core iff $e(x, S) \leq 0 \ \forall S \subseteq N$.
- For the Shapley values, $e(S, \Phi)$ is $-\frac{1}{2}$ times the weight of the edges going from $S$ to $N \setminus S$.
- Hence the Shapley value is in the core if and only if there is no negative cut $(S, N \setminus S)$. 
Can the core be empty?

The core is nonempty if and only if the graph $G$ has no negative cut. If $G$ has no negative cut, the vector of Shapley values is in the core (by the previous proof). We have seen that if the core is non-empty, then the vector of Shapley values is in the core.
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Can the core be empty?

- **NEGATIVE-CUT**: Given a weighted graph \((G, w)\), determine whether there is a negative cut in \(G\).
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- W-MAX-CUT: Given a weighted graph \((G, w)\) with non-negative weights and an integer \(k\), determine whether there is a cut of size at least \(k\) in \(G\), is NP-complete.
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- Let \((G, w)\) with non-negative weights and an integer \(k\). \(G'\) is obtained as the disjoint union of \(G\) and the graph \((\{a, b\}, \{(a, b)\})\). Define \(w'\) as \(w'(e) = w(e)\) for \(e \in E(G)\) and \(w((a, b)) = -k\).
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- \( G \) has a a cut of size at least \( k \) iff \( G' \) has a negative cut.
Theorem

The following problems are NP-complete:

- Given \((G, w)\) and an imputation \(x\), is it not in the core of \(\Gamma(G, w)\)?
- Given \((G, w)\), is the vector of Shapley values of \(\Gamma(G, w)\) not in the core of \(\Gamma(G, w)\)?
- Given \((G, w)\), is the core of \(\Gamma(G, w)\) empty?
Complexity of core related problems

Theorem

Given \((G, w)\), when all weights are non-negative, we can test in polynomial time

1. whether the core is non-empty.
2. whether an imputation \(x\) is in the core of \(\Gamma(G, w)\).
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Theorem

Given \((G, w)\), when all weights are non-negative, we can test in polynomial time

- whether the core is non-empty.
- whether an imputation \(x\) is in the core of \(\Gamma(G, w)\).

The first question is trivial as the vector of Shapley values belong to the core. The second problem can be solved by a reduction to MAX-FLOW.
1 Induced subgraph games

2 Minimum cost spanning tree games

3 References
MST Games

Minimum cost spanning tree games

A game is described by a weighted complete graph $(G, w)$ with $n + 1$ vertices. $V(G) = \{v_0, \ldots, v_n\}$. The weight of edge $(i, j) \in E$ is denoted by $w_{i, j}$. We assume $w_{i, j} \geq 0$.

In the game $\Gamma(G, w) = (N, c)$, the set of players is $N = \{v_1, \ldots, v_n\}$, and the cost $c$ of a coalition $C \subseteq N$ is $c(C) = \text{the weight of a minimum spanning tree of } G[S \cup \{v_0\}]$. Self-loops are not allowed. The cost of a singleton coalition $\{i\}$ is $c(\{i\}) = w_{0, i}$.

Observe that $v(\emptyset) = 0$ and $v(N) = w(T)$ where $T$ is a MST of $G$. 
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Properties of valuations
Core emptiness
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MST Games

MST games model situations where a number of users must be connected to a common supplier, and the cost of such connection can be modeled as a minimum spanning tree problem. The representation is succinct as long as the number of bits required to encode edge weights is polynomial in $|N|$: using an adjacency matrix to represent the graph requires only $n^2$ entries.
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Completeness?

All simple games can be represented as MST games? NO

Consider the game $\Gamma = (N, c)$, where $N = \{1, 2, 3\}$ and $c(C) = \begin{cases} 0 & \text{if } |C| \leq 1 \\ 1 & \text{if } |C| = 2 \\ 6 & \text{if } |C| = 3 \end{cases}$

Assume that $\Gamma(G, w)$ realizes $\Gamma$. $V(G) = \{0, 1, 2, 3\}$.

By the first condition $w_i = 0$, for $i \in \{1, 2, 3\}$.

Thus, a coalition with $|C| = 2$ has a MST with zero cost and the second condition cannot be met.
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- Is this a complete representation?
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Properties of valuations

- **monotone** if $v(C) \leq v(D)$ for $C \subseteq D \subseteq N$.
- **superadditive** if $v(C \cup D) \geq v(C) + v(D)$, for every pair of disjoint coalitions $C, D \subseteq N$.
- **subadditive** $v(C \cup D) \leq v(C) + v(D)$, for every pair of disjoint coalitions $C, D \subseteq N$.
- **supermodular** $v(C \cup D) + v(C \cap D) \geq v(C) + v(D)$.
- A game $(N, v)$ is convex iff $v$ is supermodular.
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  \( c \) is **subadditive**.
Consider a MST game $\Gamma(G, w)$. Let $T^*$ be a MST of $(G, w)$ obtained using Prim's algorithm. The vector $x = (x_1, \ldots, x_n)$ that allocates to player $i \in N$ the weight of the first edge $i$ encounters on the (unique path) from $v_i$ to $v_0$ in $T^*$ belongs to the core of $\Gamma$. Such an allocation is called standard core allocation.
Can the core be empty?

A standard allocation $x$ belongs to the core.
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- Clearly $\sum_{i=1}^{n} x_i = w(T^*) = c(N)$. 
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- Clearly \( \sum_{i=1}^{n} x_i = w(T^*) = c(N) \).
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- Analyzing carefully both executions it can be shown that $x_j \leq y_j$ as the edges considered in one partition are a subset of the other.
How fair are standard core allocations?

- Most of the cost is charged to player 1.
- How to find more appropriate core allocations?
More appropriate core allocations?

- There are many proposals to try to get more appropriate core allocations.

Granot and Huberman [1984] propose the weak demand allocation and strong demand allocation procedures, which rectify standard allocations by transferring costs (whenever possible) from one node to their children.

Norde, Moretti, and Tijs [2001] show how to find a population monotonic allocation scheme (PMAS), which is an allocation scheme that provides a core element for the game and all its subgames and which, moreover, satisfies a monotonicity condition in the sense that players have to pay less in larger coalitions.
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Theorem

The following problem is NP-complete:

Given \((G, w)\) and an imputation \(x\), is it not in the core of \(\Gamma(G, w)\)?

The proof follows by a reduction from EXACT COVER BY 3-SETS [Faigle et al., Int. J. Game Theory 1997].
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