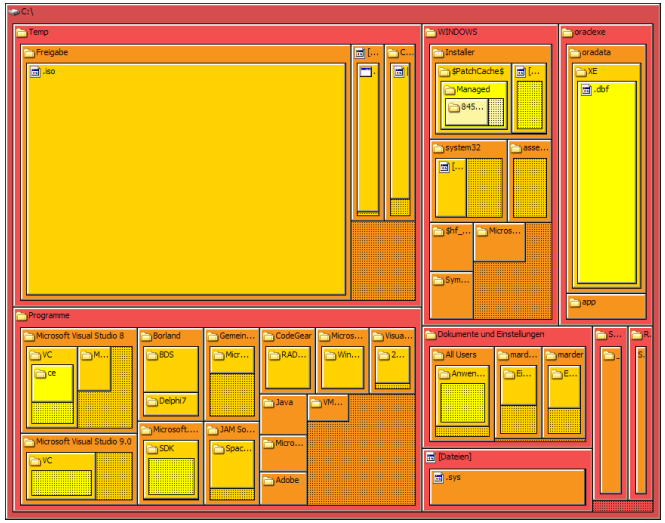
Trees

Jordi Cortadella and Jordi Petit
Department of Computer Science

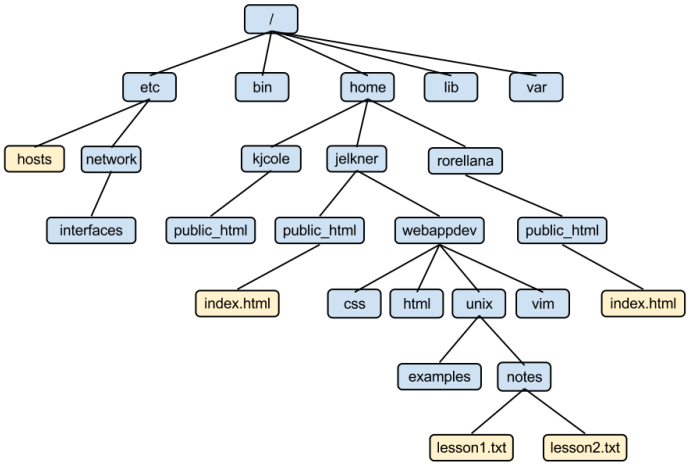
Trees

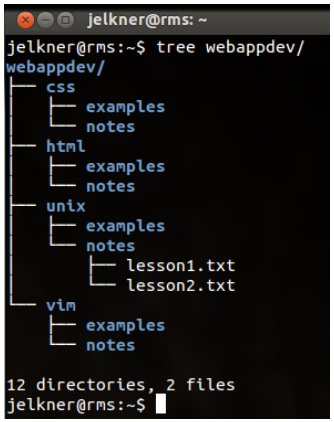
Data are often organized hierarchically



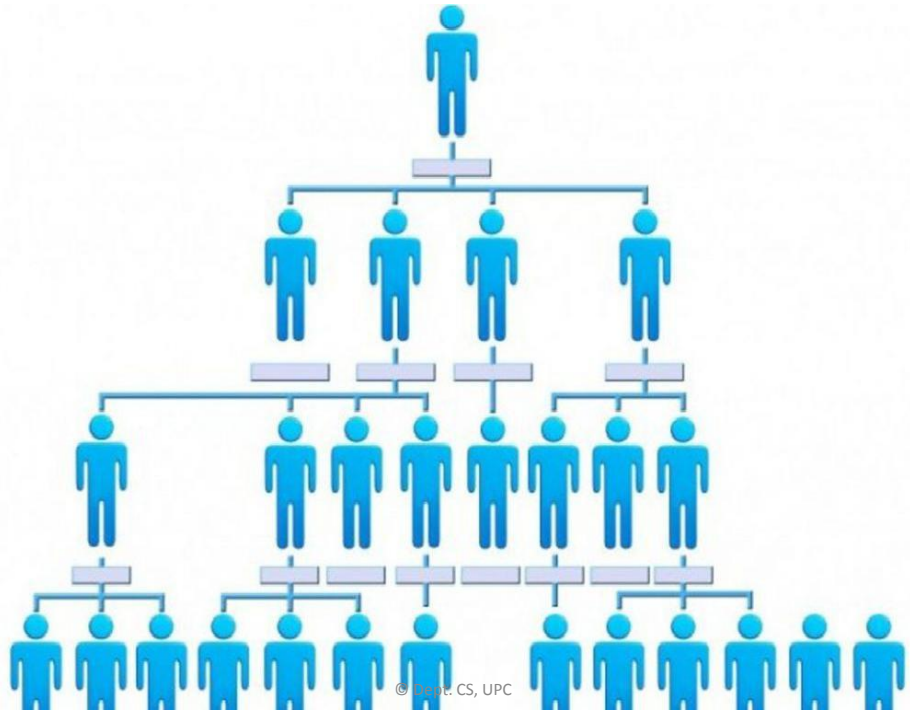
source: https://en.wikipedia.org/wiki/Tree_structure

