A Bimodal Analysis of Knowability

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Our goal is to analyse and clarify the conception of knowability expressed by the verificationist knowability principle:

\[ F \rightarrow \Diamond KF \quad (VK) \]

We analyse this principle as a scheme in a logical framework with the alethic modalities \( \Box \) (necessary), \( \Diamond \) (possible), and the epistemic modality \( K \).

Modalities \( \Box/\Diamond \) represent, in an abstract way, the process of discovery.
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Principle of Verificationist Knowability:

\[ F \rightarrow \Diamond KF \]

yields

Omniscience Principle:

\[ F \rightarrow KF \quad (OMN) \]
Theorem 1. 
[Church-Fitch] VK as schema yields OMN.

Proof. 
Consider with $F = p \land \neg Kp$ (which we will call Moore):

\[(Moore) \rightarrow \Diamond K(Moore). \quad (VK(Moore))\]

1. $K(Moore) \rightarrow Kp$ - since $K(X \land Y) \rightarrow KX$;
2. $K(Moore) \rightarrow (p \land \neg Kp) \rightarrow \neg Kp$ - factivity of knowledge;
3. $K(Moore) \rightarrow \bot$;
4. $\Diamond K(Moore) \rightarrow \Diamond \bot$ - from 3, by modal reasoning;
5. $\neg \Diamond K(Moore)$ - from 4, since $\Diamond \bot \rightarrow \bot$;
6. $\neg Moore$ - from 5 and $VK(Moore)$;
7. $p \rightarrow \neg Kp$ - from 6 since $\neg (X \land Y)$ yields $(X \rightarrow \neg Y)$;
8. $p \rightarrow Kp$ - since $\neg \neg X$ yields $X$ in classical logic.
Given this, a stronger result is provable.

**Corollary 2.**

\[ VK = OMN. \]

**Proof.**

It remains to show that \( OMN \) yields \( VK \).

By \( OMN, F \rightarrow KF \). Since \( KF \rightarrow \Diamond KF \) for reflexive modality \( \Box \),

\[ F \rightarrow \Diamond KF. \]
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VK is not intuitively valid, even under circumstances acceptable to the verificationist.

For instance, *even if it was raining when Holmes asked Watson to check for the rain, but the rain had stopped by the time Watson looked outside, Watson’s answer will be “no rain.”*

A correct verification procedure applied to a true \( F \) yields a positive result only on the assumption that \( F \) stays true during the verification.

What is missing from \( VK \) is the assumption that \( F \) is *stable.*
More formally, a proposition $F$ is **stable** in a given model, if it satisfies

$$F \rightarrow \Box F.$$ 

Note that for a reflexive modality $\Box$, $F$ is stable if and only if $F \leftrightarrow \Box F$.

A stable sentence can be false at some (or even all) states of a given model, but once it is true at a state, it remains true at all $\Box$-accessible states.
There are no reasons to believe that without the assumption that \( F \) is stable, principle \( VK \) is valid in all situations acceptable for the verificationist.

Consider a correct, i.e., knowledge- and knowability-producing verification procedure, \( \mathcal{V} \), applied to a true proposition \( F \). However, during (or, perhaps, due to) verification, \( F \) changes its truth value and \( \mathcal{V} \) eventually certifies that \( F \) is false. Then \( VK \) fails despite the fact that a correct verification procedure has been applied to a true proposition and terminates with a definitive answer.
Consider the bi-modal model, $\mathcal{M}_1$.

- $\mathcal{M}_1$ has three states $W = \{u, v, w\}$ and two accessibility relations, $R_\square$ (arrows) and $R_K$ (ovals).
- $F$ holds at $u$ but not at $v$ or $w$.
- We can think of $\mathcal{M}_1$ as modeling the process of verification or investigation, moving from ignorance to knowledge.

Model $\mathcal{M}_1$ where $VK$ fails.
F is not stable at \( u \).

\( \neg KF \) holds at all states, hence 

\( u \models \neg \Diamond KF \).

Hence 

\( u \not\models F \rightarrow \Diamond KF \).

Figure 1. Model \( \mathcal{M}_1 \) where \( VK \) fails.
The non-stability of $F$ prevents $F$ from being knowable in the sense of $VK$, despite a correct verification procedure.

This suggests that $VK$ does not properly express the idea that all truths are knowable.

