Lógica en la Informática / Logic in Computer Science
January 20th, 2020. Time: 2h30min. No books or lecture notes.

Note on evaluation: eval(propositional logic) = max{ eval(Problems 1,2,3), eval(partial exam) }.
eval(first-order logic) = eval(Problems 4,5,6).

1) Let F and G be arbitrary propositional formulas. Prove your answers using only the definitions of propositional logic.
A) Is it true that if F is satisfiable then $(F \wedge G) \vee F$ is also satisfiable?
B) Is it true that if an interpretation I is not a model of F then I is not a model of $(F \wedge G) \vee F$?
C) Is there any interpretation I such that $I \models (F \wedge G) \vee F$ and $I \not\models F$?

Answer:

A) Yes. F satisfiable implies	by definition of satisfiable
exists I such that $I \models F$ which implies	by definition of $\models$
exists I such that $eval_I(F) = 1$ which implies	by definition of max and $eval$
exists I such that $max(eval_I(F \wedge G), eval_I(F)) = 1$ which implies	by definition of $eval(\vee)$
exists I such that $eval_I(F \wedge G) \vee F) = 1$ which implies	by definition of $\models$
exists I such that $I \models (F \wedge G) \vee F$ which implies	by definition of satisfiable
$(F \wedge G) \vee F$ is satisfiable.	

B) Yes. I is not a model of F implies	by definition of model
$I \not\models F$ which implies	by definition of $\models$
$eval_I(F) = 0$ which implies	by definition of max and $eval$
$max(min(eval_I(F), eval_I(G)), eval_I(F)) = 0$ which implies	by definition of $eval(\wedge)$
$max(eval_I(F \wedge G), eval_I(F)) = 0$ which implies	by definition of $eval(\vee)$
$eval_I((F \wedge G) \vee F) = 0$ which implies	by definition of $\models$
$I \not\models (F \wedge G) \vee F$ which implies	by definition of model
I is not a model of $(F \wedge G) \vee F$.	

C) No, because by B) $I \not\models F$ implies $I \not\models (F \wedge G) \vee F$.

2a) Let F be the propositional formula $(\neg p \wedge (p \vee (q \wedge r))) \vee (q \wedge r)$. Write the smallest and simplest possible clause set S that is logically equivalent to F.
2b) Write the clauses needed for encoding into CNF without auxiliary variables the formula $a \leftrightarrow (x \vee y)$. Do the same for the formula $a \leftrightarrow (x \wedge y)$.
2c) Write the Tseitin transformation of the formula F of 2a) in terms of $\leftrightarrow$ formulas like the ones given in 2b) (no need to write the final clauses). Use auxiliary variables a_0 (for the root), $a_1, a_2, \ldots$
Answer: 2a): $\{\ q,\ r\ \}$. 2b): $\{\ \neg a \vee x \vee y,\quad \neg x \vee a,\quad \neg y \vee a\ \}\quad \{\ \neg x \vee \neg y \vee a,\quad \neg a \vee x,\quad \neg a \vee y\ \}$
2c): $a_0,\quad a_0 \leftrightarrow a_1 \vee a_4,\quad a_1 \leftrightarrow \neg p \wedge a_2,\quad a_2 \leftrightarrow p \vee a_3,\quad a_3 \leftrightarrow q \wedge r,\quad a_4 \leftrightarrow q \wedge r.$

3) We want to do *model counting*, that is, given a set of clauses S built over a set of n propositional symbols $\mathcal{P}$, determine how many different models $I\colon \mathcal{P} \to \{0, 1\}$ it has. Explain very briefly:
3a) How would you do this without a SAT solver? How would you do this using a SAT solver? In which cases using the SAT solver is likely to be faster?
3b) What is the computational cost of this in the worst case (polynomial?, exponential?)?
3c) Answer the same questions for the case where S is Horn.