Filesystems

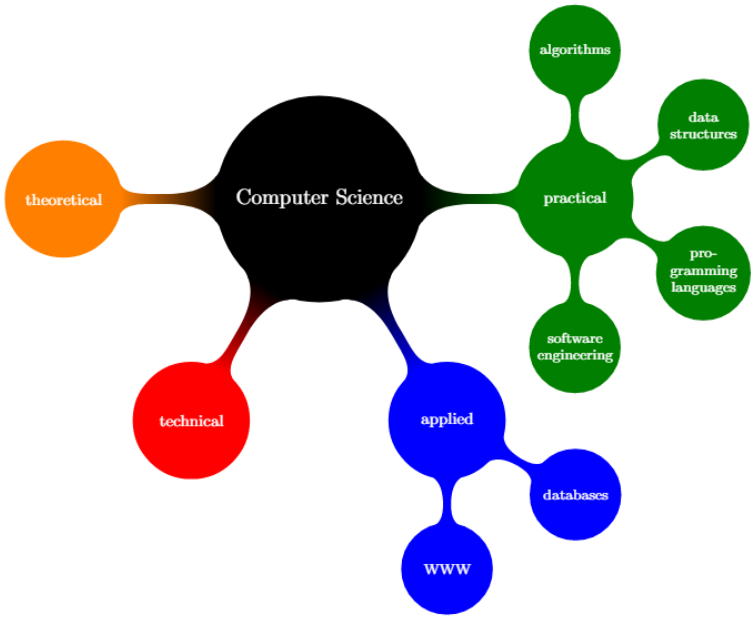




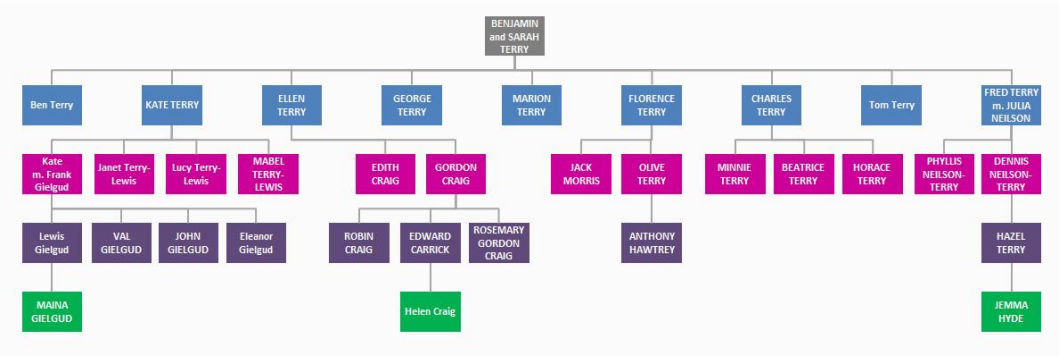
Company structure

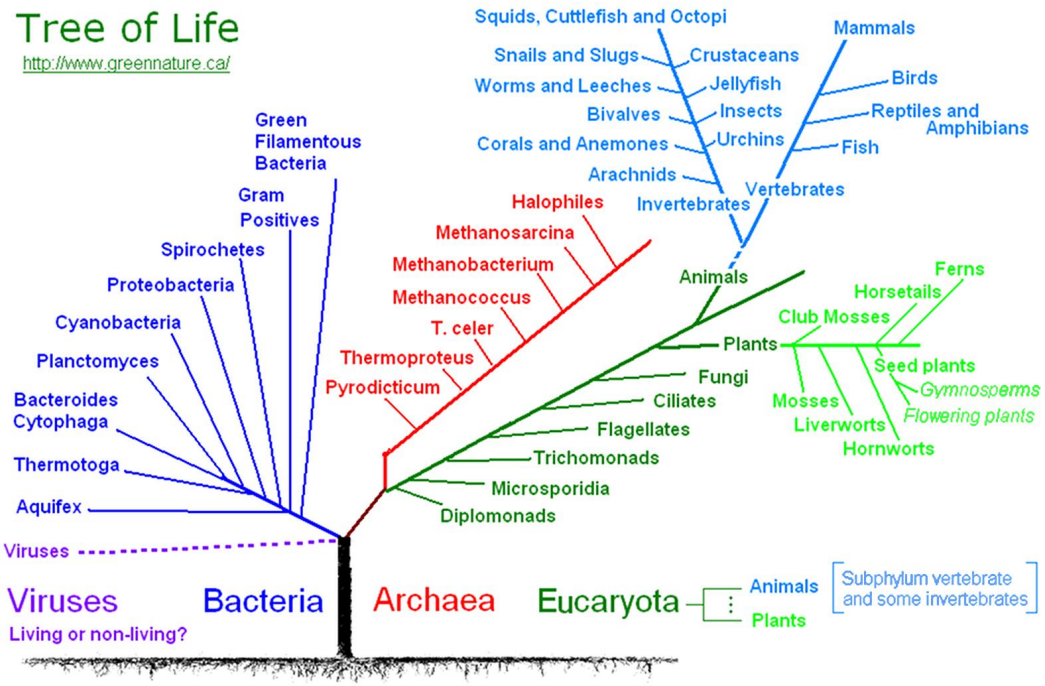


Mind maps

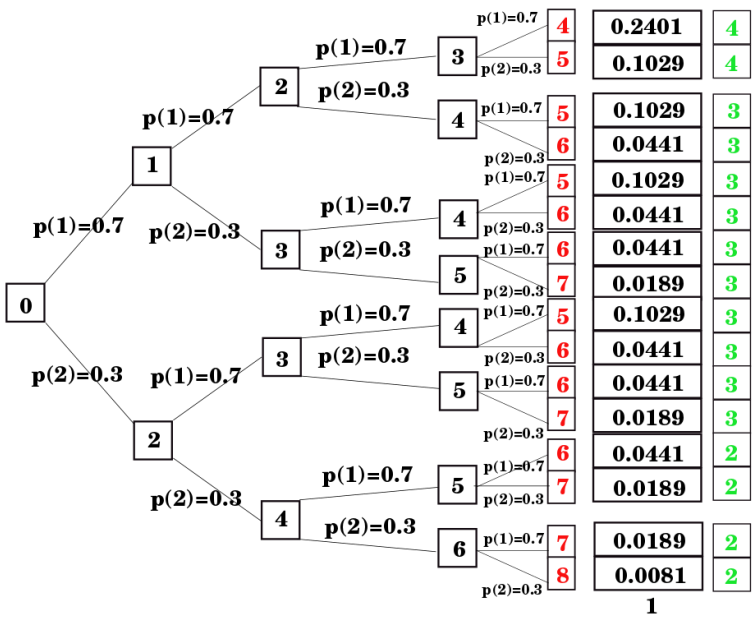


Genealogical trees





Probability trees



