Topic 7. Complexity

Data Structures and Algorithms

FIB

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Classes

Algorithm analysis studies the amount of resources that an algorithm needs to solve a problem.

Complexity theory considers all possible algorithms that solve the same problem.

- Algorithm analysis focuses on algorithms, whereas complexity theory focuses on problems
- We will study some basic tools to classify problems according to their complexity

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Decision problems

In order to better classify problems, we will consider their decision versions.

Definition

A decision problem is a problem where one has to determine whether an instance satisfies a certain property.

Decision problems

Lots of problems seen so far are or can be made decisional.

Some decision problems on graphs:

- **connectivity**: given a graph, determine whether it is connected
- **reachability**: given a graph $G = (V, E)$ and two vertices $i, j \in V$, determine whether there is a path from i to j in G
- **shortest path**: given a graph $G = (V, E)$, two vertices $i, j \in V$ and a natural number k, determine whether there is a path between i and j in G of length at most k
- **longest path**: given a graph $G = (V, E)$, two vertices $i, j \in V$ and a natural number k, determine whether there is a path between i and j in G of length at least k
- **3-colorability**: given a graph, determine whether it is 3-colorable

Decision problems

Some problems do not make sense in their decision version.

Decision n-queens problem (1st version)

Given a natural number n, determine whether we can place n queens on an $n \times n$ board so that no two queens threaten each other.

It is known that there are solutions for all $n \neq 2, 3$. Hence, the following algorithm decides the problem in time $\Theta(1)$.

```
QUEENS(n)
    if n = 2 o n = 3 then
        return FALSE
    else
        return TRUE
```

Decision problems

What is interesting is not whether there is a solution, but finding one.

Decision n-queens problem (2nd version)

Given a natural number n and k values $r_1, \ldots r_k$, with $k \leq n$, determine whether we can place n queens on an $n \times n$ board so that no two queens threaten each other and for all i such that $1 \leq i \leq k$, the queen in row i is in column r_i.

This version, despite being decisional, allows one to find a solution with

$$(n-1) + (n-2) \cdots + 2 = \sum_{i=2}^{n-1} i = \frac{n(n-1)}{2} - 1 \in \Theta(n^2)$$

executions of the algorithm that solves it.

Decision problems

Some other decision problems:

1. **primality**: given a natural number, determine whether it is a prime
2. **traveling salesperson problem (TSP)**: given n cities, the distances among them and a number of kilometers k, determine whether there is a route of at most k kilometers that visits each city exactly once and goes back to the origin

A decision problem is a set consisting of an infinite number of instances.

If a problem consists of a finite number of instances, it can be solved by a constant-time algorithm (e.g. 8-queens).

Decision problems

A decision problem is formally represented as a set.

If T is a property that can be checked on the elements of an instance set E, we can formulate the following decision problem:

Problem A
Given $x \in E$, determine whether $T(x)$ holds.

Formally, A can be described as the set:

$$A = \{ x \in E \mid T(x) \}.$$

Decision problems

The problem instances will belong to some concrete domains such as:

- natural numbers
- tuples of natural numbers
- graphs
- weighted dags
- Boolean formulas

In each case, we will consider a size or length function.

Size function

Given $x \in E$, where E is a domain, the size of x, represented as $|x|$, is the number of symbols of a standard encoding of x.

Decision problems

Given a problem A defined over an input set E, we will distinguish between

- positive instances: the ones belonging to A
- negative instances: the ones belonging to $E - A$

Primality

The primality problem can be described informally

> **Primality** (PRIMES)
> Given a natural number x, determine whether x is prime.

Or formally as the set of positive inputs:

$$\text{PRIMES} = \{x \in \mathbb{N} \mid x \text{ is prime } \}.$$

A size function for the natural numbers counts the number of digits of its binary representation:

$$|x| = \text{ number of digits of } x \text{ in binary} = \lfloor \log_2 x \rfloor + 1.$$

Decision problems

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Decision problems

Once we can describe problems as mathematical objects (decision problems as sets), we can group them into classes according to their complexity.

- We will consider classes of problems that can be solved using a certain amount of resources
- A class groups problems in the same way as a problem groups instances
- We have to distinguish between three levels of abstraction:
    - Instances ——> strings of characters
    - Problems —-> sets of instances
    - Classes ——-> sets of problems

Polynomial and exponential time

Let us assume that $t : \mathbb{N} \to \mathbb{R}^+$ is a function.

Algorithms of cost t

We say that an algorithm $\mathcal{A}$ has cost t if its worst-case cost belongs to $O(t)$.

Problems decidable in time t

If an algorithm $\mathcal{A}$ takes inputs from a set E and has a binary output, we write

$$\mathcal{A} : E \to \{0, 1\}.$$

We say that a decision problem A is decidable in time t if there exists an algorithm $\mathcal{A} : E \to \{0, 1\}$ of cost t such that, for all $x \in E$:

$$x \in A \Rightarrow \mathcal{A}(x) = 1$$

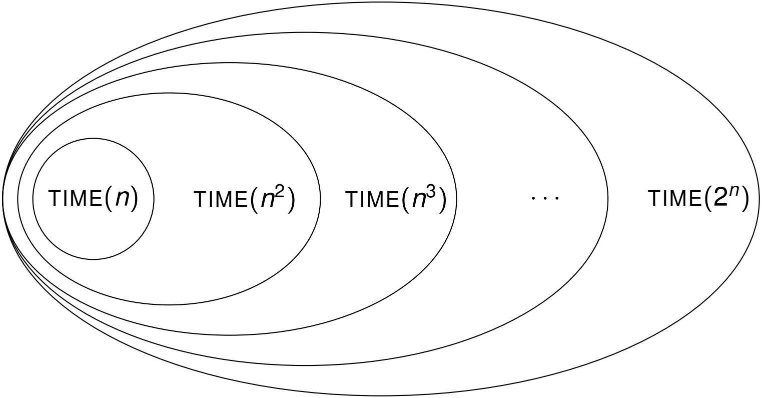
$$x \notin A \Rightarrow \mathcal{A}(x) = 0$$

Polynomial and exponential time

Class TIME(t)

Given a function $t : \mathbb{N} \rightarrow \mathbb{R}^+$, we group the problems decidable in time t:

$$\text{TIME}(t) = \{A \mid A \text{ is decidable in time } t\}.$$



Polynomial and exponential time

We remind that there is a huge difference between having a polynomial or an exponential algorithm for a problem. In Topic 1 we saw two tables showing:

- quantitative differences (table 1)
- qualitative differences (table 2)

between polynomials and exponentials.

Polynomial and exponential time

Table 1 (Garey/Johnson, *Computers and Intractability*)

Comparison between polynomial and exponential functions.

cost	10	20	30	40	50
n	0.00001 s	0.00002 s	0.00003 s	0.00004 s	0.00005 s
n^2	0.0001 s	0.0004 s	0.0009 s	0.0016 s	0.0025 s
n^3	0.001 s	0.008 s	0.027 s	0.064 s	0.125 s
n^5	0.1 s	3.2 s	24.3 s	1.7 min	5.2 min
2^n	0.001 s	1.0 s	17.9 min	12.7 days	35.7 years
3^n	0.059 s	58 min	6.5 years	3855 cents.	2×10^8 cents.

Polynomial and exponential time

Table 2 (Garey/Johnson, *Computers and Intractability*)

Effect of technological improvements on polynomial and exponential algorithms.

cost	current technology	technology $\times 100$	technology $\times 1000$
n	N_1	$100N_1$	$1000N_1$
n^2	N_2	$10N_2$	$31.6N_2$
n^3	N_3	$4.64N_3$	$10N_3$
n^5	N_4	$2.5N_4$	$3.98N_4$
2^n	N_4	$N_4 + 6.64$	$N_4 + 9.97$
3^n	N_5	$N_5 + 4.19$	$N_5 + 6.29$

Polynomial and exponential time

Class P

We define the class P as the union of all polynomial-time classes:

$$\mathrm{P} = \bigcup_{k>0} \mathsf{TIME}(n^k).$$

That is, a problem belongs to P if it is decidable in time n^k for some k.

Class EXP

We define the class EXP as the union as the union of all exponential classes:

$$\mathrm{EXP} = \bigcup_{k>0} \mathsf{TIME}(2^{n^k}).$$

That is, a problem is in EXP if it is decidable in time 2^{n^k} for some k.

Examples

Problems in P:

- **connectivity**
- **reachability**
- **primality**
- **shortest path**
- **2-colorability**

Problems in EXP (not known to be in P):

- **longest path**
- **3-colorability**
- **travelling salesperson problem**

Other problems in EXP:

- **generalized chess**, **checkers** and **go**

Theorem

$\mathrm{P} \subsetneq \mathrm{EXP}$.

