Proof-theoretic conservations of weak weak intuitionistic constructive set theories

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1. Introduction


L. G. (1977, 1982): Constructive interpretations of Set Theory that are compatible with Recursive Analysis.

H. Friedman (1977): Intuitionistic extensional set theories $T_1 \subseteq T_2 \subseteq T_3 \subseteq T_4$ whose proof-theoretic strengths are between (those of) classical first and second order arithmetic: $|PA| = |HA| = |T_1| < |T_2| < |T_3| < |T_4| = |HA_2| = |PA_2|$, which justifies the designation weak.

Despite proof-theoretic weakness, these intuitionistic set theories have great expressive power.

J. Myhill [1975], H. Friedman [1977]: Constructively meaningful principles of *weak* Set Theory.
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Example

Zermelo's power-set axiom

\[ \text{Pow} \equiv \exists \mathbb{P}(x) = \{ y : y \subset x \} \]

is replaced in \( T_1, T_2, T_3, T_4 \) by the exponentiation axiom

\[ \text{Exp} \equiv \exists (x^y) = \{ f : f \subset x \times y \land (\forall u \in x) (\exists! v \in y) (\langle u, v \rangle \in f) \} \]

In classical set theory:

\[ \text{Pow} \iff \text{Exp} \]

However, intuitionistically \( \text{Exp} \) is weaker than \( \text{Pow} \).

Hint: think of \( x^y \) as (possibly enumerable) set of constructive functions from \( x \) to \( y \) (e.g., algorithms).

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- In classical set theory: \( \text{Pow} \iff \text{Exp} \).
- However intuitionistically \( \text{Exp} \) is weaker than \( \text{Pow} \).
  - Hint: think of \( \times y \) as (possibly enumerable) set of constructive functions from \( x \) to \( y \) (e. g. algorithms).
Apart from Exp, theories $T_1, T_2, T_3$ also include:

1. $\text{Ext}$ (Cantor's axiom of extensionality).
2. $\Delta^0$-Sep (restricted separation schema).

Arguably $\Delta^0$-Sep is predicative.

Other axioms to be discussed later.

$T_1, T_2, T_3$ regarded as being constructive.

$T_4$ contains full separation and is not really constructive.
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Conservative extensions, i-conservation

Suppose $\text{HA} \subset S$, an intuitionistic theory with nice proof theory. Let $|S| = \sup\{\alpha \mid S \vdash \text{TI Ar}(\alpha)\} = \text{proof-theoretic strength of } S$. So for any arithmetical sentence $A$, $\text{HA + TI Ar}(\text{<|S|}) \vdash A \implies S \vdash A$.

Definition: Call $S$ $i$-conservative iff $S$ is a conservative extension of $\text{HA + TI Ar}(\text{<|S|})$, i.e. for any arithmetical sentence $A$, $\text{HA + TI Ar}(\text{<|S|}) \vdash A \iff S \vdash A$.

In particular if $|S| = \varepsilon_0$, then $S$ is $i$-conservative $\iff S$ is conservative extension of $\text{HA}$. Meaning: if $S$ is $i$-conservative, then its arithmetical part is based on standard intuitionistic principles only.
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- Meaning: if $S$ is **i-conservative**, then its arithmetical part is based on standard intuitionistic principles only.
Problem

H. Friedman [1977]: Are $T_1$, $T_2$, $T_3$, $T_4$ conservative extensions of $HA$, $\Sigma_1^{AC}(i)$, $ID(i)$, $HA_2$, respectively? 

Dropping $T_4$, consider equivalent question: Are $T_1$, $T_2$, $T_3$ $i$-conservative?

Solution L. G. [1982, 1988]: Yes, $T_1$, $T_2$, $T_3$ are $i$-conservative.

Note that $|T_1| = \varepsilon_0$, $|T_2| = \phi_{\varepsilon_0}(0)$, $|T_3| = \text{Howard ordinal } \phi_{\varepsilon_\Omega+1}(0)$.

Hence in particular $T_1$ is conservative extension of $HA$.
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More on $T_1 - T_3$

1. $T_1$ includes only set-restricted arithmetic induction ($\text{Ind}_0$).
2. $T_2$ includes full arithmetic induction ($\text{Ind}$) and full Relative Dependent Choice ($\text{RDC}$).
3. $T_3$ includes both $\text{Ind}$ and $\text{RDC}$, as well as full $\in$-induction ($\text{Ind}_\in$).
More on $T_1 - T_3$

1. $T_1$, $T_2$, $T_3$ all include $\text{Fnd}$ (Foundation) and $\text{SC}$ (Strong Collection):

\begin{align*}
\text{Fnd} & \equiv \text{Trans}(x) \land (\forall y \in x) (y \subset z \rightarrow y \in z) \rightarrow x \subset z \\
\text{SC} & \equiv (\forall x \in a) (\exists y \varphi(x, y) \rightarrow (\exists z ((\forall x \in a) (\exists y \in z) \varphi(x, y) \land (\forall y \in z) (\exists x \in a) \varphi(x, y))))
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§2. Recent developments

We consider Basic+Ext and its extensions.