What, then, are the natural frame conditions for $VK$?

The answer to this yields an alternative proof that $VK$ is equivalent to $OMN$. 
In an epistemic setting it seems natural to expect an epistemic state to have some combination of features that makes it different from all other states.

We consider models in which states can be defined: for each state $u$ there is a proposition $F_u$ (sometimes called *nominal*) such that

$$u \models F_u \text{ and for all } v \neq u, v \not\models F_u.$$

We call such models *definable states models* or *definable models*. Each model can be made definable by adding fresh atomic nominals: the old formulas all retain their truth value. Standard soundness/completeness theorems extend to definable models automatically.
In $\mathcal{M}_1$ the following formulas can be regarded as nominals:

- $F_u = F$ (\textit{u is the state at which F holds});
- $F_v = \neg F \land \neg K \neg F$ (\textit{v is the state at which both F and K$\neg$F are false});
- $F_w = \neg F \land K \neg F$ (\textit{w is the state in which $\neg F$ holds and is known}).

Figure 1. Model $\mathcal{M}_1$ where VK fails.
Let us call a state \( u \) **omniscient** if it is a singleton with respect to \( R_K \):

\[
u R_K v \text{ yields } u = v
\]

At an omniscient state, any true proposition is known, hence also knowable.

At a non-omniscient state, nominals, though true, are not knowable.
**Theorem 3.**
At a non-omniscient state, no nominal is knowable, i.e., $u \not\models \Box K F_u$ for all non-omniscient $u$’s.

**Proof.**
Consider an arbitrary non-omniscient state $u$. Then there is a state $v$ such that $uR_K v$ and $v \neq u$. Let $F_u$ be a nominal for $u$. Then $u \not\models \Box K F_u$. Indeed, $w \not\models K F_u$ for each $w$: for $w = u$ - since $v \not\models F_u$ and $uR_K v$, for $w \neq u$ - since $w \not\models F_u$. ■
A model is omniscient if each of its states is omniscient.

A model is a *model for a principle (schema) $P$* if all instances of $P$ hold in this model.

The definable models in which verificationist knowability principles $VK$ hold are exactly the omniscient ones.
**Theorem 4.**
A definable model $\mathcal{M}$ is a model of $\text{VK}$ if and only if $\mathcal{M}$ is omniscient.

**Proof.**
Any omniscient model is a model of $\text{VK}$: for an omniscient model, $u \models F$ yields $u \models KF$. Due to reflexivity of $\square$, $u \models KF \rightarrow \Diamond KF$, hence $u \not\models \Diamond KF$.

Consider a definable model $\mathcal{M}$ with a non-omniscient state $u$. The following instance of $\text{VK}$:

$$F_u \rightarrow \Diamond KF_u$$

fails at $u$. First, $u \models F_u$, by the definition of $F_u$. Second, $u \not\models \Diamond KF_u$ by Theorem 3. ■
Theorem 4 may be regarded as a semantical version of the Church-Fitch "knowability paradox".

It shows that contrary to common opinion there is nothing wrong with the Church-Fitch proof, \( VK \) really is equivalent to \( OMN \). Strictly speaking, there is no ‘paradox’ just an unexpected result.

Note that Moore sentences play no role in this proof.

Hence the knowability paradox is not intrinsically related to the Church-Fitch proof or \( VK(Moore) \), but rather to the structure of the verificationist knowability principles \( VK \) itself.
Assuming $VK$ trivializes the epistemic picture: all states are epistemically distinguishable and everything which is true is known.

$VK$-endorsing verificationism really has no room for ignorance, or the idea of investigation or verification as a process that reduces one’s ignorance.

Such a picture of knowledge seems not to make sense of any kind of inquiry; it cannot accept a scenario where one asks whether $F$ is true or not, engages in research, and with evidence settles the question.