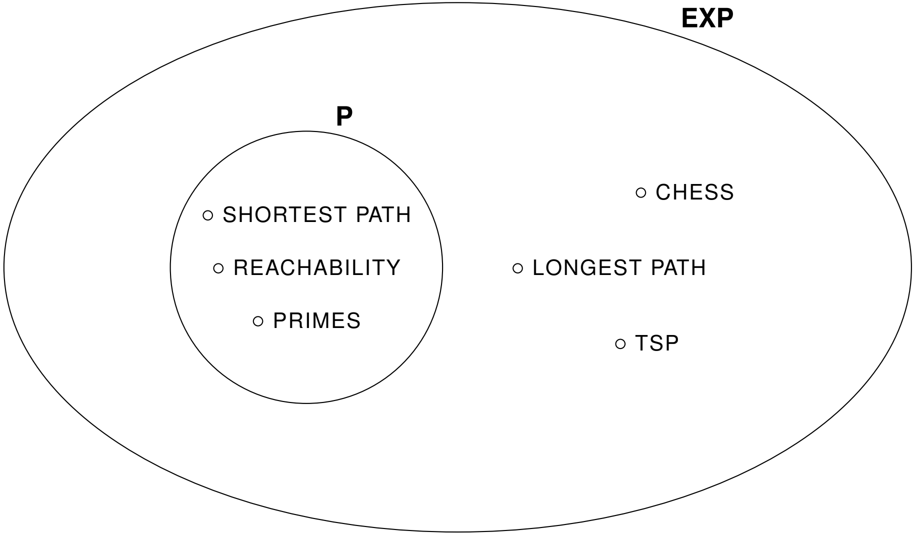
Strict inclusion in the theorem can be divided into two parts:

1. $\mathrm{P} \subseteq \mathrm{EXP}$. Obvious from the definitions:

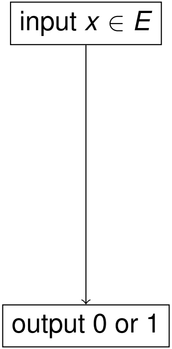
$$\mathrm{P} = \bigcup_{k>0} \mathsf{TIME}(n^k) \subseteq \bigcup_{k>0} \mathsf{TIME}(2^{n^k}) = \mathrm{EXP}$$

2. $\mathrm{P} \neq \mathrm{EXP}$. Proved using the diagonalization technique

Polynomial and exponential time



- Algorithms seen so far are deterministic: they follow a unique computation path from the input to the output
- The execution of an algorithm $\mathcal{A} : E \rightarrow \{0, 1\}$ on a domain E can be seen as a path:



Nondeterminism

A nondeterministic algorithm can reach a result via different paths. Its behavior is more similar to a tree.

Nondeterministic algorithms (informal idea)

An algorithm $\mathcal{A} : E \to \{0, 1\}$ is *nondeterministic* if it can use a new function

$$\text{CHOOSE}(y)$$

that, for an input x and $y \leq x$, splits the computation into y branches, and returns a distinct value between 0 and y on each branch.

- **Computation tree**: The computation starts in a deterministic way until the first CHOOSE instruction; for every value returned by CHOOSE, an independent computation branch is generated with the corresponding value
- **Returned value**: We say that $\mathcal{A}$ returns 1 if some branch returns 1; otherwise, $\mathcal{A}$ returns 0
- **Cost**: The cost of $\mathcal{A}$ is that of the branch with highest cost

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Nondeterminism

Example: Composites

The problem

$$\text{COMPOSITES} = \{x \mid \exists y \;\; 1 < y < x \text{ and } y \text{ divides } x\}$$

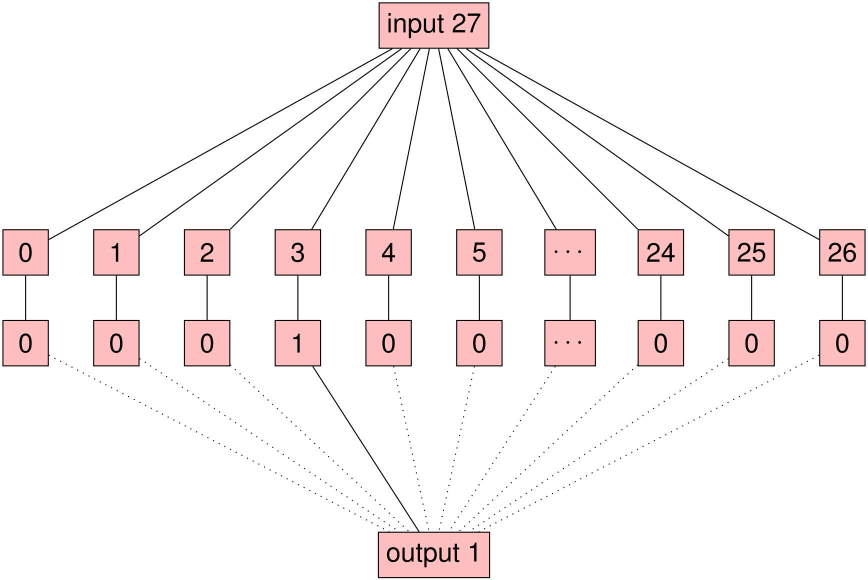
has a trivial exponential deterministic algorithm

> input x
> **for** $y = 2$ **until** $x - 1$
>   **if** y divides x **then**
>   **return** 1
> **return** 0

and a polynomial nondeterministic algorithm

> input x
> $y \leftarrow \text{CHOOSE}(x - 1)$
> **if** $y > 1$ **and** y divides x **then**
>   **return** 1
> **return** 0

Nondeterminism



Nondeterminism

- In the previous example, we say that 3 is a witness of the fact that 27 is not a prime
- That is, in the problem COMPOSITES there exist:
  - Possible witnesses ($y < x$) of the fact that x is composite
  - A polynomial-time verifier algorithm that, given x and y, checks whether y divides x

Nondeterminism

Unlike COMPOSITES, the problem GENERALIZED CHESS has no short witnesses that allow one to check that a player has a winning strategy.

But there are a lot of problems for which it is easy to find short witnesses. For all of them, there are polynomial nondeterministic algorithms.

Nondeterminism

Example: 3-colorability

The 3-colorability problem, represented by the set

$$3\text{-COLOR} = \{\ G \mid G \text{ is 3-colorable }\}$$

has an exponential-time brute-force algorithm

input $G = (V, E)$
$n \leftarrow |V|$
for each tuple $(c_1, \ldots, c_n)$ where $\forall i \leq n \;\; c_i \in \{0, 1, 2\}$
    if $(c_1, \ldots, c_n)$ is a 3-coloring of G **then**
        return 1
return 0

Example: 3-colorability

and a polynomial nondeterministic algorithm

input $G = (V, E)$
$n \leftarrow |V|$
for $i = 1$ **until** n
    $c_i \leftarrow \text{CHOOSE}(2)$
if $(c_1, \ldots, c_n)$ is a 3-coloring of G **then**
    return 1
else
    return 0

Nondeterminism

The formal definition of nondeterministic polynomial algorithms distinguishes:

- the witness computation
- the deterministic computations

Nondeterminism

Decidability in nondeterministic polynomial time

Let Σ be an alphabet and A a decision problem defined over inputs of a set E. We say that A is decidable in nondeterministic polynomial time if there exist

- a polynomial algorithm $\mathcal{V} : E \times \Sigma^* \to \{0, 1\}$ (called verifier) and
- a polynomial $p(n)$

such that for all $x \in E$, we have

$$x \in A \Rightarrow \mathcal{V}(x, y) = 1 \text{ for some } y \in \Sigma^* \text{ such that } |y| = p(|x|)$$

$$x \notin A \Rightarrow \mathcal{V}(x, y) = 0 \text{ for all } y \in \Sigma^* \text{ such that } |y| = p(|x|)$$

If $x \in A$, the y such that $\mathcal{V}(x, y) = 1$ are called witnesses or certificates.

Nondeterminism

In order to know that a problem A is decidable in nondeterministic polynomial time we will have to check that:

1. positive inputs have polynomial-sized witnesses
   (witnesses have to be defined)
2. witnesses can be verified in polynomial time
   (a verifier has to be designed)

Nondeterminism

Composites

Let us consider the problem

$$\text{COMPOSITES} = \{x \mid \exists y \quad 1 < y < x \ \text{ and } y \text{ divides } x\}$$

1. The witnesses for x are all $y \neq 1, x$ that divide x
2. The polynomial is $p(n) = n$
3. The verifier is

$\mathcal{V}(x, y)$
    if $(1 < y < x)$ **and** $(y$ divides $x)$ **then**
        return 1
    else
        return 0

COMPOSITES is decidable in nondeterministic polynomial time because

$$x \in \text{COMPOSITES} \Leftrightarrow \mathcal{V}(x, y) = 1 \text{ for some } y \text{ s.t. } |y| = p(|x|)$$

3-colorability

Let us consider the problem

$$\text{3-COLOR} = \{\ G \mid G \text{ is 3-colorable}\ \}$$

1. The witnesses for $G = (V, E)$ are all 3-colorings C of G of the form $C = (c_1, c_2, \ldots, c_n)$, where $n = |V|$ and $c_i \in \{0, 1, 2\}$ for all $i \leq n$
2. The polynomial (with reasonable encodings of G and C) can be $p(n) = n$
3. The verifier is

$\mathcal{V}(G, C)$
  if C is a 3-coloring of G **then**
    return 1
  else
    return 0

All problems decidable in nondeterministic polynomial time are grouped in one class.

Class NP

We define the class NP (from *nondeterministic polynomial time*) as:

$$\mathrm{NP} = \{A \mid A \text{ is decidable in nondeterministic polynomial time}\}.$$

How does NP compare to P and EXP?

Nondeterminism

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We define the class NP (from *nondeterministic polynomial time*) as:

$$\mathrm{NP} = \{A \mid A \text{ is decidable in nondeterministic polynomial time}\}.$$

How does NP compare to P and EXP?

Main difference between P and NP:

- solutions to problems in P can be found in polynomial time
- solutions to problems in NP can be verified in polynomial time

Example: Perfect squares and composites

1. SQUARES $= \{x \in \mathbb{N} \mid \exists y \quad 1 \leq y < x \quad \text{and} \quad x = y^2\}$
2. COMPOSITES $= \{x \in \mathbb{N} \mid \exists y \quad 1 < y < x \quad \text{and} \quad y \text{ divides } x\}$

Example: 2 and 3-colorability

1. 2-COLOR $= \{ G \mid G \text{ is 2-colorable} \}$
2. 3-COLOR $= \{ G \mid G \text{ is 3-colorable} \}$

Nondeterminism

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Nondeterminism

Theorem

$\mathrm{P} \subseteq \mathrm{NP}$.