K. Sato [2009] (basic results):

$$\|\text{Basic} + \text{Ext}\| = \varepsilon_0,$$

$$\|\text{Basic} + \text{Ext} + \Delta_0^-\text{-Sep}\| = \Gamma_0.$$

Remember: these are theories with classical logic.

Problem: What about intuitionistic counterparts Basic ($i$) + Ext and Basic ($i$) + Ext + $\Delta_0^-\text{-Sep}$?

Solution:

$$\|\text{Basic} (i) + \text{Ext}\| = \|\text{Basic} (i) + \text{Ext} + \Delta_0^-\text{-Sep}\| = \varepsilon_0.$$ Moreover Basic ($i$) + Ext + $\Delta_0^-\text{-Sep}$ is $i$-conservative, and hence conservative extension of HA.
K. Sato [2009]: classical *weak weak* set theory **Basic** and beyond. We consider **Basic+Ext** and its extensions.

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*What about intuitionistic counterparts $\text{Basic}^{(i)} + \text{Ext}$ and $\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep}$?*

### Solution

\[ |\text{Basic}^{(i)} + \text{Ext}| = |\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep}| = \varepsilon_0. \]

Moreover $\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep}$ is $i$-conservative, and hence conservative extension of $\text{HA}$. 
Generalizations

Note that Basic includes Clps (Collapsing):

\[
\text{Clps} \equiv \text{Ord}(x) \land (\forall s, t \in x) (\langle s, t \rangle \in r \leftrightarrow \langle s, t \rangle / \in r') \land \text{WF}(x, r) \rightarrow (\exists f, y) \text{TrClps}(f, x, r, y).
\]

but the rest is much weaker than Friedman's $T_1$.

Also consider Sato's strengthening Anti-Reg that is not in Basic (clearly Fnd and Anti-Reg are incompatible):

\[
\text{Anti-Reg} \equiv \text{Ord}(x) \land (\forall s, t \in x) (\langle s, t \rangle \in r \leftrightarrow \langle s, t \rangle / \in r') \rightarrow (\exists f, y) \text{TrClps}(f, x, r, y).
\]

Theorem Basic $(i) + \text{Ext} + \Delta^0_0 - \text{Sep} + \text{Exp} + \text{SC} + \text{Fnd}$ and Basic $(i) + \text{Ext} + \Delta^0_0 - \text{Sep} + \text{Exp} + \text{SC} + \text{Anti-Reg}$ are both conservative extensions of HA.

Stronger results to be discussed later.
Note that **Basic** includes **Clps** (Collapsing):

- **Clps** $\equiv \text{Ord}(x) \land (\forall s, t \in x) (\langle s, t \rangle \in r \leftrightarrow \langle s, t \rangle \notin r') \land \WF(x, r) \rightarrow (\exists f, y) \text{TrClps}(f, x, r, y)$

but the rest is much weaker than Friedman’s $T_1$. 

**Theorem**

**Basic** $(i) + \text{Ext} + \Delta^0_0 - \text{Sep} + \text{Exp} + \text{SC} + \text{Fnd}$ and **Basic** $(i) + \text{Ext} + \Delta^0_0 - \text{Sep} + \text{Exp} + \text{SC} + \text{Anti-Reg}$ are both conservative extensions of HA.
Note that **Basic** includes **Clps** (Collapsing):

- \[ \text{Clps} \equiv \text{Ord}(x) \land (\forall s, t \in x) (\langle s, t \rangle \in r \leftrightarrow \langle s, t \rangle \notin r') \land \text{WF}(x, r) \rightarrow (\exists f, y) \text{TrClps}(f, x, r, y) \]

but the rest is much weaker than Friedman’s \( T_1 \).

Also consider Sato’s strengthening **Anti-Reg** that is not in **Basic** (clearly **Fnd** and **Anti-Reg** are incompatible):
Generalizations

- Note that **Basic** includes **Clps** (Collapsing):
  - \( \text{Clps} \equiv \text{Ord}(x) \land (\forall s, t \in x) (\langle s, t \rangle \in r \leftrightarrow \langle s, t \rangle \notin r') \land \text{WF}(x, r) \rightarrow (\exists f, y) \text{TrClps}(f, x, r, y) \)

  but the rest is much weaker than Friedman’s \( T_1 \).

- Also consider Sato’s strengthening **Anti-Reg** that is not in **Basic** (clearly **Fnd** and **Anti-Reg** are incompatible):
  - \( \text{Anti-Reg} \equiv \text{Ord}(x) \land (\forall s, t \in x) (\langle s, t \rangle \in r \leftrightarrow \langle s, t \rangle \notin r') \rightarrow (\exists f, y) \text{TrClps}(f, x, r, y) \).
Generalizations

- Note that **Basic** includes **Clps** (Collapsing):
  
  \[
  \text{Clps} \equiv \text{Ord}(x) \land (\forall s, t \in x) (\langle s, t \rangle \in r \iff \langle s, t \rangle \notin r') \land \\
  \text{WF}(x, r) \to (\exists f, y) \text{TrClps}(f, x, r, y)
  \]

  but the rest is much weaker than Friedman’s \( T_1 \).