It seems $VK$ is not consistent with even this most schematic description of scientific inquiry. Indeed, such a scenario refutes it!
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▶ Moore, for knowability principles is like a crash test for vehicles. It is a test under disaster conditions.
▶ It does not cause the paradox, but it reveals the structural weakness in VK.
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- **K(Moore)** is inconsistent in any logic of knowledge, since it yields both \( Kp \) and \( \neg Kp \), so **K(Moore)** \( \rightarrow \perp \).
- Since \( \perp \) implies anything

\[
K(Moore) \leftrightarrow \perp.
\]

- Since in any normal modal logic

\[
\perp \leftrightarrow \diamond \perp,
\]

\[
K(Moore) \leftrightarrow \diamond K(Moore).
\]

- We see that the diamond actually disappears from \( \perp \), which is a well known phenomenon:

\[
Moore \rightarrow \diamond K(Moore),
\]

\[
Moore \rightarrow K(Moore),
\]

\[
Moore \rightarrow \perp.
\]
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Stable Knowability

- If we make the stability assumption explicit, no paradox results.
- Consider the principle:

\[ (F \rightarrow \Box F) \land F \rightarrow \lozenge KF. \]

For reflexive \( \Box \), \((F \rightarrow \Box F) \land F\) is logically equivalent to \(\Box F\), which prompts the principle of stable knowability, all stable truths are knowable:

\[ \Box F \rightarrow \lozenge KF. \quad (SK) \]
Monotonic Knowability

- A stable principle stronger than $SK$ is the principle of *monotonic knowability*:

\[ \Box F \rightarrow \Diamond \Box KF. \]  \hspace{1cm} (MK)

- $MK$ appears as a result of translating intuitionistic knowability into the classical bi-modal language, which we discuss below.
SK escapes the knowability paradox: it allows non-omniscient (and meaningful) definable models and hence does not yield the omniscience principle OMN.

Due to Theorem 4, to make this point it suffices to provide a non-omniscient frame in which SK holds in all models.
**Theorem 5.**

*Any definable model with the frame in Figure 2 is an SK-model:*

**Proof.**

Indeed, if $\Box X$ is false at each state, then $SK(X)$ is vacuously true. If $\Box X$ holds at some state, then $w \Vdash X$ as well, hence $w \Vdash KX$. Since $w$ is $\Box$-accessible from each state, $\Diamond KX$ holds at each state. ■

![Figure 2. Non-omniscient frame for SK.](image)
The above argument prompts a general characterization of definable $SK$-models: every state has a $\square$-accessible omniscient state.

**Corollary 6.**

*Schema $SK$ does not yield OMN.*

**Proof.**

By Theorem 5, all instances of $SK$ hold in $\mathcal{M}_1$, but $OMN(F)$ does not. Hence no combination of instances of $SK$ can yield omniscience with respect to $F$. □

![Figure 1. Model $\mathcal{M}_1$ where $OMN$ fails.](image-url)
A simple repetition of the Church-Fitch argument with $SK(Moore)$ only proves that $\neg \Box (p \land \neg Kp)$, i.e., $\Diamond OMN$. One can equivalently rewrite $\neg \Box (p \land \neg Kp)$ as

$$\Box p \rightarrow \Diamond Kp$$

stating informally that *if p holds at all possible states, then it is possible it becomes known at one of them.*

If $p$ holds at all possible states, then assuming knowability of $p$ is quite plausible.
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Given its intuitionistic inspiration, the natural logical perspective to have on $VK$ is intuitionistic.

Indeed in intuitionistic logic the Church-Fitch construction yields only $p \rightarrow \neg\neg Kp$ (the proof is below), which, read intuitionistically, is not absurd.

Some, most notably Dummett, argue that for the verificationist it is superior to $VK$ as a representation of their sense of knowability.
Let us read verificationist knowability as an intuitionistic principle.

By intuitionistic modal logic here we understand standard modal system with reflexive modality $\Box$ and intuitionistic logic. The ‘possibility’ modality $\Diamond$ is viewed as the dual of $\Box$. 
Theorem 7.
With intuitionistic modal logic, schemas $VK$ and $F \rightarrow \neg\neg KF$ are equivalent.

Proof.
1. $K(Moore) \rightarrow Kp$;
2. $K(Moore) \rightarrow (p \land \neg Kp) \rightarrow \neg Kp$;
3. $K(Moore) \rightarrow \bot$;
4. $\Box K(Moore) \rightarrow \Box \bot \rightarrow \bot$;
5. $\neg \Box K(Moore)$;
6. $\neg Moore$.
Since intuitionistically $\neg(X \land Y)$ yields $X \rightarrow \neg Y$, we conclude
7. $p \rightarrow \neg \neg Kp$. 

Proof Cont.
Here is the proof of the other direction.