Parse trees

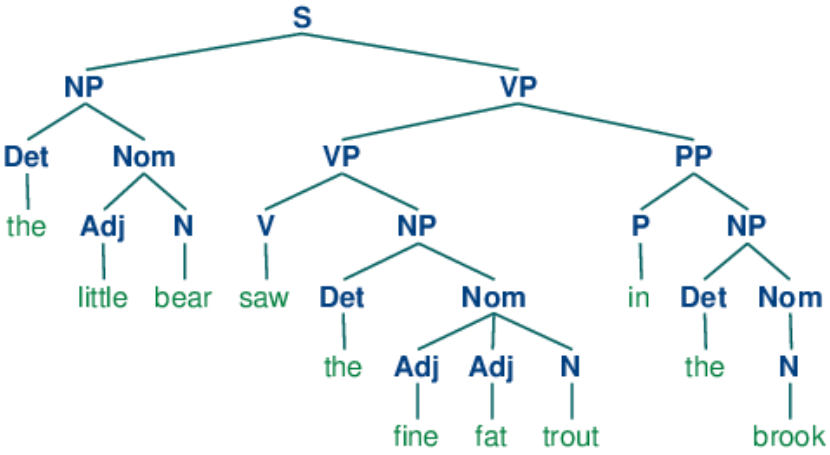
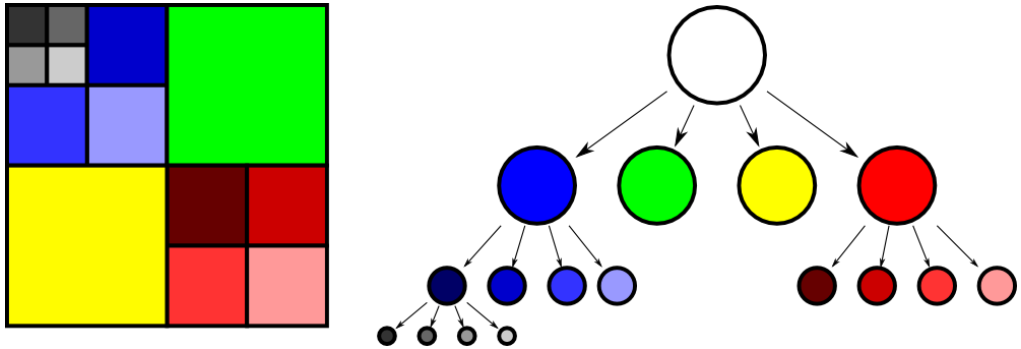
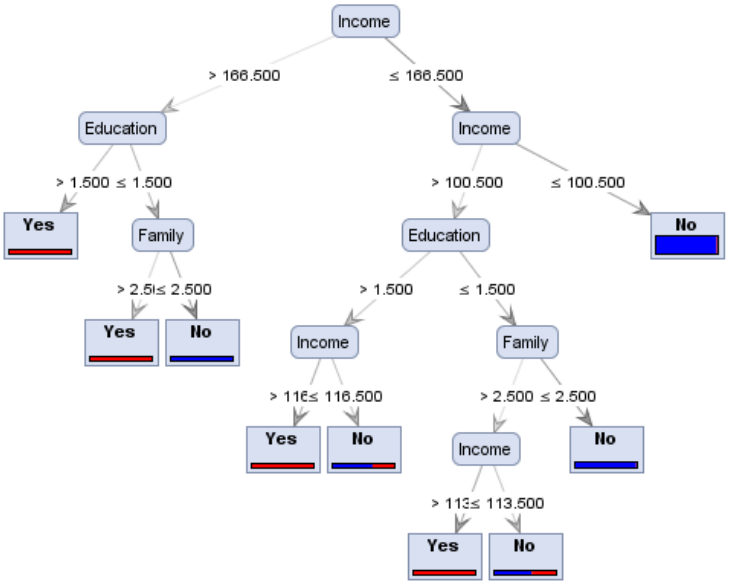


Image representation (quad-trees)



Decision trees

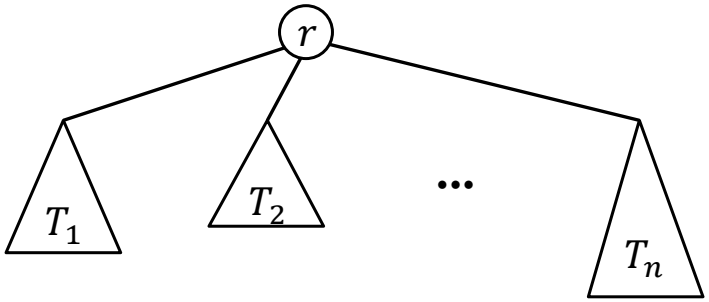


source: http://www.simafore.com/blog/bid/94454/A-simple-explanation-of-how-entropy-fuels-a-decision-tree-model

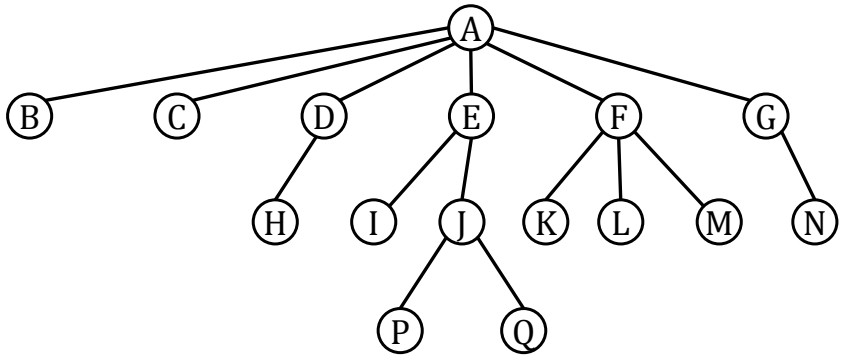
Tree: definition

- Graph theory: a tree is an undirected graph in which any two vertices are connected by exactly one path.