Proof

Any deterministic algorithm is nondeterministic (but does not use CHOOSE).

Equivalently, for all $A \in \mathrm{P}$, we can create verifiers $\mathcal{V}$ such that for any x:

$$x \in A \Rightarrow \mathcal{V}(x, y) = 1 \text{ for all } y \in \Sigma^* \text{ such that } |y| = |x|$$

$$x \notin A \Rightarrow \mathcal{V}(x, y) = 0 \text{ for all } y \in \Sigma^* \text{ such that } |y| = |x|$$

To find $\mathcal{V}(x, y)$, it is only needed to simulate $\mathcal{A}(x)$ and return the same value 0 or 1 (independently of y). Hence, $A \in \mathrm{NP}$.

Nondeterminism

Differences between NP and EXP:

- problems in NP have solutions verifiable in polynomial time
- problems in EXP can have exponentially large solutions
- in order to solve problems in NP there is a standard algorithm that searches for a witness, but this is not the case for EXP problems

Nondeterminism

Theorem

$\mathrm{NP} \subseteq \mathrm{EXP}$.

Proof

Let $A \in \mathrm{NP}$. Hence, there is a polynomial $p(n)$ and a verifier $\mathcal{V}$ such that

$$x \in A \Rightarrow \mathcal{V}(x, y) = 1 \text{ for some } y \in \Sigma^* \text{ such that } |y| = p(|x|)$$

$$x \notin A \Rightarrow \mathcal{V}(x, y) = 0 \text{ for all } y \in \Sigma^* \text{ such that } |y| = p(|x|)$$

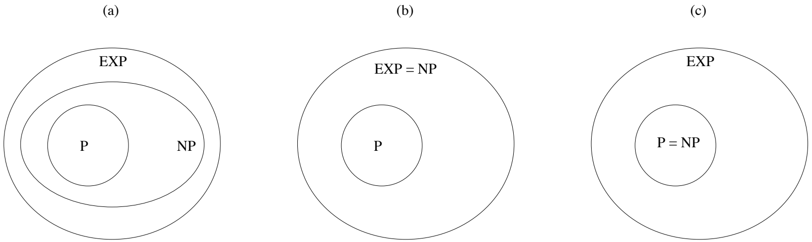
We can consider an exponential algorithm for A that looks for a witness:

```
input x
for all y such that |y| = p(|x|)
    if V(x, y) = 1 then
        return 1
return 0
```

It is easy to see that the previous algorithm is exponential and decides A. Hence, $A \in \mathrm{EXP}$.

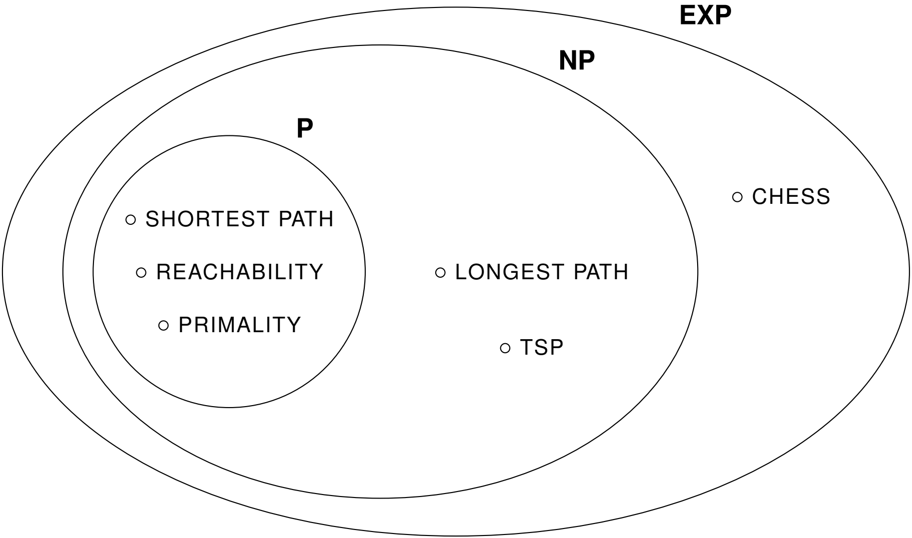
Nondeterminism

- We know that $\mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{EXP}$
- We also know that $\mathrm{P} \neq \mathrm{EXP}$
- Thus, we can assure that either $\mathrm{P} \neq \mathrm{NP}$ or $\mathrm{NP} \neq \mathrm{EXP}$ (or both), and we are left with three possibilities:



We will take (a) as our working hypothesis.

Nondeterminism



The cup of tea story

Concept of reduction

Reductions

Let A and B be two decision problems with input sets E and E', respectively. We say *A reduces to B in polynomial time* if there exists a polynomial-time algorithm $\mathcal{F}$ such that

$$x \in A \Rightarrow \mathcal{F}(x) \in B$$

$$x \notin A \Rightarrow \mathcal{F}(x) \notin B$$

In this case, we write $A \leq^p B$ (or $A \leq^p B$ *via* $\mathcal{F}$) and we say that $\mathcal{F}$ is a polynomial-time reduction from A to B.

Examples of reductions

Parity

Let us consider the language of even numbers

$$\text{EVEN} = \{x \in \mathbb{N} \mid \exists y \in \mathbb{N} \quad x = 2y\}$$

and that of odd numbers

$$\text{ODD} = \{x \in \mathbb{N} \mid \exists y \in \mathbb{N} \quad x = 2y + 1\}$$

As one can see, EVEN reduces to ODD via an algorithm $\mathcal{F}$ that adds 1 to the input: $\mathcal{F}(x) = x + 1$. It is obvious that for all x:

$$x \in \text{EVEN} \Leftrightarrow \mathcal{F}(x) \in \text{ODD}.$$

In this case, one can also reduce ODD to EVEN using the same algorithm $\mathcal{F}$. That is, ODD $\leq^p$ EVEN via $\mathcal{F}$.

Examples of reductions

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In this case, one can also reduce ODD to EVEN using the same algorithm $\mathcal{F}$. That is, ODD $\leq^p$ EVEN via $\mathcal{F}$.

Examples of reductions

Partitions

Consider the following two problems:

Partition
Given natural numbers $x_1, x_2, \ldots, x_n$, determine whether they can be divided into two groups having the same sum.

Knapsack
Given natural numbers $x_1, x_2, \ldots, x_n$ and a capacity $C \in \mathbb{N}$, determine whether there is a selection of the x_i's that sums exactly C.

Formally:

$$\text{PARTITION} = \{(x_1, \ldots, x_n) \mid \exists I \subseteq \{1, \ldots, n\} \quad \sum_{i \in I} x_i = \sum_{i \notin I} x_i\}$$

$$\text{KNAPSACK} = \{(x_1, \ldots, x_n, C) \mid \exists I \subseteq \{1, \ldots, n\} \quad \sum_{i \in I} x_i = C\}$$

Examples of reductions

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Examples of reductions

Partitions

The algorithm

$\mathcal{F}(x_1, \ldots, x_n)$
    $S \leftarrow \sum_{i=1}^{n} x_i$
    if S is odd **then**
        return $(x_1, \ldots, x_n, S+1)$
    else
        return $(x_1, \ldots, x_n, S/2)$

is a polynomial-time reduction from PARTITION to KNAPSACK:

$$(x_1, \ldots, x_n) \in \text{PARTITION} \Leftrightarrow \mathcal{F}(x_1, \ldots, x_n) \in \text{KNAPSACK}.$$

Exercise

We define the following collection of coloring problems:

k-**Colorability** (k-COLOR)
Given an undirected graph G, determine whether the vertices in G can be colored with at most k colors, so that each pair of adjacent vertices of get different colors.

Prove that, for all $k \geq 1$, it holds that:

$$k\text{-COLOR} \leq^p (k+1)\text{-COLOR}.$$

Examples of reductions

Definition

A Hamiltonian path in a graph G is a path in G containing all of its vertices without repetitions.

Exercise

We define the Hamiltonian path problem (HP) i and the Hamiltonian path problem between two points (HP_2) as:

- $\mathrm{HP} = \{G \mid G \text{ has a Hamiltonian path}\}$
- $\mathrm{HP}_2 = \{ (G, u, v) \mid G \text{ has a Hamiltonian path with endpoints } u, v\}$

Propose:

1. a reduction proving $\mathrm{HP} \leq^p \mathrm{HP}_2$
2. a reduction proving $\mathrm{HP}_2 \leq^p \mathrm{HP}$

Properties

Properties: Reflexivity

For all A, $A \leq^p A$.

We can consider the algorithm that computes the identity function:

$\mathcal{F}(x)$
return x

It is obvious that, for all x

$$x \in A \Leftrightarrow \mathcal{F}(x) = x \in A.$$

Properties

Properties: Transitivity

For all A, B, C, if $A \leq^p B$ and $B \leq^p C$, then $A \leq^p C$.

If

- $A \leq^p B$ via $\mathcal{F}$ and
- $B \leq^p C$ via $\mathcal{G}$,

then the composition $\mathcal{G} \circ \mathcal{F}$ ($\mathcal{F}|\mathcal{G}$ in UNIX *pipe* notation) proves that $A \leq^p C$.