1. \( p \rightarrow \neg
\neg Kp \);
2. \( \square
\neg Kp \rightarrow \neg Kp \) - by reflexivity of \( \square \);
3. \( \neg
\neg Kp \rightarrow \neg \square
\neg Kp \) - since \( X \rightarrow Y \) yields \( \neg Y \rightarrow \neg X \);
4. \( p \rightarrow \neg \square
\neg Kp \) - from 1 and 3, by syllogism;
5. \( p \rightarrow \Diamond Kp \) - since here \( \Diamond \) is \( \neg \square \neg \). \( \blacksquare \)
By *Intuitionistic Knowability* we mean the following principle:

\[ p \rightarrow \neg\neg Kp. \quad (IK) \]
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In the 1930s Gödel found a fair embedding of intuitionistic logic into modal logic with the S4-style modality.

One translates intuitionistic formulas by means of the rule *box every sub-formula*.

By $g(F)$ we denote the Gödel translation of formula $F$.

The Gödel translation provides a complete characterization of intuitionistic validity: a formula $F$ is intuitionistically valid if and only if its translation $g(F)$ is valid in S4.
Gödel’s motivation resulted from the provability reading of the □ modality, hence boxing a formula G forces a constructive reading of it as *G is provable* rather than classically as *G is true*.

The Kripke semantics for intuitionistic and modal logics reveals that on the semantic level, the Gödel translation specifies intuitionistic logic as a fragment of the classical modal logic S4 satisfying the stable condition: *what is true remains true*. 
Notice, the Gödel translation of IK is

\[ g(IK) = \Box(\Box p \rightarrow \Box \neg \Box \neg \Box K \Box p) = \Box(\Box p \rightarrow \Box \Diamond \Box K \Box p). \]

As a schema, \( g(IK) \) is equivalent to the principle of monotonic knowability:

\[ \Box F \rightarrow \Diamond \Box K F. \quad (MK) \]
Theorem 8.

\[ g(IK) = MK. \]

Proof.

It is an easy exercise in modal logic to show that for reflexive \( \Box \),

\[ g(IK) \rightarrow MK \]

which yields that as a schema, \( g(IK) \) implies \( MK \).

To show the converse, assume \( MK \) and consider \( MK(\Box F) \) which is

1. \( \Box \Box F \rightarrow \Diamond \Box K \Box F; \)
2. \( \Box F \rightarrow \Diamond \Box K \Box F \), from 1 by transitivity of \( \Box \);
3. \( \Box (\Box F \rightarrow \Diamond \Box K \Box F) \), from 2 by Necessitation;
4. \( \Box \Box F \rightarrow \Box \Diamond \Box K \Box F \), from 3 by distributivity;
5. \( \Box F \rightarrow \Box \Diamond \Box K \Box F \), from 4 by transitivity of \( \Box \);
6. \( \Box (\Box F \rightarrow \Box \Diamond \Box K \Box F) \), which is nothing but \( g(IK) \). □
**Theorem 9.**

*MK is strictly stronger than SK.*

**Proof.**

It is easy to see that *MK* logically implies *SK* for a reflexive $\Box$.

To show that *SK* as a schema does not yield *MK* consider model $\mathcal{M}_2$ in Figure 3.

Any instance of *SK* holds in $\mathcal{M}_2$. Indeed, if $\Box X$ holds at some node, then $w \vDash X$, hence $w \vDash KX$ and $\Diamond KX$ holds at each node.

However *MK* fails at $u$. Indeed, $u \not\vDash \Box F$. On the other hand, $u \not\vDash KF$, hence $\Box KF$ fails at both $u$ and $w$. Therefore, $u, w \not\vDash \Diamond \Box KF$. Hence $\Box F \rightarrow \Diamond \Box KF$ fails at $u$.

$\blacksquare$
MK incorporates certain specifically intuitionistic assumptions about □ and K.

Both MK and SK say that, given F is stable, there is a possible state in which verification confirms F.

g(IK)/MK also assumes that once F is known it stays known, i.e., □KF. MK reflects the assumption that in an intuitionistic universe knowledge is never lost.

SK does not make such an assumption. M₂ is not an intuitionistic universe: KF holds at w, but is not stable there, i.e., w ⊭ □KF.
VK is expressed in a classical language, but it is supposed to express a constructive idea, and it is often taken to be the ‘◇K’ which makes VK express its constructive content.