- Recursive definition (CS). A non-empty tree T consists of:
  - a root node r
  - a list of trees $T_1, T_2, \dots, T_n$ that hierarchically depend on r.

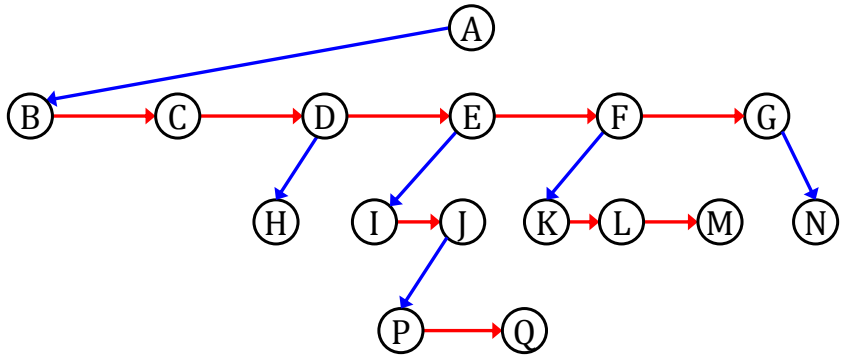


Tree: nomenclature



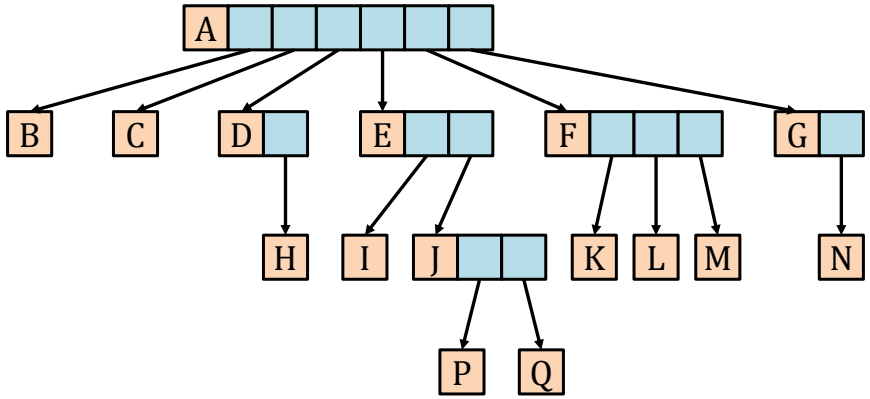
- A is the **root** node.
- Nodes with no children are **leaves** (e.g., B and P).
- Nodes with the same parent are **siblings** (e.g., K, L and M).
- The **depth** of a node is the length of the path from the root to the node. Examples: depth(A)=0, depth(L)=2, depth(Q)=3.

Tree: representation with linked lists



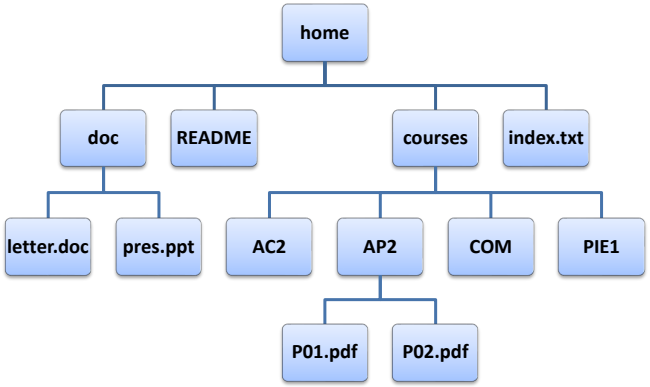

```
struct TreeNode {
  Type element;
  list<TreeNode> children; // Linked list of children
};
```


Tree: representation with vectors




```
struct TreeNode {
  Type element;
  vector<TreeNode> children; // Vector of children
};
```


Print a tree




```
home
  doc
    letter.doc
    pres.ppt
  README
  courses
    AC2
    AP2
      P01.pdf
      P02.pdf
    COM
    PIE1
  index.txt
```

```
struct Tree {
  string name;
  vector<Tree> children;
};
```