- Also consider Sato’s strengthening **Anti-Reg** that is not in **Basic** (clearly **Fnd** and **Anti-Reg** are incompatible):
  
  \[
  \text{Anti-Reg} \equiv \text{Ord}(x) \land (\forall s, t \in x) (\langle s, t \rangle \in r \iff \langle s, t \rangle \notin r') \to \\
  (\exists f, y) \text{TrClps}(f, x, r, y)
  \]

**Theorem**

Theorem Basic \( i \) + Ext + \( \Delta_0 \) - Sep + Exp + SC + Fnd and Basic \( i \) + Ext + \( \Delta_0 \) - Sep + Exp + SC + Anti-Reg are both conservative extensions of \( \text{HA} \).

Stronger results to be discussed later.
Generalizations

- Note that **Basic** includes **Clps** (Collapsing):
  
  \[
  \text{Clps} \equiv \text{Ord}(x) \land (\forall s, t \in x) ((s, t) \in r \leftrightarrow (s, t) \notin r') \land \\
  \text{WF}(x, r) \rightarrow (\exists f, y) \text{TrClps}(f, x, r, y)
  \]

  but the rest is much weaker than Friedman's \( T_1 \).

- Also consider Sato's strengthening **Anti-Reg** that is not in **Basic** (clearly **Fnd** and **Anti-Reg** are incompatible):
  
  \[
  \text{Anti-Reg} \equiv \text{Ord}(x) \land (\forall s, t \in x) ((s, t) \in r \leftrightarrow (s, t) \notin r') \rightarrow \\
  (\exists f, y) \text{TrClps}(f, x, r, y).
  \]

### Theorem

\[
\begin{align*}
\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \text{Exp} + \text{SC} + \text{Fnd} & \text{ and} \\
\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \text{Exp} + \text{SC} + \text{Anti-Reg} & \text{are both conservative extensions of HA.}
\end{align*}
\]
Note that **Basic** includes **Clps** (Collapsing):

\[
\text{Clps} \equiv \text{Ord}(x) \land (\forall s, t \in x) (\langle s, t \rangle \in r \leftrightarrow \langle s, t \rangle \notin r') \land \\
\text{WF}(x, r) \rightarrow (\exists f, y) \text{TrClps}(f, x, r, y)
\]

but the rest is much weaker than Friedman’s $T_1$.

Also consider Sato’s strengthening **Anti-Reg** that is not in **Basic** (clearly **Fnd** and **Anti-Reg** are incompatible):

\[
\text{Anti-Reg} \equiv \text{Ord}(x) \land (\forall s, t \in x) (\langle s, t \rangle \in r \leftrightarrow \langle s, t \rangle \notin r') \rightarrow \\
(\exists f, y) \text{TrClps}(f, x, r, y).
\]

**Theorem**

**Basic**$^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \text{Exp} + \text{SC} + \text{Fnd}$ and **Basic**$^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \text{Exp} + \text{SC} + \text{Anti-Reg}$

are both conservative extensions of HA.

- Stronger results to be discussed later.
Theorem K. Sato [2009]: $|\text{Basic} + \text{Ext} + \Delta_0\text{-Sep}| \geq \Gamma_0$ (in fact $= \Gamma_0$).

Proof. Crucial inconstructive argument: all ordinals are comparable.

Let $\text{Ord}(\alpha) \land \text{Ord}(\beta)$. We show by $\text{WF}(\alpha + 1) \land \text{WF}(\beta + 1)$ via $\Delta_0\text{-Sep}$ that $(\forall \gamma \in \alpha + 1) (\forall \delta \in \beta + 1) (\gamma \in \delta \lor \gamma = \delta \lor \gamma \ni \delta)$.

Now by the IH we have: $(\forall \gamma' \in \gamma) (\forall \delta' \in \delta) (\gamma' \in \delta' \lor \gamma' = \delta' \lor \gamma' \ni \delta')$.

If $(\exists \gamma' \in \gamma) (\gamma' = \delta \lor \gamma' \ni \delta)$ then $\delta \in \gamma$.

Thus $\delta \in \gamma$ or $\gamma \subset \delta$, and similarly $\gamma \in \delta$ or $\delta \subset \gamma$.

But then $\delta \in \gamma \lor (\gamma \subset \delta \land \delta \subset \gamma) \lor \gamma \in \delta$, which by Ext yields $\delta \in \gamma \lor \delta = \gamma \lor \delta \ni \gamma$.

Hence $\alpha \in \beta \lor \alpha = \beta \lor \alpha \ni \beta$, as desired.

And hence all countable well-orderings are mutually comparable, since $\text{Clps}$ postulates that they can be collapsed to ordinals.
Theorem

K. Sato [2009]: $|\text{Basic} + \text{Ext} + \Delta_0\text{-Sep}| \geq \Gamma_0$ (in fact $= \Gamma_0$).

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Let $\text{Ord}(\alpha) \land \text{Ord}(\beta)$. We show by $\text{WF}(\alpha + 1) \land \text{WF}(\beta + 1)$ via $\Delta_0$-Sep that $(\forall \gamma \in \alpha + 1) (\forall \delta \in \beta + 1) (\gamma \in \delta \lor \gamma = \delta \lor \gamma \ni \delta)$.

