Propositional Logic: exercises

1. Prove that $p \wedge \neg p$ is unsatisfiable
2. Prove that $p \vee \neg p$ is a tautology
3. Write the truth table of the following two formula $(p \wedge \neg(q \vee r))$ and $(\neg p \vee (q \vee r))$. Say for each one if it is a tautology, satisfiable or contradiction. Say if one is a logical consequence of the other
4. Let F and G be two formula. Is it true that $F \vee G$ is a tautology iff one of them is a tautology?
5. Find three formula $F1$, $F2$, and $F3$ such that $F1 \wedge F2 \wedge F3$ is unsatisfiable and such that the conjunction of any pair of them is satisfiable
6. Prove that $(\neg q \vee \neg r \vee p) \wedge (p \vee q) \wedge (r \vee p)$ and $p \vee q$ are not logically equivalent
7. Prove that $(p \vee q) \wedge (r \vee p) \wedge (\neg q \vee \neg r \vee p) \equiv p$
8. if Si $F \to G$ is a tautology and F is a tautology, then G is a tautology?
9. Let F and G be two formula. Is it true that $F \models G$ or $F \models \neg G$?
10. Are there any F and G such that $F \models G$ and $F \models \neg G$?
11. Write a satisfying interpretation and a non-satisfying interpretation for the following formula,
$$(p \wedge (q \vee r) \wedge \neg(p \wedge q) \wedge \neg(p \wedge r)) \vee (p \wedge \neg r)$$
12. Consider the following formula,
$$F = (p \to q) \vee (q \to r)$$
$$G = (p \vee q \vee r) \wedge (\neg p \to q) \wedge (\neg r \to p)$$
For each pair of formulas in the set $\{F, G, \neg F, \neg G\}$ indicate whether one is a logical consequence of the other.
13. Prove using truth tables that $p \to (q \wedge r) \models (\neg q \vee \neg r) \to \neg p$
14. Let $F = \neg(\neg p \wedge \neg q \wedge \neg r) \wedge (\neg p \vee q \vee r)$ and $G = q \vee r$. Say which one of the following properties hold: $F \models G$, $G \models F$, $F \equiv G$, $F \wedge \neg G$ is a tautology, $F \wedge \neg G$ is a contradiction, $\neg F \vee G$ is a tautology, $\neg F \vee G$ is a contradiction.
15. Is the following statement true? (provide a justification for your answer). If $F \to G$ is satisfiable and F is satisfiable, then G is satisfiable.
16. Say wether the following statements are true or not:
    - For every formula F there is a logically equivalent formula G in CNF
    - For every formula F there is a unique logically equivalent formula G in CNF
    - If F is a tautology, there is a unique equivalent formula in CNF.
    - If F is a contradiction, there is a unique equivalent formula in CNF.
17. Transform the following sentence into CNF using two methods: *i*) via distributivity, *ii*) via Tseitin.
$$F = \neg p \wedge q \vee \neg(\neg r \vee \neg q)$$

18. A boolean formula is in disjunctive normal form (DNF) if it is a disjunction of one or more conjunctions of one or more literals (e.g. $(x \wedge \neg y \wedge z) \vee (\neg x \wedge z) \vee (w \wedge y \wedge \neg z)$). Contradictory conjunctions (i.e, containing a literal and its negation $x_i \wedge \neg x_i$,) are not permitted. Say wether the following statements are true or not:

    - For every formula F there is a logically equivalent formula G in DNF
    - For every formula F there is a unique logically equivalent formula G in DNF
      Solution: False. $p \vee q \equiv (p \wedge \neg r) \vee (q \wedge \neg r) \vee (p \wedge r) \vee (q \wedge r)$
    - If F is a tautology, there is a unique equivalent formula in DNF.
    - If F is a contradiction, there is a unique equivalent formula in DNF.

19. Show that there is a polynomial algorithm that says if a DNF formula is satisfiable or not.

    Solution: To satisfy a DNF formula, all we need is to satisfy one conjunction of literals and the disjunction is satisied. Traverse the firs conjunction and assign 0 to all variables that appear with $\neg$ sign, assign 1 to all variables without $\neg$ sign, assign arbitrary values to all variables that do not appear in the clause at all.

20. Show how to transform a CNF formula into an equivalent DNF formula using the distributive law. What is the size of the resulting DNF formula ?