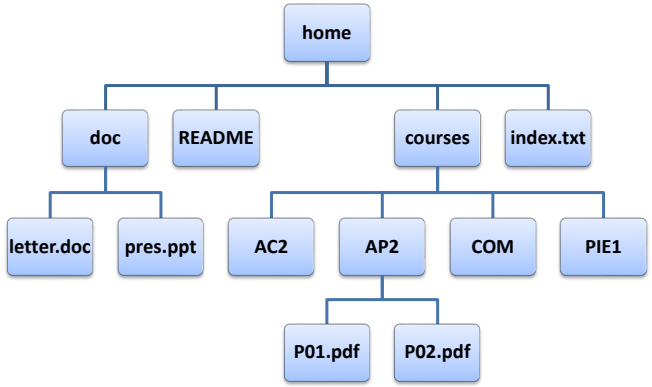
```
print(const Tree& T, int depth=0);
```

Print a tree

```
/** Prints a tree indented according to depth.
  * Pre: The tree is not empty. */
print(const Tree& T, int depth) {

  // Print the root indented by 2*depth
  cout << string(2*depth, ' ') << T.name << endl;

  // Print the children with depth + 1
  for (const Tree& child: T.children)
      print(child, depth + 1);
}
```

This function executes a ***preorder*** traversal of the tree: each node is processed *before* the children.

Print a tree (postorder traversal)




```
    letter.doc
    pres.ppt
  doc
  README
    AC2
      P01.pdf
      P02.pdf
    AP2
    COM
    PIE1
  courses
  index.txt
home
```

Postorder traversal: each node is processed after the children.

Print a tree (postorder traversal)

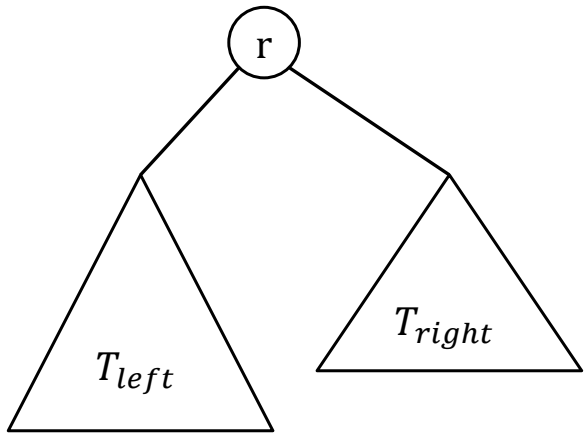
```
/** Prints a tree (in postorder) indented according to depth.
  * Pre: The tree is not empty. */
printPostOrder(const Tree& T, int depth) {

  // Print the children with depth + 1
  for (const Tree& child: T.children)
      printPostOrder(child, depth + 1);

  // Print the root indented by 2*depth
  cout << string(2*depth, ' ') << T.name << endl;
}
```

This function executes a ***postorder*** traversal of the tree: each node is processed *after* the children.

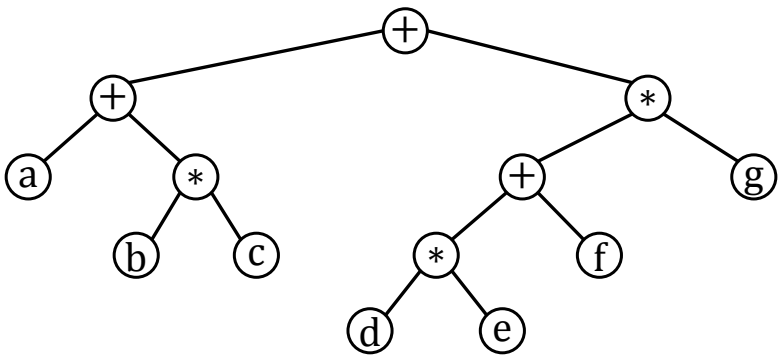
Binary trees

Nodes with at most two children.

```
struct BinTree {
  Type element;
  BinTree* left;
  BinTree* right;
};
```



Example: expression trees



Expression tree for: **a + b∗c + (d∗e + f) ∗ g**

Postfix representation: **a b c ∗ + d e ∗ f + g ∗ +**

How can the postfix representation be obtained?