Now by the IH we have: $(\forall \gamma' \in \gamma) (\gamma' \in \delta \lor \gamma' = \delta \lor \gamma' \ni \delta)$ and $(\forall \delta' \in \delta) (\gamma \in \delta' \lor \gamma = \delta' \lor \gamma \ni \delta')$.

If $(\exists \gamma' \in \gamma) (\gamma' = \delta \lor \gamma' \ni \delta)$ then $\delta \in \gamma$.

Thus $\delta \in \gamma$ or $\gamma \subset \delta$, and similarly $\gamma \in \delta$ or $\delta \subset \gamma$.

But then $\delta \in \gamma \lor (\gamma \subset \delta \land \delta \subset \gamma) \lor \gamma \in \delta$, which by $\text{Ext}$ yields $\delta \in \gamma \lor \delta = \gamma \lor \delta \ni \gamma$.

Hence $\alpha \in \beta \lor \alpha = \beta \lor \alpha \ni \beta$, as desired.

And hence all countable well-orderings are mutually comparable, since $\text{Clps}$ postulates that they can be collapsed to ordinals.
Theorem

K. Sato [2009]: \(|\text{Basic + Ext + } \Delta_0\text{-Sep}| \geq \Gamma_0 \ (\text{in fact } = \Gamma_0)\).
§3. More on Basic + Ext + $\Delta_0$-Sep

**Theorem**

*K. Sato [2009]:* $|\text{Basic} + \text{Ext} + \Delta_0\text{-Sep}| \geq \Gamma_0$ (*in fact* $= \Gamma_0$).

**Proof.**

Crucial inconstructive argument: all ordinals are comparable.

Let $\text{Ord}(\alpha) \land \text{Ord}(\beta)$.

We show by WF($\alpha + 1) \land \text{WF}(\beta + 1)$ via $\Delta_0$-Sep that ($\forall \gamma \in \alpha + 1)$ ($\forall \delta \in \beta + 1)$ ($\gamma \in \delta \lor \gamma = \delta \lor \gamma \ni \delta$).

Now by the IH we have: ($\forall \gamma' \in \gamma)$ ($\gamma' \in \delta \lor \gamma' = \delta \lor \gamma' \ni \delta$) and ($\forall \delta' \in \delta)$ ($\gamma \in \delta' \lor \gamma = \delta' \lor \gamma \ni \delta'$).

If ($\exists \gamma' \in \gamma)$ ($\gamma' = \delta \lor \gamma' \ni \delta$) then $\delta \in \gamma$.

Thus $\delta \in \gamma$ or $\gamma \subset \delta$, and similarly $\gamma \in \delta$ or $\delta \subset \gamma$.

But then $\delta \in \gamma \lor (\gamma \subset \delta \land \delta \subset \gamma) \lor \gamma \in \delta$, which by Ext yields $\delta \in \gamma \lor \delta = \gamma \lor \delta \ni \gamma$.

Hence $\alpha \in \beta \lor \alpha = \beta \lor \alpha \ni \beta$, as desired.

And hence all countable well-orderings are mutually comparable, since Clps postulates that they can be collapsed to ordinals.
§3. More on **Basic + Ext + Δ₀-Sep**

**Theorem**

*K. Sato [2009]*: \(|\text{Basic} + \text{Ext} + Δ₀-\text{Sep}| ≥ Γ₀ (in fact = Γ₀).

**Proof.**

Crucial **inconstructive** argument: all ordinals are comparable. *)
Theorem

K. Sato [2009]: $|\text{Basic} + \text{Ext} + \Delta_0\text{-Sep}| \geq \Gamma_0$ (in fact $= \Gamma_0$).

Proof.

Crucial *inconstructive* argument: all ordinals are comparable. *)
Let $\text{Ord} (\alpha) \land \text{Ord} (\beta)$.

*) And hence all countable well-orderings are mutually comparable, since Clps postulates that they can be collapsed to ordinals.
§3. More on Basic + Ext + $\Delta_0$-Sep

Theorem

K. Sato [2009]: $|\text{Basic } + \text{ Ext } + \Delta_0\text{-Sep}| \geq \Gamma_0$ (in fact $= \Gamma_0$).

Proof.

Crucial inconstructive argument: all ordinals are comparable. *)

Let $\text{Ord}(\alpha) \land \text{Ord}(\beta)$. We show by $\text{WF}(\alpha + 1) \land \text{WF}(\beta + 1)$ via $\Delta_0$-Sep that $(\forall \gamma \in \alpha + 1) (\forall \delta \in \beta + 1) (\gamma \in \delta \lor \gamma = \delta \lor \gamma \ni \delta)$.

*) And hence all countable well-orderings are mutually comparable, since Clps postulates that they can be collapsed to ordinals.
$\S 3$. More on $\text{Basic} + \text{Ext} + \Delta_0\text{-Sep}$

**Theorem**

*K. Sato [2009]:* $|\text{Basic} + \text{Ext} + \Delta_0\text{-Sep}| \geq \Gamma_0$ (*in fact* $= \Gamma_0$).