We will consider that $\mathcal{G} \circ \mathcal{F}(x) = \mathcal{G}(\mathcal{F}(x))$.

Properties

Exercise

Prove that

$$3\text{-}\textsc{color} \leq^p k\text{-}\textsc{color}$$

for all $k \geq 4$ by two different methods:

1. using transitivity of reductions
2. providing an explicit reduction

Properties

Corollary

Reductions form a preorder.

Question

Observe that, although reductions form a preorder, they do not form a partial order due to the fact that they do not satisfy antisymmetry:

- $\forall A, B \quad A \leq^p B \wedge B \leq^p A \Rightarrow A = B$

Properties

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Properties

Closure of P under reductions

For all A, B, if $A \leq^p B$ and $B \in \mathrm{P}$, then $A \in \mathrm{P}$.

If

- $\mathcal{B}$ is a polynomial algorithm for B and
- $\mathcal{F}$ is a polynomial algorithm that proves $A \leq^p B$,

then the composition $\mathcal{F} \circ \mathcal{B}$ is a polynomial algorithm for A:

1. $\mathcal{B} \circ \mathcal{F}$ is polynomial since it is a composition of polynomial-time algorithms
2. $\mathcal{B} \circ \mathcal{F}(x)$ accepts $\Leftrightarrow$ $\mathcal{B}$ accepts $\mathcal{F}(x) \Leftrightarrow \mathcal{F}(x) \in B \Leftrightarrow x \in A$

Properties

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Properties

Notation: Polynomial equivalence

Given two decision problems A, B, we write $A \equiv^p B$ if $A \leq^p B$ and $B \leq^p A$.

Problem: Equivanlence classes of P

1. Prove that $\equiv^p$ is an equivalence relation (reflexive, symmetric, and transitive)
2. Prove that for all A, B, if $A \in \mathrm{P}$ and $B \neq \emptyset, \Sigma^*$, then $A \leq^p B$
3. Obtain the partition of P into equivalence classes induced by relation $\equiv^p$

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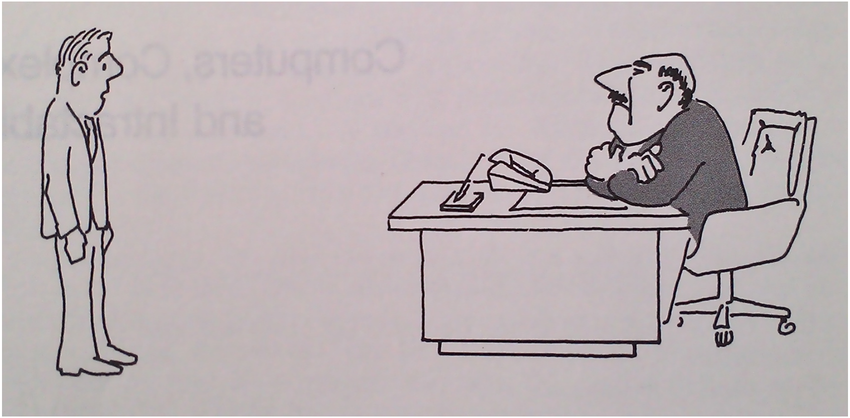
NP-completeness theory

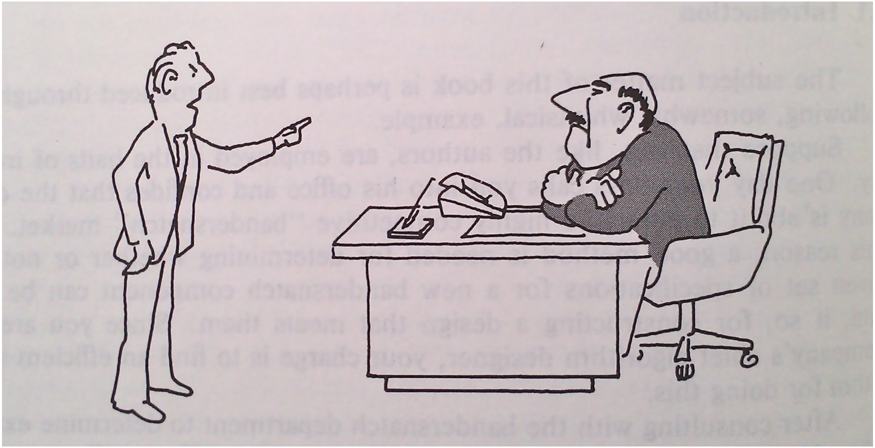
– I can't find an efficient algorithm, I guess I'm just too dumb.



Garey & Johnson, *Computers and Intractability*

NP-completeness theory

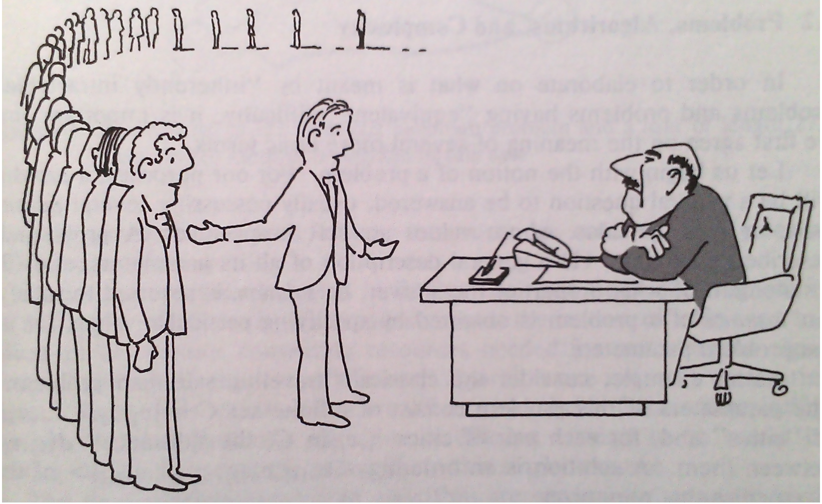
– I can't find an efficient algorithm because no such algorithm is possible!



Garey & Johnson, *Computers and Intractability*

NP-completeness theory

– I can't find an efficient algorithm, but neither can all these famous people.



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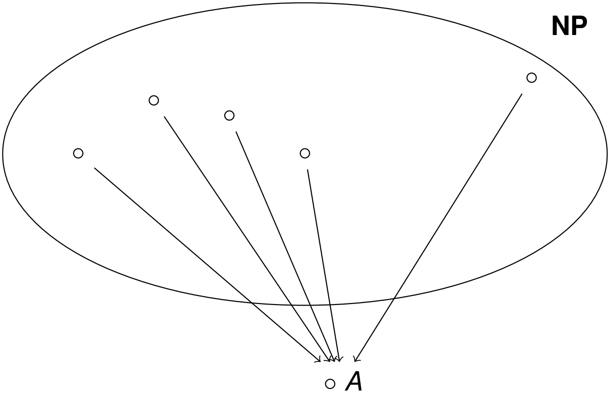
Definition

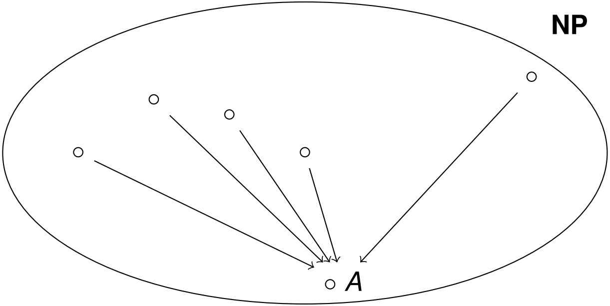
A problem A is NP-hard if for any problem $B \in \mathrm{NP}$ it holds that $B \leq^p A$.



NP-completeness theory

Definition

A problem A is NP-complete if it is NP-hard and $A \in \mathrm{NP}$.



NP-completeness theory

Any NP-complete problem "represents" the whole NP class in relation to P.

More formally...

Proposition

Let A be an NP-complete problem. Then, $\mathrm{P} = \mathrm{NP}$ if and only if $A \in \mathrm{P}$.

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Any two NP-complete problems are equivalent.

More formally...

Definition

We write $A \equiv^p B$ when $A \leq^p B$ and $B \leq^p A$.

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If A and B are NP-complete, then $A \equiv^p B$.

Since A and B are NP-complete, we have

1. $A \in \mathrm{NP}$ and
2. B is NP-hard

and then, $A \leq^p B$.

Symmetrically, we can argue that $B \leq^p A$. Therefore, $A \equiv^p B$.

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NP-completeness theory

But...do NP-complete problems exist?

Boolean formulas

- A Boolean formula (BF) is a formula over Boolean variables with no quantifiers
- We will use the connectives:
  $\vee$ (disjunction), $\wedge$ (conjunction) and $\neg$ (negation)

For example,

$$F(x, y, z) = (x \vee y \vee \neg z) \wedge \neg(x \wedge y \wedge z)$$

is a Boolean formula.

Conjunctive Normal Form (CNF)

- A literal is a positive or negative variable (x, $\neg x$)
- A clause is a disjunction of literals ($x \vee \neg y \vee z$)
- A Boolean formula is in CNF if it is a conjunction of clauses

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Satisfiability

A Boolean formula is satisfiable if there exists an assignment from variables to $\{0,1\}$ under which the formula evaluates to true. For example,

$$F(x,y,z) = (x \vee \neg y \vee z) \wedge (\neg x \vee \neg z)$$

is satisfiable with $x = 1$, $y = 0$, $z = 0$. We write $F(100) = 1$.