Example: expression trees

```
struct ExprTree {
  char op; // operand or operator
  ExprTree* left;
  ExprTree* right;
};

using Expr = ExprTree*;
```

Expressions are represented by strings in postfix notation in which the characters ‘a’…‘z’ represent operands and the characters ‘+’ and ‘∗’ represent operators.

```
/** Builds an expression tree from a correct
  * expression represented in postfix notation. */
Expr buildExpr(const string& expr);

/** Generates a string with the expression in
  * infix notation. */
string infixExpr(const Expr T);

/** Evaluates an expression taking V as the value of the
  * variables (e.g., V[‘a’] contains the value of a). */
int evalExpr(const Expr T, const map<char,int>& V);
```


Example: expression trees

```
Expr buildExpr(const string& expr) {
  stack<Expr> S;
  // Visit the chars of the string sequentially
  for (char c: expr) {
    if (c >= ‘a’ and c <= ‘z’) {
      // We have an operand in {‘a’…‘z’}. Create a leaf node.
      S.push(new ExprTree{c, nullptr, nullptr});
    } else {
      // c is an operator (‘+’ or ‘*’)
      Expr right = S.top();
      S.pop();
      Expr left = S.top();
      S.pop();
      S.push(new ExprTree{c, left, right});
    }
  }
  // The stack has only one element and is freed after return
  return S.top();
}
```


Example: expression trees

```
/** Returns a string with an infix representation of T. */
string infixExpr(const Expr T) {

  // Let us first check the base case (an operand)
  if (T->left == nullptr) return string(1, T->op);

  // We have an operator. Return ( T->left T->op T->right )
  return “(“ +
             infixExpr(T->left) +
             T->op +
             infixExpr(T->right) +
         “)”;
}
```

Inorder traversal: node is visited *between* the left and right children.

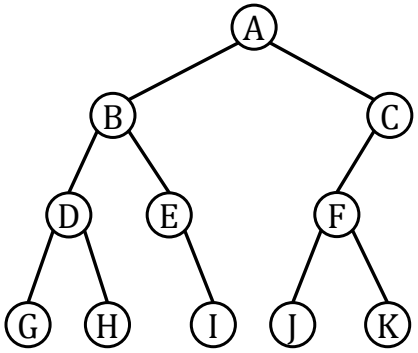
Example: expression trees

```
/** Evaluates an expression taking V as the value of the
  * variables (e.g., V[‘a’] contains the value of a). */
int evalExpr(const Expr T, const map<char,int>& V) {
  if (T->left == nullptr) return V.at(T->op);
  int l = evalExpr(T->left, V);
  int r = evalExpr(T->right, V);
  return T->op == ‘+’ ? l+r : l*r;
}

/** Example of usage of ExprTree. */
int main() {
  Expr T = buildExpr(“abc*+de*f+g*+”);
  cout << infixExpr(T) << endl;
  cout << “Eval = ”
       << evalExpr(T, {{‘a’,3}, {‘b’,1}, {‘c’,0}, {‘d’,5},
                       {‘e’,2}, {‘f’,1}, {‘g’,6}})
       << endl;
  freeExpr(T); // Not implemented yet
}
```


Exercises

- Design the function `freeExpr`.
- Modify `infixExpr` for a nicer printing:
  - Minimize number of parenthesis.
  - Add spaces around + (but not around ∗).
- Extend the functions to support other operands, including the unary – (e.g., –a/b).

Tree traversals



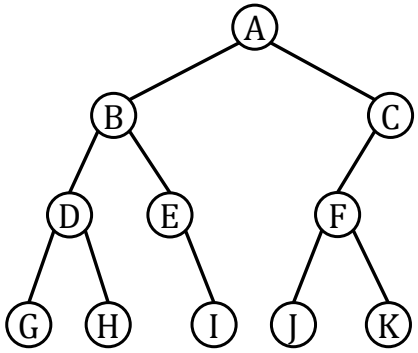
Traversal: algorithm to visit the nodes of a tree in some specific order.

The actions performed when visiting each node can be a parameter of the traversal algorithm.

```
struct TreeNode {
  Tinfo info;
  TreeNode* left;
  TreeNode* right;
};

using Tree = TreeNode*;
```

```
using visitor = void (int &);

// This function matches the type visitor
void print(int& i) {
  cout << i << endl;
}

void traversal(Tree T, visitor v);
```

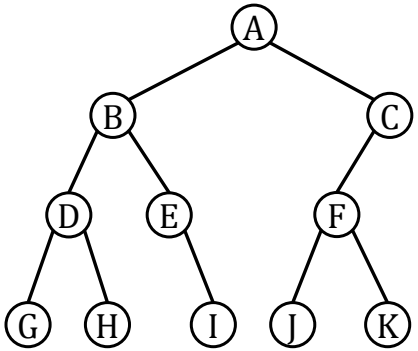

Tree traversals



Preorder: A B D G H E I C F J K

```
void preorder(Tree T, visitor v) {
  if (T != nullptr) {
    v(T->elem);
    preorder(T->left, v);
    preorder(T->right, v);
  }
}
```