**Proof.**

Crucial *inconstructive* argument: all ordinals are comparable. *)

Let $\text{Ord}(\alpha) \land \text{Ord}(\beta)$. We show by $\text{WF}(\alpha + 1) \land \text{WF}(\beta + 1)$ via $\Delta_0\text{-Sep}$ that $(\forall \gamma \in \alpha + 1) (\forall \delta \in \beta + 1) (\gamma \in \delta \lor \gamma = \delta \lor \gamma \ni \delta)$. Now by the IH we have: $(\forall \gamma' \in \gamma) (\gamma' \in \delta \lor \gamma' = \delta \lor \gamma' \ni \delta)$ and $(\forall \delta' \in \delta) (\gamma \in \delta' \lor \gamma = \delta' \lor \gamma \ni \delta')$. 

*) And hence all countable well-orderings are mutually comparable, since Clps postulates that they can be collapsed to ordinals.
§3. More on Basic + Ext + $\Delta_0$-Sep

**Theorem**

*K. Sato [2009]*: $|\text{Basic} + \text{Ext} + \Delta_0\text{-Sep}| \geq \Gamma_0$ (*in fact $= \Gamma_0$*).

**Proof.**

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Let $\text{Ord}(\alpha) \land \text{Ord}(\beta)$. We show by $\text{WF}(\alpha + 1) \land \text{WF}(\beta + 1)$ via $\Delta_0\text{-Sep}$ that $(\forall \gamma \in \alpha + 1) (\forall \delta \in \beta + 1) (\gamma \in \delta \lor \gamma = \delta \lor \gamma \ni \delta)$.

Now by the IH we have: $(\forall \gamma' \in \gamma) (\gamma' \in \delta \lor \gamma' = \delta \lor \gamma' \ni \delta)$ and $(\forall \delta' \in \delta) (\gamma \in \delta' \lor \gamma = \delta' \lor \gamma \ni \delta')$. If $(\exists \gamma' \in \gamma) (\gamma' = \delta \lor \gamma' \ni \delta)$ then $\delta \in \gamma$.
Theorem

K. Sato [2009]: $|\text{Basic} + \text{Ext} + \Delta_0\text{-Sep}| \geq \Gamma_0$ (in fact $= \Gamma_0$).

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Crucial inconstructive argument: all ordinals are comparable. *)

Let $\text{Ord}(\alpha) \land \text{Ord}(\beta)$. We show by $\text{WF}(\alpha + 1) \land \text{WF}(\beta + 1)$ via $\Delta_0\text{-Sep}$ that $(\forall \gamma \in \alpha + 1) (\forall \delta \in \beta + 1) (\gamma \in \delta \lor \gamma = \delta \lor \gamma \ni \delta)$. Now by the IH we have: $(\forall \gamma' \in \gamma) (\gamma' \in \delta \lor \gamma' = \delta \lor \gamma' \ni \delta)$ and $(\forall \delta' \in \delta) (\gamma \in \delta' \lor \gamma = \delta' \lor \gamma \ni \delta')$. If $(\exists \gamma' \in \gamma) (\gamma' = \delta \lor \gamma' \ni \delta)$ then $\delta \in \gamma$. Thus $\delta \in \gamma$ or $\gamma \subset \delta$, and similarly $\gamma \in \delta$ or $\delta \subset \gamma$. 

§3. More on Basic + Ext + $\Delta_0$-Sep

Theorem

K. Sato [2009]: $|\text{Basic} + \text{Ext} + \Delta_0$-Sep$| \geq \Gamma_0$ (in fact $= \Gamma_0$).

Proof.

Crucial inconstructive argument: all ordinals are comparable. *)

Let Ord($\alpha$) $\land$ Ord($\beta$). We show by WF($\alpha + 1$) $\land$ WF($\beta + 1$) via $\Delta_0$-Sep that $(\forall \gamma \in \alpha + 1)(\forall \delta \in \beta + 1)(\gamma \in \delta \lor \gamma = \delta \lor \gamma \ni \delta)$. Now by the IH we have: $(\forall \gamma' \in \gamma)(\gamma' \in \delta \lor \gamma' = \delta \lor \gamma' \ni \delta)$ and $(\forall \delta' \in \delta)(\gamma \in \delta' \lor \gamma = \delta' \lor \gamma \ni \delta')$. If $(\exists \gamma' \in \gamma)(\gamma' = \delta \lor \gamma' \ni \delta)$ then $\delta \in \gamma$. Thus $\delta \in \gamma$ or $\gamma \subset \delta$, and similarly $\gamma \in \delta$ or $\delta \subset \gamma$.

But then $\delta \in \gamma \lor (\gamma \subset \delta \land \delta \subset \gamma) \lor \gamma \in \delta$, which by Ext yields $\delta \in \gamma \lor \delta = \gamma \lor \delta \ni \gamma$. 

* And hence all countable well-orderings are mutually comparable, since Clps postulates that they can be collapsed to ordinals.
§3. More on Basic + Ext + $\Delta_0$-Sep

**Theorem**

*K. Sato [2009]:* $|\text{Basic + Ext} + \Delta_0\text{-Sep}| \geq \Gamma_0$ (*in fact $= \Gamma_0$*).