We define

$$\text{SAT} = \{\ F \mid F \text{ is a satisfiable Boolean formula}\ \}$$

$$\text{CNF-SAT} = \{\ F \mid F \text{ is a satisfiable BF in CNF}\ \}$$

Cook-Levin Theorem (1971)

CNF-SAT is NP-complete.

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Cook-Levin Theorem (1971)

CNF-SAT is NP-complete.

In order to prove Cook-Levin theorem, we need to show:

1. CNF-SAT $\in \mathrm{NP}$
2. CNF-SAT is NP-hard

(1) CNF-SAT $\in \mathrm{NP}$

- The witnesses are functions from Boolean variables to $\{0, 1\}$.
- In any reasonable encoding of a formula F with n variables, $n \leq |F|$. Since a witness α has n bits, $|\alpha| = n \leq |F|$.
- Hence, choosing $p(n) = n$, we have that $|\alpha| \leq p(|F|)$.
- We can verify whether an assignment α satisfies F in polynomial time:
  - replace variables by their values given by α
  - evaluate the connectives bottom up

Example

If we consider the following BF in CNF

$$F(x, y, z) = (x \vee \neg y \vee z) \wedge (x \vee \neg z)$$

and the assignment $\alpha = 100$ (that is, $x = 1$, $y = 0$, $z = 0$), the verifier would evaluate:

- $F(\alpha) = (1 \vee \neg 0 \vee 0) \wedge (1 \vee \neg 0)$ (replace values)
- $F(\alpha) = (1 \vee 1 \vee 0) \wedge (1 \vee 1)$ (negations)
- $F(\alpha) = 1 \wedge 1$ (disjunctions)
- $F(\alpha) = 1$ (conjunctions)

Lemma

Given an algorithm $\mathcal{A} : E \to \{0, 1\}$ with worst-case polynomial-space cost, we can find a BF in CNF $F_{\mathcal{A}}$ in polynomial time such that for all $y \in E$:

$$F_{\mathcal{A}}(y) = 1 \Leftrightarrow \mathcal{A}(y) = 1$$

(2) CNF-SAT is NP-hard.

Let $A \in \mathrm{NP}$. Then, there is a polynomial q and a verifier $\mathcal{V}$ s.t. for all x:

$$x \in A \Leftrightarrow \exists y \quad |y| = q(|x|) \wedge \mathcal{V}(x, y) = 1.$$

Let $\mathcal{V}_x(y)$ be a new verifier, for a fixed x, such that

$$\mathcal{V}_x(y) = 1 \Leftrightarrow |y| = q(|x|) \wedge \mathcal{V}(x, y) = 1.$$

Then,

$$x \in A \Leftrightarrow \exists y \;\; F_{\mathcal{V}_x}(y) \;\; \Leftrightarrow F_{\mathcal{V}_x}(y) \in \text{CNF-SAT}.$$

Hence, $A \leq^p$ CNF-SAT.

NP-completeness theory

Finding a first NP-complete problem (CNF-SAT) makes it possible to find others via reductions.

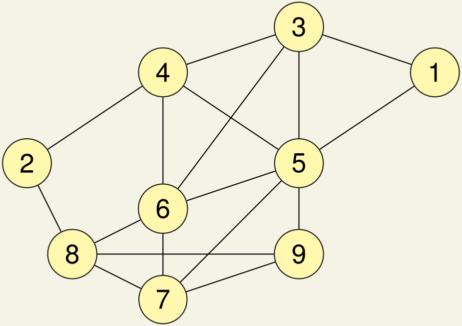
NP-complete problems

Clique problem

We say that H is a complete subgraph of G if it contains all possible edges among its vertices, i.e., if H is isomorphic to K_i for some i. Now define

$$\text{CLIQUE} = \{ (G, k) \mid G \text{ has a complete subgraph with } k \text{ vertices} \}.$$

Given graph G



NP-complete problems

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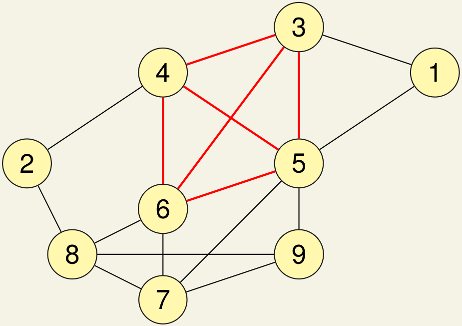


observe that $(G, 4) \in \text{CLIQUE}$ but $(G, 5) \notin \text{CLIQUE}$.

NP-complete problems

Theorem

CLIQUE is NP-complete

In order to prove that CLIQUE is NP-complete we have to see that:

1. CLIQUE $\in \mathrm{NP}$
2. CLIQUE is NP-hard

(1) CLIQUE $\in \mathrm{NP}$

Let (G, k) be an instance of CLIQUE.

- Witnesses are the vertices of a k-sized complete subgraph of G (in the previous example, the set $C = \{3, 4, 5, 6\}$)
- The polynomial $p(n) = n$ is enough because a witness C satisfies $|C| \leq |(G, k)| = p(|(G, k)|)$
- We can verify in polynomial time whether a set C is a witness: any pair of vertices in C should have an edge in G ($\binom{n}{2} \leq n^2$ checks)

NP-complete problems

Theorem

CLIQUE is NP-complete

In order to prove that CLIQUE is NP-complete we have to see that:

1. CLIQUE $\in \mathrm{NP}$
2. CLIQUE is NP-hard

(2) CLIQUE is NP-hard

We will prove that CNF-SAT $\leq^p$ CLIQUE. Then,

- Since CNF-SAT is NP-hard, any $S \in \mathrm{NP}$ satisfies $S \leq^p$ CNF-SAT
- By transitivity, any $S \in \mathrm{NP}$ satisfies $S \leq^p$ CLIQUE
- Hence, CLIQUE is NP-hard

NP-complete problems

We can express the previous property in general.

Proposition

Let A be an NP-complete problem and B a problem such that $B \in \mathrm{NP}$ and $A \leq^p B$. Then, B is also NP-complete.

- Since A is NP-hard, any $S \in \mathrm{NP}$ satisfies $S \leq^p A$
- By transitivity, any $S \in \mathrm{NP}$ satisfies $S \leq^p B$
- Hence, B is NP-hard

NP-complete problems

CNF-SAT $\leq^p$ CLIQUE

Let F be a Boolean formula in CNF with:

- clauses $C_1, \ldots, C_m$
- literals $l_1, \ldots, l_r$

We define the reduction algorithm $\mathcal{R}(F) = (G, m)$, where $G = (V, E)$ is:

- $V = \{(i, j) \mid \; l_i \text{ appears in } C_j \}$
  (Vertices represent occurrences of literals in clauses)
- $E = \{ \{(i, j), (k, l)\} \mid j \neq l \wedge \neg l_i \neq l_k \}$
  (Edges represent pairs of literals that can be simultaneously true)

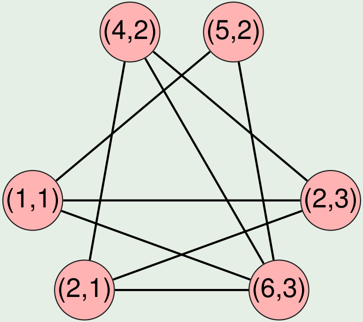
NP-complete problems

Example

$F(x_1, x_2, x_3) = C_1 \wedge C_2 \wedge C_3$, where

- $C_1 = (x_1 \vee x_2)$, $C_2 = (\neg x_1 \vee \neg x_2)$, $C_3 = (x_2 \vee \neg x_3)$
- $l_1 = x_1$, $l_2 = x_2$, $l_3 = x_3$, $l_4 = \neg x_1$, $l_5 = \neg x_2$, $l_6 = \neg x_3$

The reduction $\mathcal{R}(F) = (G, 3)$, where G is the graph



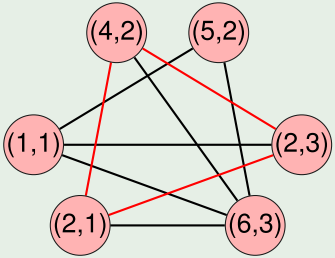
NP-complete problems

In general, we have that $F \in$ CNF-SAT $\Leftrightarrow (G, m) \in$ CLIQUE:

$\Rightarrow$ Let α be an assignment satisfying F. Hence, there are m literals that α simultaneously satisfies and hence they form a complete subgraph in G.

$\Leftarrow$ If G has a complete subgraph with m vertices, each vertex belongs to a different clause. Hence, we can simultaneously satisfy one literal in each clause, thus satisfying F.

Previous example with $l_2 = 1, l_4 = 1$



NP-complete problems

Definitions

- H is an independent subset of G if it consists of isolated vertices
- H is a vertex cover of G if it has an endpoint of any edge in G

Exercise

Given the following problems:

- CLIQUE $= \{ (G, k) \mid$ G has a complete subgraph with k vertices $\}$
- IS $= \{ (G, k) \mid$ G has an independent subset of k vertices $\}$
- VC $= \{ (G, k) \mid$ G has a vertex cover of k vertices $\}$

prove that

1. CLIQUE $\leq^p$ IS
2. IS $\leq^p$ VC
3. VC $\leq^p$ CLIQUE

Lots of NP-complete problems have “particular cases” that are in P.

For example, in CNF-SAT we can fix the number of literals per clause in order to obtain an infinite family of problems.

***k*-Bounded Satisfiability** (*k*-SAT)
Given a Boolean formula in CNF over *n* variables and at most *k* literals per clause, determine whether it is satisfiable.

We will see how to classify *k*-SAT for the different values of *k*.