    Solution:

    - $(x1 \vee x2 \vee x3) \Longrightarrow (x1) \vee (x2) \vee (x3)$
    - $(x1 \vee x2 \vee x3) \wedge (x4 \vee x5 \vee x6) \Longrightarrow (x1 \wedge x4) \vee (x2 \wedge x4) \vee (x3 \wedge x4)_(x1 \wedge x5) \vee (x2 \wedge x5) \vee (x3 \wedge x5)_(x1 \wedge x6) \vee (x2 \wedge x6) \vee (x3 \wedge x6)$
    - $(x1 \vee x2 \vee x3) \wedge (x4 \vee x5 \vee x6) \wedge (x7 \vee x8 \vee x9) \Longrightarrow (x1 \wedge x4 \wedge x7) \vee (x1 \wedge x4 \wedge x8) \vee \ldots \vee (x3 \wedge x6 \wedge x9)$

    A CNF formula with m size-3 clauses is converted into a DNF formula with 3^m disjunctions

21. Prove using resolution that $p \rightarrow (q \wedge r) \models (\neg q \vee \neg r) \rightarrow \neg p$

22. Use resolution to prove that $p \rightarrow q$ is a logical consequence of $((t \rightarrow q) \wedge (\neg r \rightarrow \neg s) \wedge (p \rightarrow u) \wedge (\neg t \rightarrow \neg r) \wedge (u \rightarrow s))$

23. Prove with resolution that the following formulas are tautologies:

    - $p \rightarrow (q \rightarrow p)$
    - $(p \wedge (p \rightarrow q)) \rightarrow q$
    - $((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$
    - $((p \rightarrow q) \wedge \neg q) \rightarrow \neg q$

24. A unary clause is a clause containing only one literal (i.e. p or $\neg p$). Unit resolution is resolution restricted to pairs of clauses in which at least one of them is a unary clause. Is unit resolution correct? is refutationally complete?

25. A binary clause is a clause with no more than two literals. Binary resolution is resolution restricted to pairs of binary clauses (i.e. from $p \vee q$ and $\neg p \vee r$ we obtain $q \vee r$).

    - Is binary resolution correct? and complete? and refutationally complete?
      Solución: Es correcta porque es una caso particular de resolución, que vimos en clase que era correcta. No es completa exactamente por el mismo motivo. No es refutacionalmente correcta porque se puede construir una formula insatisfactible cuyas clausulas sean todas ternarias. La resolución binaria no podría deducir nada en este caso.

- A binary CNF formula is a CNF formula containing only binary clauses. If we apply binary resolution only to binary CNF formulas, is it a correct inference rule? is it refutationally complete?

  Solución: Es correcta por el mismo motivo que antes. Ahora si es refutacionalmente correcta. Para verlo hay que darse cuenta de que resolucion binaria nunca genera cláusulas de tamaño mayor de dos. Por lo tanto si aplicamos resolución a una formula binaria, siempre estaremos aplicando resolucion binaria.

26. Let $(x_1, x_2, ..., x_n)$ be an ordered list of propositional variables. Write Formulas (and tell their size) with the following meanings:

    - Two consecutive variables cannot take the same value.
    - $k > 1$ consecutive variables cannot take the same value.
    - No more than one variable takes value 1 (i.e. $\sum_i x_i \leq 1$)
    - No more than two variables take value 1 (i.e. $\sum_i x_i \leq 2$)
    - No more than k variables take value 1 (i.e. $\sum_i x_i \leq k$)
    - At least one variable takes value 1 (i.e. $\sum_i x_i > 0$)
    - More than one variable takes value 1 (i.e. $\sum_i x_i > 1$)
    - More than k variables take value 1 (i.e. $\sum_i x_i > k$)
    - k variables take value 1 (i.e. $\sum_i x_i = k$)
    - No zero occurs after a one.
    - No zero occurs after k consecutive ones.

27. The K-coloring problem: given a graph G and a natural number K, assign to each vertex of G a natural between 1 and K such that adjacent vertices do not have the same number. Express the problem in CNF.

28. There are N towns each one having a local radio station. We have to assing a radio frequency out of Q available ones to each radio station. To avoid interferences, towns closer than 20Km can not use the same frequency. We have a function Distance(i,j) indicating the distance between town i and j. Is it possible to assign the frequencies? Express the problem in CNF.

29. A judge must form a jury of k people. No one in the jury should know any other member. There are n candidates and Knows(i,j) is a boolean predicate telling whether i knows j. Express the problem in CNF

30. Let $G = (V, E)$ be an undirected graph. We would like to know if G is connected. Model this problem with propositional logic. Hint: a graph is connected iff for all non-empty $A \subset V$ there are edges between A and $V - A$.

31. Let $G = (V, E)$ be an undirected graph. A matching M in G is a set of edges $M \subseteq E$ such that no two edges share a common vertex. A perfect matching is a matching such that every vertex of the graph is incident to exactly one edge of the matching. We would like to know if G contains a perfect matching. Model this problem with propositional logic. Disclaimer: there are much better ways to solve this problem than using propositional logic

32. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set and $B_1, B_2, \ldots B_m$ subsets of A. Let k be a natural number. We want to find out if there exists a subset $H \subset A$ of size k such that contains at least one element from each subset B_i. Model this problem as a CNF formula.