**Proof.**

Crucial inconstructive argument: all ordinals are comparable. *)

Let $\text{Ord}(\alpha) \land \text{Ord}(\beta)$. We show by $\text{WF}(\alpha + 1) \land \text{WF}(\beta + 1)$ via $\Delta_0\text{-Sep}$ that $(\forall \gamma \in \alpha + 1)(\forall \delta \in \beta + 1)(\gamma \in \delta \lor \gamma = \delta \lor \gamma \ni \delta)$. Now by the IH we have: $(\forall \gamma' \in \gamma)(\gamma' \in \delta \lor \gamma' = \delta \lor \gamma' \ni \delta)$ and $(\forall \delta' \in \delta)(\gamma \in \delta' \lor \gamma = \delta' \lor \gamma \ni \delta')$. If $(\exists \gamma' \in \gamma)(\gamma' = \delta \lor \gamma' \ni \delta)$ then $\delta \in \gamma$. Thus $\delta \in \gamma$ or $\gamma \subset \delta$, and similarly $\gamma \in \delta$ or $\delta \subset \gamma$.

But then $\delta \in \gamma \lor (\gamma \subset \delta \land \delta \subset \gamma) \lor \gamma \in \delta$, which by Ext yields $\delta \in \gamma \lor \delta \lor \gamma \ni \gamma$. Hence $\alpha \in \beta \lor \alpha = \beta \lor \alpha \ni \beta$, as desired.

*) And hence all countable well-orderings are mutually comparable, since Clps postulates that they can be collapsed to ordinals.
§3. More on Basic + Ext + $\Delta_0$-Sep

**Theorem**

*K. Sato [2009]:* \(|\text{Basic} + \text{Ext} + \Delta_0\text{-Sep}| \geq \Gamma_0 \) (in fact \(= \Gamma_0\)).

**Proof.**

Crucial inconstructive argument: all ordinals are comparable. *)

Let \(\text{Ord} (\alpha) \land \text{Ord} (\beta)\). We show by \(\text{WF} (\alpha + 1) \land \text{WF} (\beta + 1)\) via $\Delta_0$-Sep that \((\forall \gamma \in \alpha + 1) (\forall \delta \in \beta + 1) (\gamma \in \delta \lor \gamma = \delta \lor \gamma \ni \delta)\). Now by the IH we have: \((\forall \gamma' \in \gamma) (\gamma' \in \delta \lor \gamma' = \delta \lor \gamma' \ni \delta)\) and \((\forall \delta' \in \delta) (\gamma \in \delta' \lor \gamma = \delta' \lor \gamma \ni \delta')\). If \((\exists \gamma' \in \gamma) (\gamma' = \delta \lor \gamma' \ni \delta)\) then \(\delta \in \gamma\). Thus \(\delta \in \gamma\) or \(\gamma \subset \delta\), and similarly \(\gamma \in \delta\) or \(\delta \subset \gamma\).

But then \(\delta \in \gamma \lor (\gamma \subset \delta \land \delta \subset \gamma) \lor \gamma \in \delta\), which by Ext yields \(\delta \in \gamma \lor \delta = \gamma \lor \delta \ni \gamma\). Hence \(\alpha \in \beta \lor \alpha = \beta \lor \alpha \ni \beta\), as desired.

*) And hence all countable well-orderings are mutually comparable, since Clps postulates that they can be collapsed to ordinals.
§4. Stronger results

Theorem And yet $\left| \left| \text{Basic} \left( i \right) + \text{Ext} + \Delta_0 - \text{Sep} \right| \right| = \epsilon_0$.

Actually we have: $\left| \left| \text{Basic} \left( i \right) + \text{Ext} + \Delta_0 - \text{Sep} + \Theta + \text{Fnd} \right| \right|$ and $\left| \left| \text{Basic} \left( i \right) + \text{Ext} + \Delta_0 - \text{Sep} + \Theta + \text{Cpl} \right| \right|$ are both conservative extensions of $\text{HA}$, where $\Theta = \text{Ful} + \text{AC}! + \text{SC} + \text{Enm}$ and $\Theta = \text{Cpl} \equiv r \subset x \times x \rightarrow (\exists f, y) \text{TrClps} (f, x, r, y)$, $\text{Enm} \equiv (\exists y \subset \omega) (\exists f) \text{Surj} (f, y, x)$, $\text{AC}! \equiv (\forall u \in x) \left( \exists! v \in y \psi (u, v) \rightarrow \exists f \left( \text{Func} (f, x, y) \land (\forall u \in x) \psi (u, f (u)) \right) \right)$, $\text{Ful} \equiv (\exists z) \left( (\forall r \in z) \text{Tot} (r, x, y) \land (\forall r) (\text{Tot} (r, x, y) \rightarrow (\exists s \in z) (s \subset r \land \text{Tot} (s, x, y)) \right)$. $\text{Tot} (r, x, y) \equiv r \subset x \times y \land (\forall u \in x) (\exists v \in y) (\langle u, v \rangle \in r)$.
§4. Stronger results

Theorem

And yet ⏐⏐

Basic $(i) +$ Ext $+ ∆_0 - Sep ⏐⏐ = ε_0$.

