Trees

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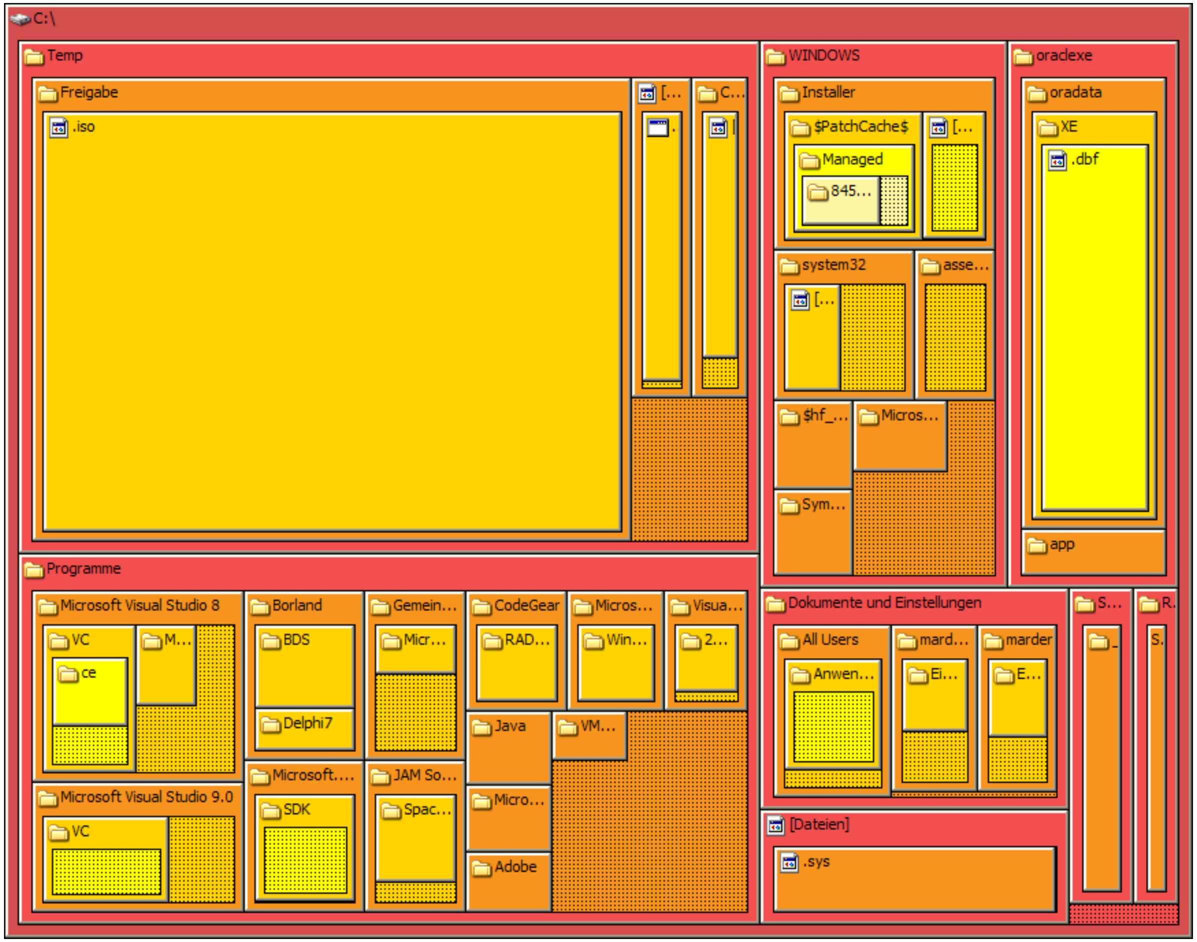
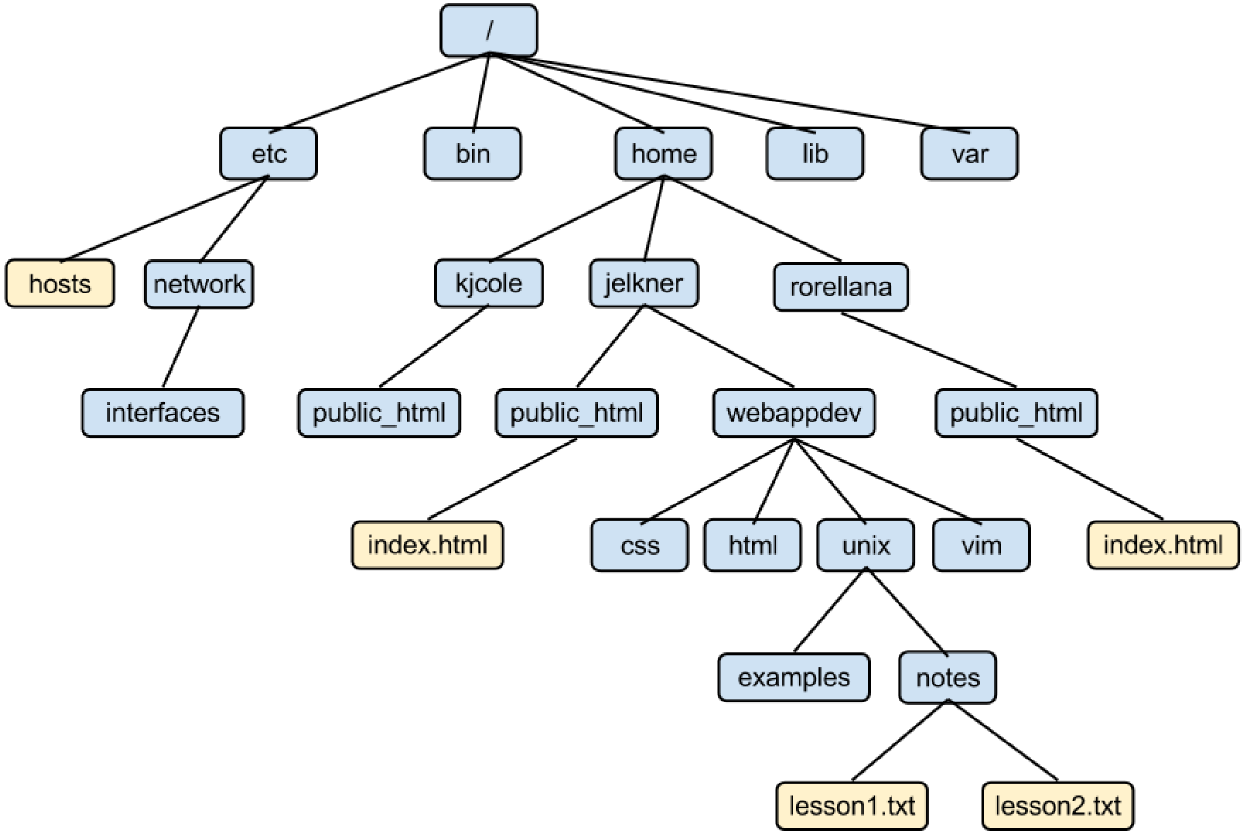
Trees