Answer:
3a: For each one of the 2^n interpretations I, check whether $I \models S$ (2^n checks linear in the size of S). Using a SAT solver, which will work better if there are not many models:
repeat: find a model I; counter++; add the clause forbidding or *blocking* I; **until** unsat.
(for example, if $I(p) = 1, I(q) = 0, I(r) = 0, \ldots$ we add the clause $\neg p \vee q \vee r \vee \ldots$).
3b: Both methods in the worst case take 2^n (exponential) time, since up to 2^n models may exist.
3c: S being Horn does not help with the worst case complexity. Same answers as 3a and 3b.

4a) Consider binary function symbols s and p and the first-order interpretations I and I' where D_I is the set of rational numbers and I' where $D_{I'}$ is the set of real numbers and where in both cases, s is interpreted as the sum and p is interpreted as the product. Write the simplest possible formula F in first-order logic with equality using only the function symbols s and p (*no other symbols*) and the equality predicate $=$, such that F is true in one of the interpretations and false in the other one. Do not give any explanations. Hint: the square root of 2 is irrational.

Answer: $F: \quad \exists y\, \exists z\ (\ (\forall x\ p(x,y) = x) \land p(z,z) = s(y,y)\ )$

4b) Consider the two first-order formulas:
F is $\forall z\, (\exists x\, p(x,z) \land \exists y\, p(z,y))$
G is $\exists x\, \exists y\, \forall z\, (p(x,z) \land p(z,y))$
Do we have $F \models G$? Prove it.

Answer: No. Let I be the interpretation with $D_I = \{a,b\}$ and where $p_I(a,a) = 1$, $p_I(a,b) = 0$, $p_I(b,a) = 0$, $p_I(b,b) = 1$. Then $I \models F$ but $I \not\models G$.

5) For each one of the following statements, indicate if it is true or false in propositional logic and also for first-order logic. Give no explanations why. Example: **A:** True in Prop Logic. True in F-O Logic. Below always F and G are formulas and I is an interpretation.
A) There are infinitely many different formulas, even if there is only one predicate symbol.
B) $F \models G$ iff $F \land \neg G$ is unsatisfiable.
C) F is a tautology iff $\neg F$ insat.
D) Given I and F, it is decidable in linear time whether $I \models F$.
E) Given I and F, it is decidable whether $I \models F$.
F) Given F, it is decidable in polynomial time whether F is satisfiable.
G) Given F, it is decidable whether F is satisfiable.

Answer:

	Prop	F-O		Prop	F-O
B:	T	T	E:	T	F
C:	T	T	F:	F	F
D:	T	F	G:	T	F

6) Formalize the following five sentences by five first-order formulas F_1, F_2, F_3, F_4, F_5.
Is $F_1 \land F_2 \land F_3 \land F_4 \land F_5$ satisfiable? Prove it.
F_1: If a person has a bad health he/she cannot run fast.
F_2: Friends of sports professionals do not smoke.
F_3: Piqué is a sports professional and Shakira is his friend.
F_4: Smokers have a bad health.
F_5: Shakira cannot run fast.

Answer:
F_1: $\forall x\, badhealth(x) \rightarrow \neg runfast(x)$
F_2: $\forall x\, (\exists y\, (prof(y) \land friend(x,y)) \rightarrow \neg smoker(x))$
F_3: $prof(pique) \land friend(pique, shakira)$
F_4: $\forall x\, (smoker(x) \rightarrow badhealth(x))$
F_5: $\neg runfast(shakira)$

$F_1 \land F_2 \land F_3 \land F_4 \land F_5$ is satisfiable. The following I is a model:
$D_i = \{p, s\}$
$pique_I = p, \quad shakira_I = s$
$friend_I(s,s) = 1, \quad friend_I(s,p) = 1, \quad friend_I(p,s) = 1, \quad friend_I(p,p) = 1$
$badhealth_I(p) = 0, \quad badhealth_I(s) = 0$
$runfast_I(p) = 1, \quad runfast_I(s) = 0$
$prof_I(p) = 1, \quad prof_I(s) = 1$
$smoker_I(p) = 0, \quad smoker_I(s) = 0$