Actually we have:

Basic $(i) +$ Ext $+ ∆_0 - Sep + Θ + Fnd$

and

Basic $(i) +$ Ext $+ ∆_0 - Sep + Θ + Cpl$

are both conservative extensions of $HA$, where

$Θ = Ful + AC! + SC + Enm$

and:

$Cpl ≡ r ⊂ x × x → (∃ f, y) TrClps (f, x, r, y)$,

$Enm ≡ (∃ y ⊂ ω) (∃ f) Surj (f, y, x)$,

$AC! ≡ (∀ u ∈ x) (∃! v ∈ y) ψ (u, v) → (∃ f (Func (f, x, y) ∧ (∀ u ∈ x) ψ (u, f (u))))$,

$Ful ≡ (∃ z) (∀ r ∈ z) (Tot (r, x, y) ∧ (∀ r) (Tot (r, x, y) → (∃ s ∈ z) (s ⊂ r ∧ Tot (s, x, y))))$,

$Tot (r, x, y) ≡ r ⊂ x × y ∧ (∀ u ∈ x) (∃ v ∈ y) (⟨ u, v ⟩ ∈ r)$.
§4. Stronger results

Theorem

And yet \(|\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep}| = \varepsilon_0\).
§4. Stronger results

Theorem

And yet \(|\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep}| = \varepsilon_0\). Actually we have:
Basic^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \Theta + \text{Fnd} and
Basic^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \Theta + \text{Cpl}
are both conservative extensions of HA, where
\Theta = \text{Ful} + \text{AC}! + \text{SC} + \text{Enm} and:
§4. Stronger results

Theorem

And yet $|\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep}| = \varepsilon_0$. Actually we have:

$\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \Theta + \text{Fnd}$ and

$\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \Theta + \text{Cpl}$

are both conservative extensions of HA, where

$\Theta = \text{Ful} + \text{AC}! + \text{SC} + \text{Enm}$ and:

$\text{Cpl} \equiv r \subseteq x \times x \rightarrow (\exists f, y) \text{TrClps} (f, x, r, y)$,
§ 4. Stronger results

Theorem

And yet $|\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep}| = \varepsilon_0$. Actually we have:

Basic$^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \Theta + \text{Fnd}$ and
Basic$^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \Theta + \text{Cpl}$

are both conservative extensions of HA, where

$\Theta = \text{Ful} + \text{AC}! + \text{SC} + \text{Enm}$ and:

$\text{Cpl} \equiv r \subset x \times x \rightarrow (\exists f, y) \text{TrClps}(f, x, r, y),$

$\text{Enm} \equiv (\exists y \subset \omega) (\exists f) \text{Surj}(f, y, x),$
§4. Stronger results

Theorem

And yet \( \lvert \text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep} \rvert = \varepsilon_0 \). Actually we have:

Basic\(^{(i)}\) + Ext + \(\Delta_0\text{-Sep} + \Theta + \text{Fnd} \) and

Basic\(^{(i)}\) + Ext + \(\Delta_0\text{-Sep} + \Theta + \text{Cpl} \)

are both conservative extensions of HA, where

\( \Theta = \text{Ful} + \text{AC}! + \text{SC} + \text{Enm} \) and:

Cpl \( \equiv r \subset x \times x \rightarrow (\exists f, y) \text{TrClps} (f, x, r, y), \)

Enm \( \equiv (\exists y \subset \omega) (\exists f) \text{Surj} (f, y, x), \)

AC! \( \equiv (\forall u \in x) (\exists! v \in y) \psi (u, v) \rightarrow \exists f (\text{Func} (f, x, y) \land (\forall u \in x) \psi (u, f (u))) \)
§4. Stronger results

**Theorem**

And yet \(|\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep}| = \varepsilon_0\). Actually we have:

\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \Theta + \text{Fnd} and

\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \Theta + \text{Cpl}

are both conservative extensions of HA, where

\[\Theta = \text{Ful} + \text{AC!} + \text{SC} + \text{Enm}\]

and:

\[\text{Cpl} \equiv r \subseteq x \times x \rightarrow (\exists f, y) \text{TrClps} (f, x, r, y),\]

\[\text{Enm} \equiv (\exists y \subseteq \omega) (\exists f) \text{Surj} (f, y, x),\]

\[\text{AC!} \equiv (\forall u \in x) (\exists! v \in y) \psi (u, v) \rightarrow \exists f (\text{Func} (f, x, y) \land (\forall u \in x) \psi (u, f (u)))\]

\[\text{Ful} \equiv \]

\[\exists z \left( (\forall r \in z) \text{Tot} (r, x, y) \land \forall r \left( (\text{Tot} (r, x, y) \rightarrow (\exists s \in z) \left( s \subseteq r \land \text{Tot} (s, x, y) \right) \right) \right), \]
§4. Stronger results