    Solución:

    - Variables: Para $1 \leq i \leq n$ y $1 \leq j \leq k$ sea x_{ij} una variable que me dice si el objeto a_i es el j-esimo elemento de H.
    - Para cada j, exactamente un elemento es el j-esimo

  - Para $1 \le j \le k$, ponemos una cláusula $x_{1j} \vee x_{2j} \vee ... \vee x_{nj}$
  - Para $1 \le j \le k$, Para $1 \le i < i' \le n$, ponemos una cláusula $\neg x_{ij} \vee \neg x_{i'j}$
- Al menos un elemento de cada B_r es escogido en alguna posicion,
  - Para $1 \le r \le m$, sea $B_r = \{a_{r_1}, a_{r_2}, ..., a_{r_{p_k}}\}$, ponemos una cláusula $(x_{r_1 1} \vee x_{r_2 1} \vee ... \vee x_{r_{p_k} 1}) \vee (x_{r_1 2} \vee x_{r_2 2} \vee ... \vee x_{r_{p_k} 2}) \vee \ldots \vee (x_{r_1 k} \vee x_{r_2 k} \vee ... \vee x_{r_{p_k} k})$

33. Three students A, B and C are accused of introducing a virus in the school lab. During the interrogation they make the following claims:

    - A says: "B did it and C is innocent".
    - B says: "If A is guilty then C is guilty too".
    - C says: "I did not do it. One of the others, or maybe both of them did it".

    Solution: To answer the following questions it is convenient to write the truth table of the following formula

    $$B \wedge \neg C \wedge (\neg A \vee C) \wedge \neg C \wedge (A \vee B)$$

    where A, B, C means being guilty. Note that the formula represents the conjunction of the three claims.

    (a) Are the three statements contradictory?

    Solution: No, because the formula is satisfiable (e.g. $\neg A \wedge B \wedge \neg C$ is a model), meaning that the three claims can be simultaneously satisfied.

    (b) Assuming that all of them are guilty, who lied during the interrogation?

    Solution: From the corresponding entry of the truth table, one can see that A and C lied (e.g. their claims are not satisfied)

    (c) Assuming that nobody lied, who is innocent and who is guilty?

    Solution: B is guilty, the other two are innocent. This is the only model for the three claims.

34. Consider a chess-board (the size is 8 by 8), and 8 queens. Is it possible to place the queens in the board in such a way that they do not attack each other?. Recall that, according to chess rules, queens can move any number of cells along rows, columns or diagonals.

    Write a propositional logic formula such that from its models, we can easily compute solutions to the previous problem.

    Solution: I use variables x_{ij} to indicate that there is a queen in cell (i, j)

    I need the following constraints (indices are bounded as appropriate):

    - There is exactly one queen in each row

    $$\forall i, \quad x_{i1} \vee ... \vee x_{i8}$$

    $$\forall i, j \neq j', \quad \neg x_{ij} \vee \neg x_{ij'}$$

    - There is exactly one queen in each column

    $$\forall j, \quad x_{1j} \vee ... \vee x_{8j}$$

    $$\forall i \neq i', j, \quad \neg x_{ij} \vee \neg x_{i'j}$$

    - For every pair of cells in the same diagonal, no two queens can be placed there $\forall i, j, i', j'$ such that $(i, j) \neq (i', j')$ and $|i - i'| = |j - j'|$

    $$\neg x_{ij} \vee \neg x_{i'j'}$$

35. Supose that you have a SAT-solver, such as Minisat, available (what you have available is the executable, not the source code). What would you do if you want to know the number of models of a CNF formula F? (This problem is usually called *Model Counting* or #SAT)

    Solution: The idea is to execute the SAT-solver sequentially. At each iteration, if the SAT-solver returns a solution we add to the formula a new clause forbidding that solution. We repeat this process until the formula becomes unsatisfiable. The number of iterations will be the number of solutions of the original formula. In algorithmic form, Assuming $SAT - Solver(F)$ returns one model of F,

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c:=0
while SAT-Solver(F)<> Nil do
  (l1, l2,...,ln):=SAT-Solver(F)
  F:= F and (no l1 or no l2 or ... or no ln)
  c++
endwhile
```

36. Supose that you have a SAT-solver, such as Minisat, available (what you have available is the executable, not the source code). Supose as well that you have an unsatifiable CNF formula F. What would you do if you want to know if it is possible to satisfy at least k clauses? and if you want to know the maximum number of clauses that can be satisfied? (this is the so-called MAX-SAT problem)

37. Let $G = (V, E)$ be an undirected graph. A *cut* of size 1 exists if there is a $A \subset V$ such that exactly one edge goes between A and $V - A$. We would like to know if G contains a cut of size larger than k (this is the so-called Max-CUT problem)

38. Let $G = (V, E)$ be an undirected graph. A *clique* of size k exists if there is a $A \subset V$ such that $|A| = k$ and all the vertices in A are pairwise connected by an edge. We would like to know if G contains a clique of size larger than or equal to k (this is the so-called Max-CLIQUE problem)