NP-complete problems

1-Bounded Satisfiability (1-SAT)
Given a Boolean formula F in CNF with n variables and 1 literal per clause, determine whether it is satisfiable.

For example,

$$F(x, y, z, t) = (x) \wedge (\neg y) \wedge (z) \wedge (\neg t).$$

1-SAT is decidable in polynomial time with the following algorithm:

input F
if F has two contradictory literals **then**
  return 0
else
  return 1

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1-SAT is decidable in polynomial time with the following algorithm:

input F
if F has two contradictory literals **then**
  return 0
else
  return 1

NP-complete problems

2-Bounded Satisfiability (2-SAT)
Given a Boolean formula F in CNF with n variables and ≤ 2 literals per clause, determine whether it is satisfiable.

For example,

$$F(x, y, z) = (x \vee y) \wedge (x \vee \neg z) \wedge (\neg x \vee y) \wedge (\neg y \vee \neg z).$$

2-SAT is decidable in polynomial time

- transforming the formula into a directed graph
- applying a paths algorithm to the graph

2-Bounded Satisfiability (2-SAT)
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2-SAT is decidable in polynomial time

- transforming the formula into a directed graph
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NP-complete problems

Sketch of the algorithm

Given a 2-CNF Boolean formula

$$F(x,y,z) = (x \vee y) \wedge (x \vee \neg z) \wedge (\neg x \vee y) \wedge (\neg y \vee \neg z)$$

it can be rewritten using implications

$$F(x,y,z) = (\neg x \Rightarrow y) \wedge (z \Rightarrow x) \wedge (x \Rightarrow y) \wedge (y \Rightarrow \neg z)$$

that are based on the equivalences

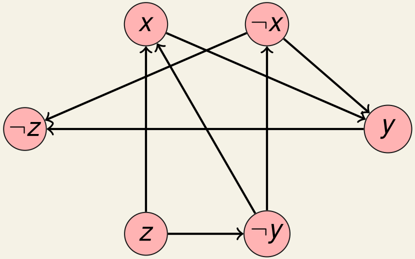
- $(a \vee b) \equiv (\neg a \Rightarrow b) \equiv (\neg b \Rightarrow a)$
- $(a) \equiv (a \vee a) \equiv (\neg a \Rightarrow a) \equiv (a \Rightarrow \neg a)$

NP-complete problems

The Boolean formula with implications

$$F(x, y, z) = (\neg x \Rightarrow y) \wedge (z \Rightarrow x) \wedge (x \Rightarrow y) \wedge (y \Rightarrow \neg z)$$

is transformed into a digraph D_F and we apply the following lemma.



Lemma

F is unsatisfiable if and only if $\exists x$ for which D_F has paths from x to $\neg x$ and from $\neg x$ to x.

NP-complete problems

3**-Bounded Satisfiability** (3-SAT)
Given a Boolean formula F in CNF with n variables and $\leq$ 3 literals per clause, determine whether it is satisfiable.

Theorem
3-SAT is NP-complete.

To prove it, we need two facts:

1. 3-SAT $\in$ NP
(similar to CNF-SAT)
2. 3-SAT is NP-hard
(we show CNF-SAT $\leq^P$ 3-SAT)

NP-complete problems

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To prove it, we need two facts:

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2. 3-SAT is NP-hard
   (we show CNF-SAT $\leq^p$ 3-SAT)

CNF-SAT $\leq^p$ 3-SAT

The following method transforms a Boolean formula in CNF into an equisatisfiable one in 3-CNF.

Given a BF F in CNF,

1. Let F' be an empty BF
2. For each clause $C = (a_1 \vee \cdots \vee a_k)$ in F:
   - if $k \leq 3$, add C to F'
   - if $k > 3$, add the clause

     $$(a_1 \vee a_2 \vee z_1) \wedge (\neg z_1 \vee a_3 \vee z_2) \wedge (\neg z_2 \vee a_4 \vee z_3) \ldots (\neg z_{k-3} \vee a_{k-1} \vee a_k)$$

     to F', where $z_1, \ldots, z_{k-3}$ are new variables.
3. Return F'

NP-complete problems

Example

Given a clause with five literals $C = (a_1 \vee a_2 \vee a_3 \vee a_4 \vee a_5)$, the reduction returns

$$C' = (a_1 \vee a_2 \vee z_1) \wedge (\neg z_1 \vee a_3 \vee z_2) \wedge (\neg z_2 \vee a_4 \vee a_5).$$

- It is obvious that if C is true with assignment α, C' can be satisfied with α and appropriate values for z_1 and z_2
- If C' is true with assignment β, some a_i will be true and C will be true with β

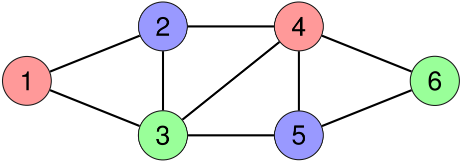
NP-complete problems

Definition

A graph $G = (V, E)$ with n vertices is *k*-colorable if there exists a total function

$$\chi : V \to \{1, \dots, k\}$$

such that $\chi(u) \neq \chi(v)$ for any edge $\{u, v\} \in E$. Function χ is a *k*-coloring.



3-coloring

NP-complete problems

With the number of colors k as an external parameter, we can formulate the coloring problem as a function of k.

k-**Colorability** (k-COLOR)
Given a graph G, determine whether it is k-colorable.

Polynomial algorithms are known for the following cases:

- 1-COLOR
- 2-COLOR

For 3-COLOR, we prove NP-completeness:

- We already showed that 3-COLOR $\in$ NP
- Next, we show that it is NP-complete via a reduction from 3-CNF-SAT

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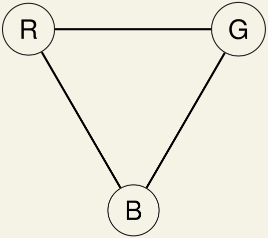
NP-complete problems

CNF-SAT $\leq^p$ 3-COLOR

Let F be a Boolean formula in CNF. We will construct a graph G that is 3-colorable if and only if F is satisfiable.

- There will be 3 special vertices called R, G, B forming a triangle:

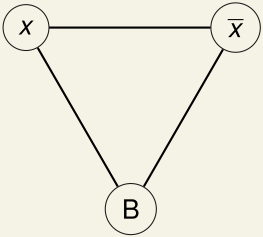


We can assume that in any coloring, vertices R, G, B have the colors:

R $\rightarrow$ red, G $\rightarrow$ green, B $\rightarrow$ blue

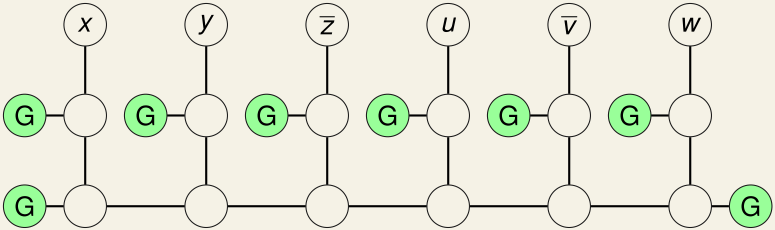
NP-complete problems

- We add a vertex for each literal. Then, we connect each literal and its negation to vertex B.



- For each clause, we add a subgraph as follows. In the case

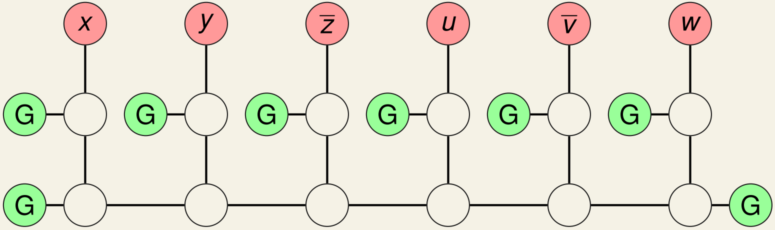
$$(x \vee y \vee \overline{z} \vee u \vee \overline{v} \vee w).$$



Property: A coloring of the upper vertices with red or green can be extended to a global 3-coloring if and only if at least one has green color.

NP-complete problems

If all of the above are red....



...we cannot complete the 3-coloring.

