Lógica en la Informática / Logic in Computer Science
January 17th, 2018. Time: 2h30min. No books or lecture notes.

Note on evaluation: eval(propositional logic) = max{ eval(Problems 1,2,3), eval(partial exam) }.
eval(first-order logic) = eval(Problems 4,5,6).

1a) Let F and G be propositional formulas. Is it true that always $F \models G$ or $F \models \neg G$? Prove it using only the definitions of propositional logic.

1b) Let F and G be propositional formulas. Is it true that $F \models G$ iff $F \wedge \neg G$ is unsatisfiable? Prove it using only the definitions of propositional logic.

2) We are interested in the optimization problem, called *minOnes*: given a set S of clauses over variables $\{x_1, \ldots, x_n\}$, finding a *minimal* model I (if it exists), that is, a model of S with the minimal possible number of ones $I(x_1)+\ldots+I(x_n)$. Explain very briefly your answers to the following questions:
2a) Does every S have a unique minimal model, or can there be several minimal models?
2b) Given the set S and an arbitrary natural number k, what is the complexity of deciding whether S has any model I with at most k ones, that is, such that $I(x_1) + \ldots + I(x_n) \leq k$?
2c) Same question as 2a, if S is a set of Horn Clauses.
2d) Same question as 2b, if S is a set of Horn Clauses.

3) We want to encode pseudo-Boolean constraints into SAT with the minimal set of clauses, and using no auxiliary variables. For $2x + 3y + 5z + 6u + 8v \leq 11$, the clauses are:

$\neg v \vee \neg u \qquad \neg v \vee \neg z \qquad \neg v \vee \neg x \vee \neg y \qquad \neg u \vee \neg z \vee \neg y \qquad \neg u \vee \neg z \vee \neg x$

Write the minimal set of clauses for $2x + 3y + 5z + 6u + 8v \geq 11$ (give no explanations).

4) For each one of the following cases, write a formula F of first-order logic without equality such that F fulfils the requirement. Keep F *as simple as you can* and give no explanations.
4a) F is unsatisfiable.
4b) F is a tautology.
4c) F is satisfiable and has no model I with $|D_I| < 3$.
4d) F is satisfiable but has no model with finite domain.
4e) F is satisfiable and all models I of F have $|D_I| = 2$.
4f) Same question as 4e, but for first-order logic with equality.

5a) Explain in a few words how to formally prove $F \not\models G$ for given first-order formulas F and G.
5b) Same question for $F \models G$.
5c) F is $\forall x\, p(a, x) \wedge \exists y\, \neg q(y)$ and G is $\exists v \exists w\, \neg q(w) \wedge p(v, a)$. Do we have $F \models G$? Prove it.
5d) F is $\forall x \exists y\, p(x, y)$ and G is $\exists y \forall x\, p(x, y)$. Do we have $F \models G$? Prove it.

6) Consider the following Prolog program and its well-known behaviour:

```
animals([dog,lion,elephant]).
bigger(lion,cat).
faster(lion,cat).
better(X,Y):- animals(L), member(X,L), bigger(X,Y), faster(X,Y).

      ?- better(U,V).
      U = lion
      V = cat
```

In Prolog, a list like `[dog,lion,elephant]` is in fact represented as a term
`f(dog,f(lion,f(elephant,emptylist)))`.
Therefore, we assume that the program also contains the standard clauses for `member` like this:

```
member( E, f(E,_) ).
member( E, f(_,L) ):- member(E,L).
```

Express the program as a set of first-order clauses P and prove that $\exists u\, \exists v\; better(u, v)$ is a logical consequence of P. Which values did the variables u and v get (by unification) in your proof? **Only write the steps and values. No explanations.**