Tree traversals

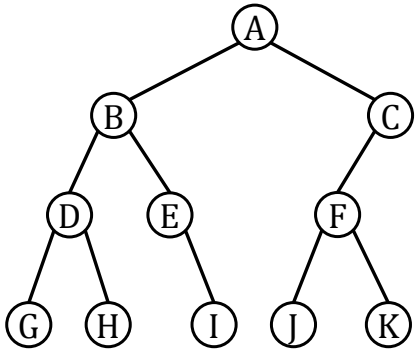


Preorder: A B D G H E I C F J K

Postorder: G H D I E B J K F C A

```
void postorder(Tree T, visitor v) {
  if (T != nullptr) {
    postorder(T->left, v);
    postorder(T->right, v);
    v(T->elem);
  }
}
```


Tree traversals



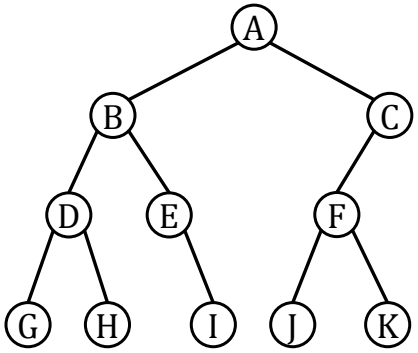
Preorder: A B D G H E I C F J K

Postorder: G H D I E B J K F C A

Inorder: G D H B E I A J F K C

```
void inorder(Tree T, visitor v) {
  if (T != nullptr) {
    inorder(T->left, v);
    v(T->elem);
    inorder(T->right, v);
  }
}
```


Tree traversals



Preorder: A B D G H E I C F J K

Postorder: G H D I E B J K F C A

Inorder: G D H B E I A J F K C

By levels: A B C D E F G H I J K

```
void byLevels(Tree T, visitor v) {
  queue<Tree> Q; Q.push(T);
  while (not Q.empty()) {
    T = Q.front(); Q.pop();
    if (T != nullptr) {
      v(T->elem);
      Q.push(T->left); Q.push(T->right);
    }
  }
}
```


EXERCISES

Traversals: Full Binary Trees

- A Full Binary Tree is a binary tree where each node has 0 or 2 children.
- Draw the full binary trees corresponding to the following tree traversals:
  - Preorder: 2 7 3 6 1 4 5; Postorder: 3 6 7 4 5 1 2
  - Preorder: 3 1 7 4 9 5 2 6 8; Postorder: 1 9 5 4 6 8 2 7 3
- Given the pre- and post-order traversals of a binary tree (not necessarily full), can we uniquely determine the tree?
  - If yes, prove it.
  - If not, show a counterexample.

Traversals: Binary Trees

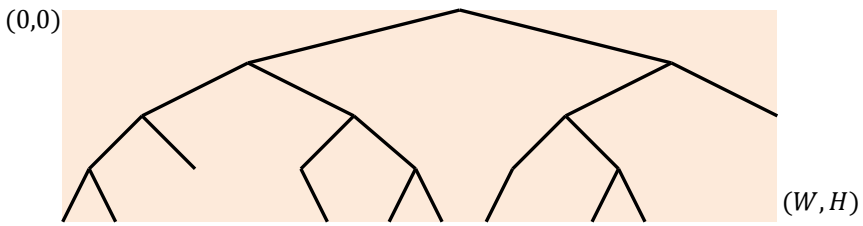
- Draw the binary trees corresponding the following traversals:
  - Preorder: 3 6 1 8 5 2 4 7 9; Inorder: 1 6 3 5 2 8 7 4 9
  - Level-order: 4 8 3 1 2 7 5 6 9; Inorder: 1 8 5 2 4 6 7 9 3
  - Postorder: 4 3 2 5 9 6 8 7 1; Inorder: 4 3 9 2 5 1 7 8 6
- Describe an algorithm that builds a binary tree from the preorder and inorder traversals.

Drawing binary trees

We want to draw the skeleton of a binary tree as it is shown in the figure. For that, we need to assign (x, y) coordinates to each tree node. The layout must fit in a pre-defined bounding box of size $W \times H$, with the origin located in the top-left corner.

Design the function

```
void draw(Tree T, double W, double H)
```

to assign values to the attributes `x` and `y` of all nodes of the tree in such a way that the lines that connect the nodes do not cross.

Suggestion: calculate the coordinates in two steps. First assign (x, y) coordinates using some arbitrary unit. Next, shift/scale the coordinates to exactly fit in the bounding box.