NP-complete problems

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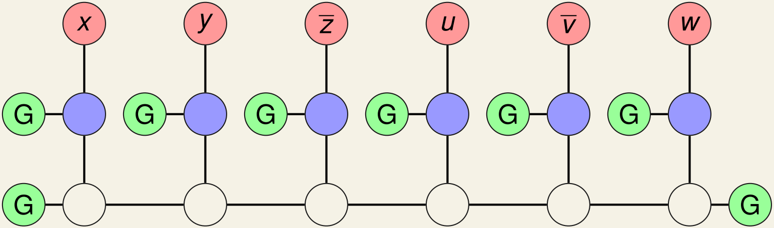

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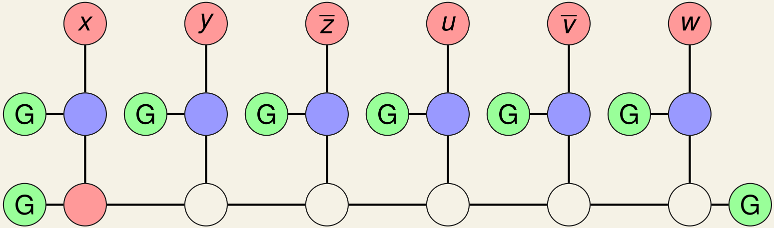

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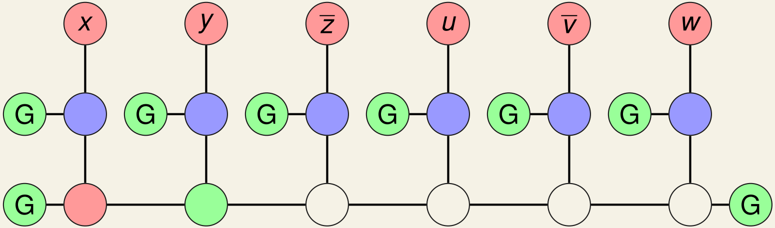

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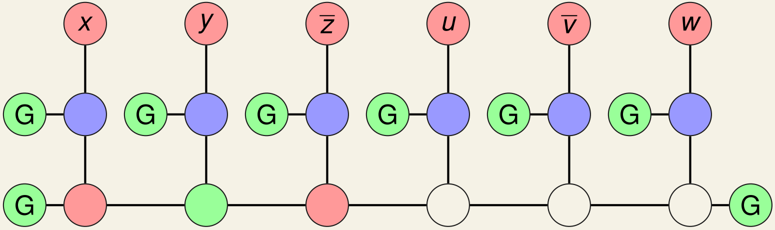

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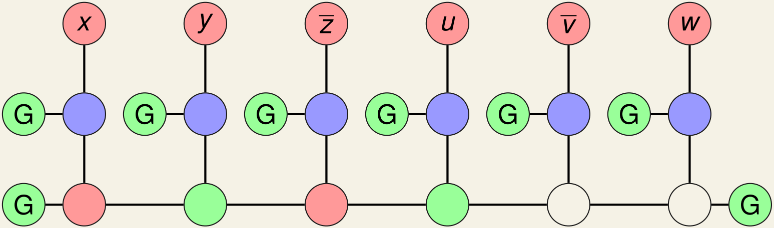

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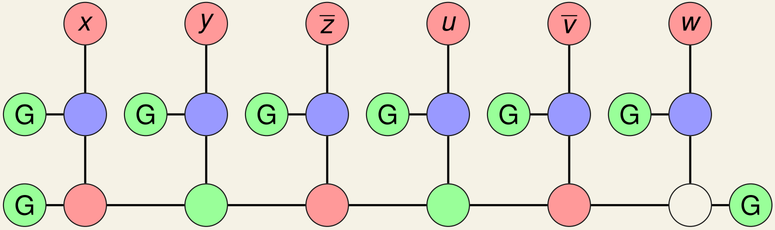

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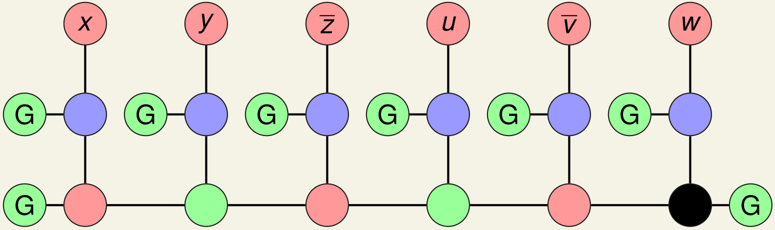

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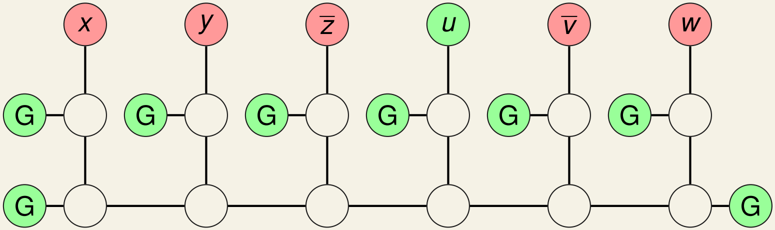
If at least one is green...



...we can obtain a global 3-coloring.

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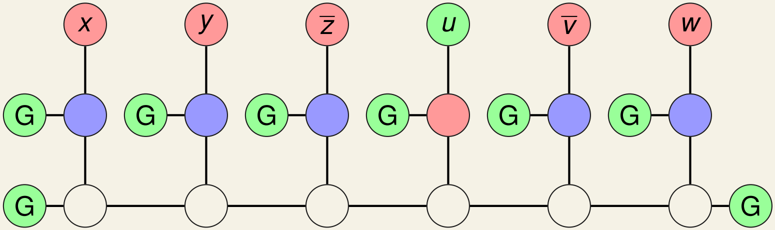

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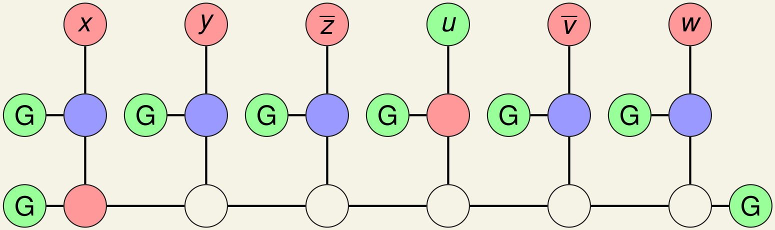

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NP-complete problems

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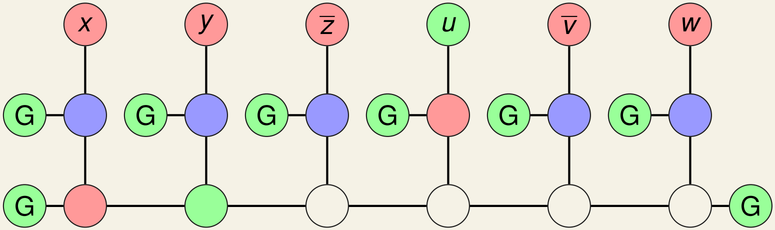

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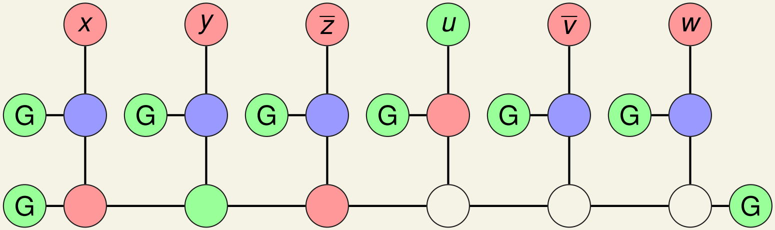

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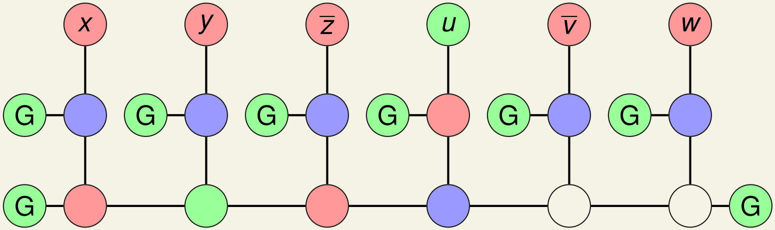

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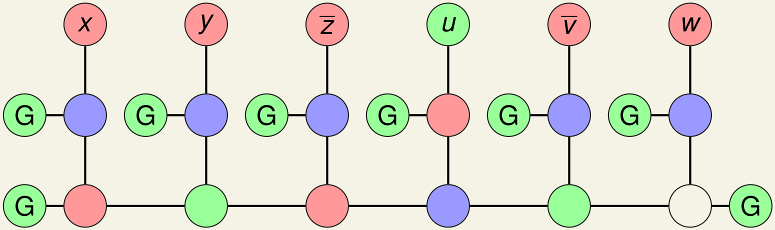

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NP-complete problems

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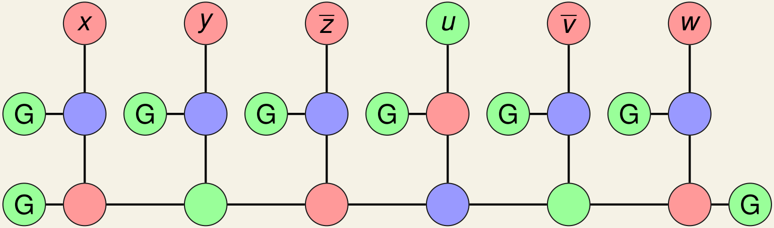
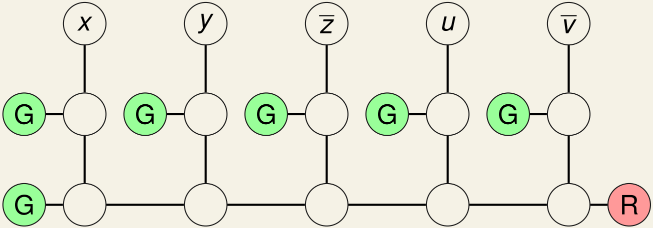


...we can obtain a global 3-coloring.

If the number of literals is odd, the rightmost vertex will be R.
For example,

$$(x \vee y \vee \overline{z} \vee u \vee \overline{v})$$



NP-complete problems

If G is the graph with all vertices and edges defined as before, then

$$F \text{ is satisfiable} \Leftrightarrow G \text{ is 3-colorable.}$$

Since G can be constructed in polynomial time, we have that

$$\text{CNF-SAT} \leq^p \text{3-COLOR}.$$

Theorem

3-COLOR is NP-complete.

NP-complete problems

For the other k-COLOR problems, we have the following.

Proposition

For all $k > 3$, 3-COLOR $\leq^p$ k-COLOR.

The reduction consists of, given a graph G, adding to it a complete subgraph with $k - 3$ vertices connected to all vertices of G.