**Theorem**

And yet \(|\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep}| = \varepsilon_0\). Actually we have:

\[
\begin{align*}
\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \Theta + \text{Fnd} \quad &\text{and} \\
\text{Basic}^{(i)} + \text{Ext} + \Delta_0\text{-Sep} + \Theta + \text{Cpl} \\
\end{align*}
\]

are both conservative extensions of HA, where

\(\Theta = \text{Ful} + \text{AC!} + \text{SC} + \text{Enm}\) and:

\[
\begin{align*}
\text{Cpl} &\equiv r \subset x \times x \rightarrow (\exists f, y) \text{TrClps} (f, x, r, y), \\
\text{Enm} &\equiv (\exists y \subset \omega) (\exists f) \text{Surj} (f, y, x), \\
\text{AC!} &\equiv (\forall u \in x) (\exists ! v \in y) \psi (u, v) \rightarrow \\
&\exists f (\text{Func} (f, x, y) \land (\forall u \in x) \psi (u, f (u))) \\
\text{Ful} &\equiv \\
(\exists z) \left( (\forall r \in z) \text{Tot} (r, x, y) \land \forall r \left( \left( \text{Tot} (r, x, y) \rightarrow (\exists s \in z) \\
(s \subset r \land \text{Tot} (s, x, y)) \right) \right) \right), \\
\end{align*}
\]

\(\text{Tot} (r, x, y) \equiv r \subset x \times y \land (\forall u \in x)(\exists v \in y)(\langle u, v \rangle \in r).\)
Remarks

Basic $i + \Delta_0$-Sep $\Theta + Fnd$ is a proper extension of Friedman's $T_1$. Within Basic $i + \Delta_0$-Sep $\text{Ful}$ implies $\text{Exp}$ (but not otherwise), both being much weaker than $\text{Pow}$.

Within Basic $i + \Delta_0$-Sep $\text{Enm}$: $\text{Cpl}$ is equivalent to $\text{Anti-Reg}$.

For brevity we use standard constructive version of $\text{Ord}$ ($x$):

$\text{Ord}(x) \equiv \text{POrd}(x) \land \emptyset \in x \land (\forall u)((\forall y \in x)(y \subset u \iff y \in u) \rightarrow x \subset u)$. 

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Proof-theoretic conservation of weak weak intuitionistic constructive set theories
Remarks

- **Basic**(i) + Ext + $\Delta_0$-Sep + $\Theta$ + Fnd is a proper extension of Friedman’s $T_1$. 
Remarks

- **Basic**(i) + Ext + Δ₀-Sep + Θ + Fnd is a proper extension of Friedman’s $T_1$.
- Within **Basic**(i) + Δ₀-Sep + SC, Ful implies Exp (but not otherwise), both being much weaker than Pow.
Basic\(^{(i)}\) + Ext + \(\Delta_0\)-Sep + \(\Theta\) + Fnd is a proper extension of Friedman’s \(T_1\).

Within Basic\(^{(i)}\) + \(\Delta_0\)-Sep + SC, Ful implies Exp (but not otherwise), both being much weaker than Pow.

Within Basic\(^{(i)}\) + \(\Delta_0\)-Sep + Enm : Cpl is equivalent to Anti-Reg.
Remarks

- **Basic\(^{(i)}\) + Ext + \(\Delta_0\)-Sep + \(\Theta\) + Fnd** is a proper extension of Friedman’s \(T_1\).
- Within **Basic\(^{(i)}\) + \(\Delta_0\)-Sep + SC**, **Ful** implies **Exp** (but not otherwise), both being much weaker than **Pow**.
- Within **Basic\(^{(i)}\) + \(\Delta_0\)-Sep + Enm**:
  - **Cpl** is equivalent to **Anti-Reg**.
- For brevity we use standard constructive version of \(\text{Ord}(x)\):
  \[
  \text{Ord}(x) \equiv P\text{Ord}(x) \land \emptyset \in x \land \\
  (\forall u)(((\forall y \in x)(y \subseteq u \iff y \in u) \rightarrow x \subseteq u)
  \]
The proofs run along the lines of L. G. [1982, 1988] in 3 steps:

1. Realizability bisimulation-interpretation of chosen extensional set theory T within suitable Feferman-style explicit intensional intuitionistic theory of functions and classes EFC.

2. Constructive cut elimination in EFC (most difficult part of proof).

3. Realizability elimination via forcing in explicit intuitionistic arithmetic EHA (along the lines of M. Beeson [1979]).

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Proof-theoretic conservations of weak weak intuitionistic constructive set theories.
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On proofs -1-

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This yields for any arithmetical sentence $A$:

1. $T \vdash A \Rightarrow EFC \vdash (A$ realizable$)$,
2. $EFC \vdash (A$ realizable$) \Rightarrow EHA \vdash (A$ realizable$)$,
3. $EHA \vdash (A$ realizable$) \Rightarrow HA \vdash A$,

as desired.
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This yields for any arithmetical sentence $A$:

1. $T \vdash A \Rightarrow EFC \vdash (A \text{ realizable})$,
2. $EFC \vdash (A \text{ realizable}) \Rightarrow EHA \vdash (A \text{ realizable})$, as desired.
This yields for any arithmetical sentence $A$:

1. $T \vdash A \Rightarrow EFC \vdash (A \text{ realizable})$,
2. $EFC \vdash (A \text{ realizable}) \Rightarrow EHA \vdash (A \text{ realizable})$,
3. $EHA \vdash (A \text{ realizable}) \Rightarrow HA \vdash A$, as desired.
Thanks for patience!