But if VK expresses a constructive idea then we need a way of marking that the truth of F is also constructive, and analogous to the Gödel embedding, that is what stability does.

SK, makes explicit the constructive meaning of VK in the classical language, without which VK says, in effect, if F is classically true, then F is (constructively) knowable.

What we observe here is the remarkably robust character of stable knowability and its ability to represent the constructive intent of verificationist knowability, and its compatibility with intuitionistic knowability.
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The history of studies of constructive, e.g., intuitionistic, logic has two distinguished traditions.

The ‘witness’, Brouwer-Heyting-Kolmogorov, semantics where \( F \) is true is understood as

\[
\text{there is a proof of } F.
\]

The Kripke semantics, according to which \( F \) is true means

\[
F \text{ holds in all epistemically possible situations.}
\]

Intuitionistic truth in the ‘universal’ setting is stable if \( F \) is true at \( u \), \( F \) stays true at all other states accessible from \( u \).
The task of reconciling these approaches was completed in the Logic of Proofs which connects the ‘existential’ and ‘universal’ intuitionistic semantics: a formula $F$ is true in the stable ‘universal’ semantics if and only if $F$ is true in the ‘existential’ semantics of proofs.

Verificationism is based on the ‘existential’ witness semantics of constructive truth. We offer a Kripke-style ‘universal’ semantics of knowability with the core stability condition leading to $SK$.

Reconciling these approaches in the context of verificationism and knowability is a natural outstanding problem.
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SK and MK do not apply to all truths, but only the stable ones. What about the idea that all truths are knowable?

Generally speaking, knowability can be thought of as a generalization of decidability. To say ‘F is knowable’ is to say that one knows of a procedure which, if carried out in an appropriate situation, would yield a decision on the truth of F.

The proper formalisation of the claim that F is knowable is the principle of total knowability:

\[ \lozenge KF \lor \lozenge K\neg F. \quad (TK) \]
TK is a meaningful principle of possible knowledge.

TK is not a ‘better’ version of VK. Indeed prominent verificationists, e.g. Dummett, explicitly reject TK.

TK represents a broadly constructive position that applies to all types of propositions, and which is distinct from VK.
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VK is strictly stronger than TK.

**Theorem 10.**
Principle TK does not yield VK.

**Proof.**
Consider again $M_1$. Since $w$ is an $R_K$ singleton for each $X$, either $w \models KX$ or $w \models K\neg X$. Since $w$ is accessible from each node, $\Diamond KX \lor \Diamond K\neg X$ holds at each node. Therefore all instances of TK hold in $M_1$.

However, $M_1 \not\models VK$. Indeed, $u \models F$ but obviously, $KF$ does not hold at any node, hence $u \not\models \Diamond KF$, and so $u \not\models VK$. □
Theorem 11. 
*Principle VK yields TK.*

Proof. 
1. $F \rightarrow \Box K F$ - an instance of VK; 
2. $\neg F \rightarrow \Box K \neg F$ - an instance of VK; 
3. $F \lor \neg F$ - propositional axiom; 
4. $\Box K F \lor \Box K \neg F$ - from 1–3, by modal reasoning. ■
**Theorem 12.**

*Principle TK yields SK.*

**Proof.**

More specifically, we establish that $TK(F)$ yields $SK(F)$.

1. $K \neg F \rightarrow \neg F$ - factivity of knowledge;
2. $F \rightarrow \neg K \neg F$ - from 1;
3. $\Box F \rightarrow \Box \neg K \neg F$ - from 2;
4. $\Box F \rightarrow \neg \Box K \neg F$ - from 3;
5. $\neg \Box K \neg F \rightarrow \Box K F$ - $TK(F)$ and $(X \lor Y) \rightarrow (\neg Y \rightarrow X)$;
6. $\Box F \rightarrow \Diamond K F$ - from 4, 5.

The latter is nothing but $SK(F)$.

|
Theorem 13.

*Principle SK does not yield TK.*

Proof.
Consider model $\mathcal{M}_3$ in Figure 4.