Corollary

For all $k > 3$, k-COLOR is NP-complete.

Hence, we have:

- k-COLOR $\in$ P for all $k \leq 2$
- k-COLOR is NP-complete for all $k \geq 3$

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NP-complete problems

What can we say about colorability of planar graphs? Let us consider the following family of problems.

k-**Planar Colorability** (k-COLOR-PL)
Given a planar graph G, determine whether it is k-colorable.

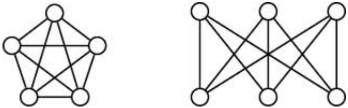
Planarity can be checked in polynomial time.

NP-complete problems

Definition

A graph is planar if it can be drawn on the plane without any edge intersection.

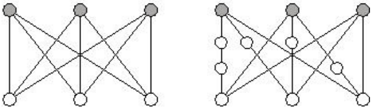
Planar graphs have applications in circuit design and graphics.



K_5 $K_{3,3}$

Kuratowski Theorem

A graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.



$K_{3,3}$ and homeomorphic graph

NP-complete problems

Kuratowski Theorem

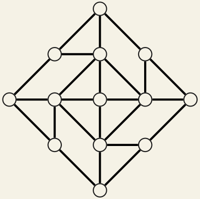
A graph is planar if and only if it does not contain a subgraph homeomorphic to K_5 or $K_{3,3}$.

Planarity test

- **Brute force**: $O(n^6)$
  - Contract edges of degree 2
  - Check whether some set of 5 vertices is K_5
  - Check whether some set of 6 vertices is $K_{3,3}$
- **Efficient**: $O(n)$
  - Apply DFS

3-COLOR $\leq^p$ 3-COLOR-PL

Given a graph G, we will considered a representation of G, possibly with edge intersections. Each intersection will be replaced by the gadget W:



W has interesting properties:

1. in any 3-coloring of W, opposite extreme points have the same color
2. any color assignment where opposite extreme points have the same color can be extended to a 3-coloring of W

NP-complete problems

3-COLOR $\leq^p$ 3-COLOR-PL

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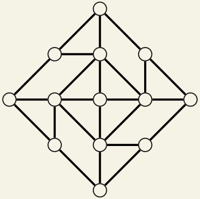

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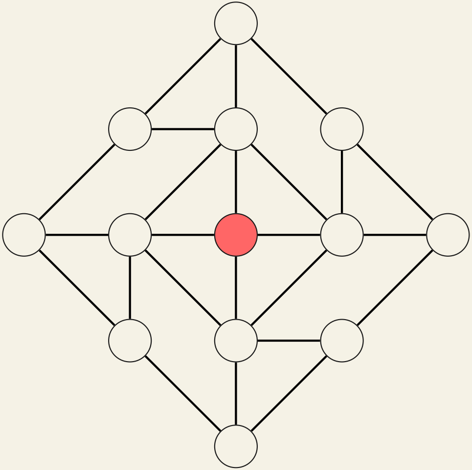
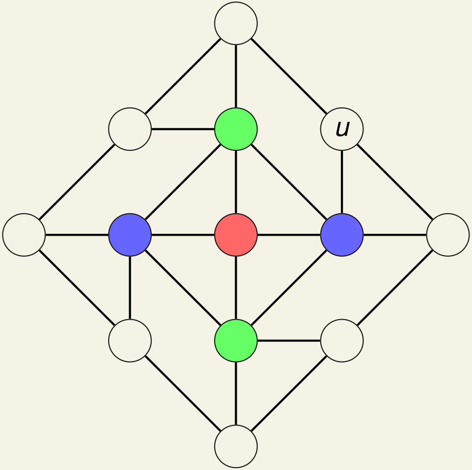
NP-complete problems



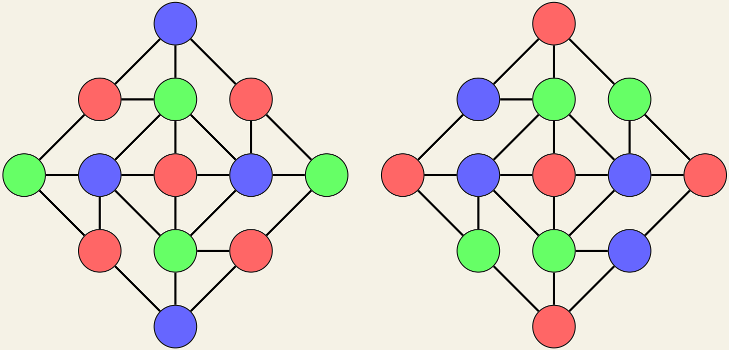
There are two colors available for vertex u.

NP-complete problems



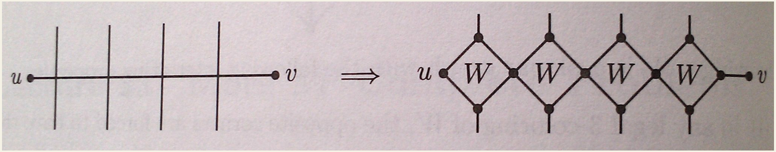
There are two colors available for vertex *u*.

This allows two colorings (up to isomorphism).



It is easy to check that they fullfill properties (1) i (2).

The graph we obtain after the replacements

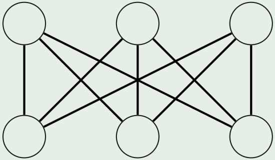


in the representation of *G*

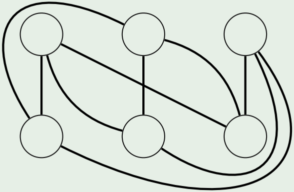
- is planar and
- is 3-colorable if and only if *G* is 3-colorable

Example

Let us assume that we have $K_{3,3}$ as input to 3-COLOR:

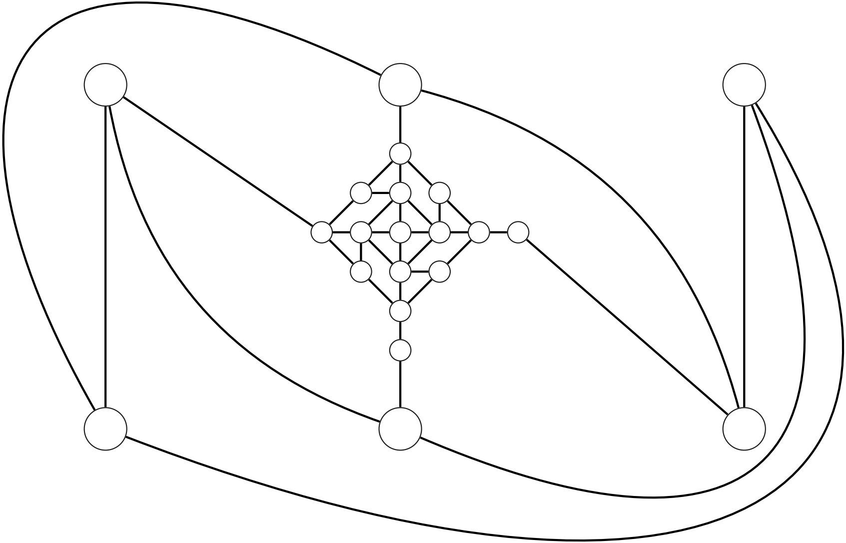


But we consider the following representation with just one intersection:



NP-complete problems

A 3-coloring for $K_{3,3}$ induces a 3-coloring for the this graph (and viceversa):



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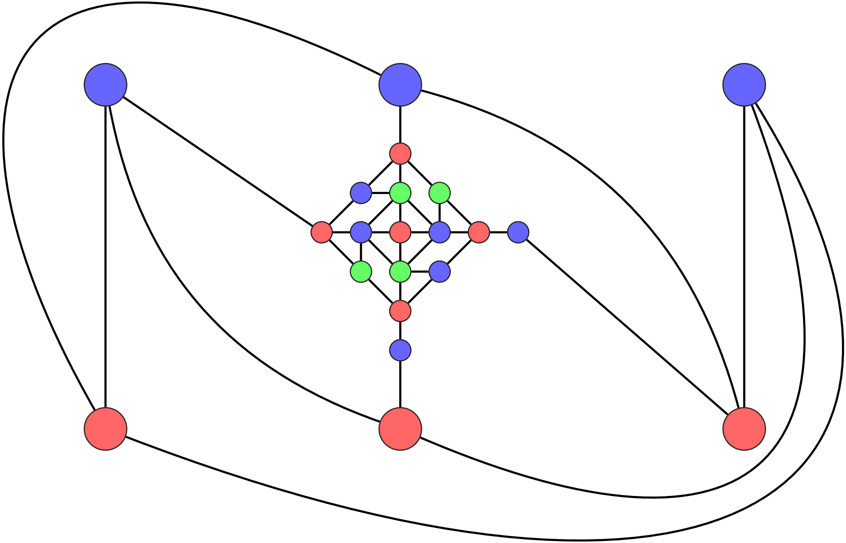

NP-complete problems

Corollary

3-COLOR-PL is NP-complete.

Hence, we have:

- k-COLOR-PL $\in$ P for all $k \leq 2$
- 3-COLOR-PL is NP-complete
- k-COLOR-PL $\in$ P for all $k \geq 4$

Corollary

3-COLOR-PL is NP-complete.

Hence, we have:

- k-COLOR-PL $\in \mathrm{P}$ for all $k \leq 2$
- 3-COLOR-PL is NP-complete
- k-COLOR-PL $\in \mathrm{P}$ for all $k \geq 4$
  (due to the 4-color theorem)

NP-complete problems

So far, we have seen the following tree of reductions.