![Figure 4. Model $\mathcal{M}_3$ where SK holds but TK fails.](image)

It is easy to see that in $\mathcal{M}_3$ all instances of SK hold. Indeed, for any proposition $X$, if $\Box X$ holds at some node, then $X$ holds in the whole of $\mathcal{M}_3$, hence both $\text{KX}$ and $\Diamond \text{KX}$ hold in $\mathcal{M}_3$. Therefore $\Box X \rightarrow \Diamond \text{KX}$ hold in $\mathcal{M}_3$. It now suffices to show that $\text{TK}(F)$ fails in $\mathcal{M}_3$. Indeed, $u \not\models \text{KF}$ and $u \not\models \text{K}\neg F$ and hence $u \not\models \Diamond \text{KF}$ and $u \not\models \Diamond \text{K}\neg F$, so $u \not\models \text{TK}(F)$. ☐
Informally, the scenario represented by model $\mathcal{M}_3$ says that $F$ is not really known at any epistemic state. No correct verification process can produce a definitive result concerning $F$ hence $TK$ fails. However, if verification is meticulous, i.e., can observe all epistemically possible states, then verification sees that $F$ cannot stay true at any node. This makes $SK$ vacuously true at each node: unlike $TK$, $SK$ does not take any knowability obligations with respect to non-stable truths.
Theorem 14.
Principle VK yields MK.

Proof.
By Theorem 1, VK yields $F \rightarrow KF$. By necessitation and distributivity, $\Box F \rightarrow \BoxKF$. By reflexivity, $\BoxKF \rightarrow \Diamond \BoxKF$, hence $\Box F \rightarrow \Diamond \BoxKF$. $\blacksquare$
Theorem 15.
*Principle MK does not yield TK.*

**Proof.**
All instances of $MK$ hold in model $M_3$. As before, for any proposition $X$, if $\Box X$ holds at some node, then $X$ holds in the whole of $M_3$, hence, $KX$, $\Box KX$, and $\Diamond \Box KX$ hold in $M_3$. As shown in Theorem 13, $TK$ fails in $M_3$. □

*Figure 4.* Model $M_3$ where $MK$ holds but $TK$ fails.
Theorem 16.
Principle TK does not yield MK.

Proof.
Consider model $\mathcal{M}_2$ in Figure 3. According to Theorem 9, $MK$ fails in $\mathcal{M}_2$. On the other hand, each instance of $TK$ holds in $\mathcal{M}_2$. Indeed, $w$ is $\Box$-accessible from each node and $w$ is omniscient. ■
Figure 5 provides the diagram of relationships between the knowability principles.

```
MK \approx IK
```

- **OMN** represents omniscient models only.
- **VK** represents non-omniscient models.
- **MK \approx IK** indicates that models that are not omniscient.

The diagram is labeled as the "Knowability Diamond".
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  Stable Knowability
  Intuitionistic Knowability
  Knowability and the Gödel Embedding
  Stable Knowability and Constructive Semantics

Knowability Framework
  Total Knowability
  Knowability Diamond

Conclusion
Van Benthem distinguishes two basic approaches to avoiding the knowability paradox: weakening the logic of the Church-Fitch proof (turning down the radio so as not to hear the bad news), or weakening $VK$ itself (censoring the news, you hear it fine but its not so interesting). He argues that “what one really wants is a new systematic viewpoint” from which to approach the paradox.

We argue that we achieve this. Yes, we weaken $VK$, but what we leave out was not news (like turning off Fox News, there is no loss in truthful information).
We give an alternative proof that $VK$ yields the unacceptable $OMN$, which shows the problem lies with $VK$.

We show the legitimate scope of $VK$ is stable instances only.

With stability made explicit the knowability paradox disappears.

Intuitionistic knowability yields stability requirement.

There is no need to adopt a non-classical logic, or give up any epistemic principle.
We also offer three options for the verificationist:

1. Stable knowability $SK$. This is a conservative approach which attempts to preserve the format of $VK$ by limiting it to its epistemically justified stable version $SK$.

2. Monotonic knowability $MK$, which reflects specifically intuitionistic reading of knowability. $MK$ is stronger than $SK$, but more restrictive; it stipulates that once a proposition becomes known it stays known at all further steps.

3. Knowability in a general setting reflected by total knowability $TK$. This approach attempts to preserve the idea that all truths are knowable.

None of them yield $OMN$. 
The framework also opens the door to the study of knowability from a logical point of view.

Such a framework allows us to begin systematically studying a concept which pervades epistemology.

The possibility or impossibility of knowing is a central topic in discussions of skepticism, *a priori* knowledge, verificationism/constructivism.
Thank you!