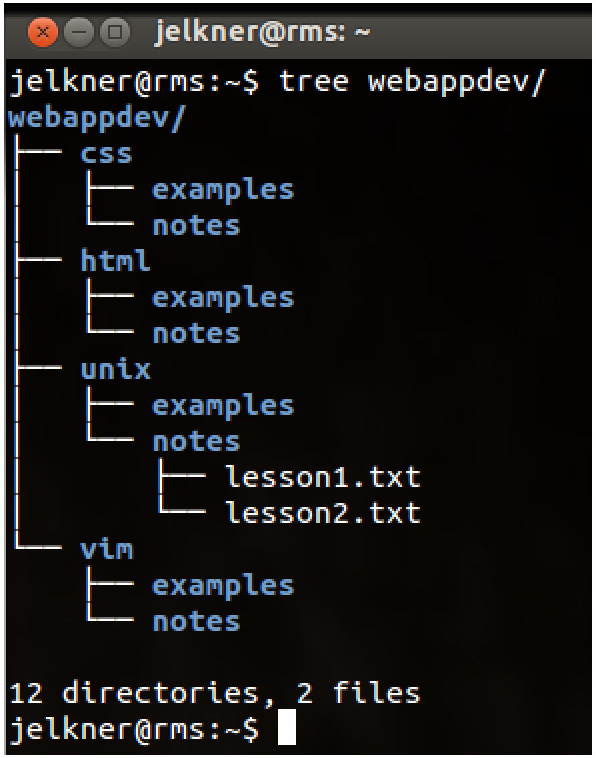
Data are often organized hierarchically



source: https://en.wikipedia.org/wiki/Tree_structure

Filesystems

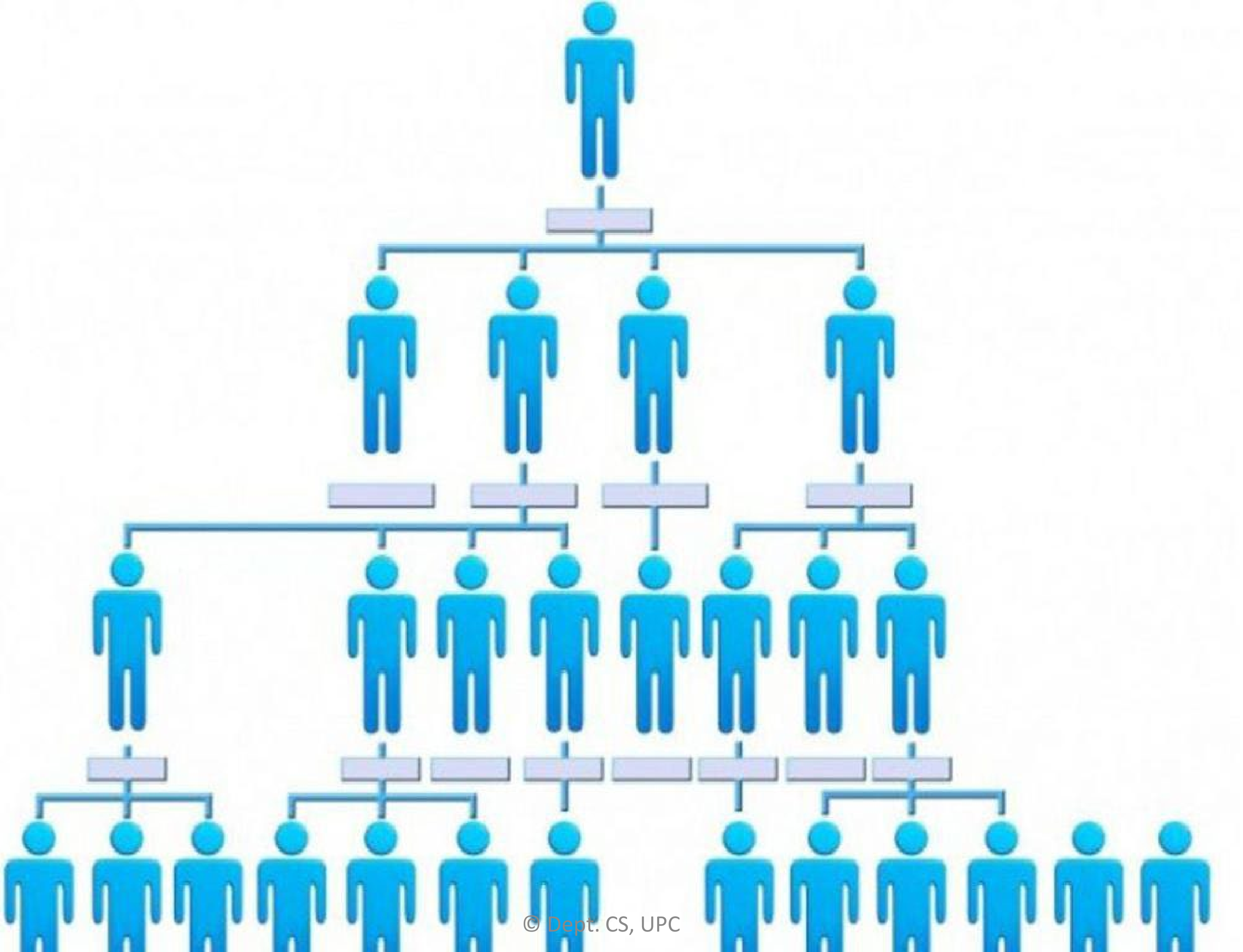




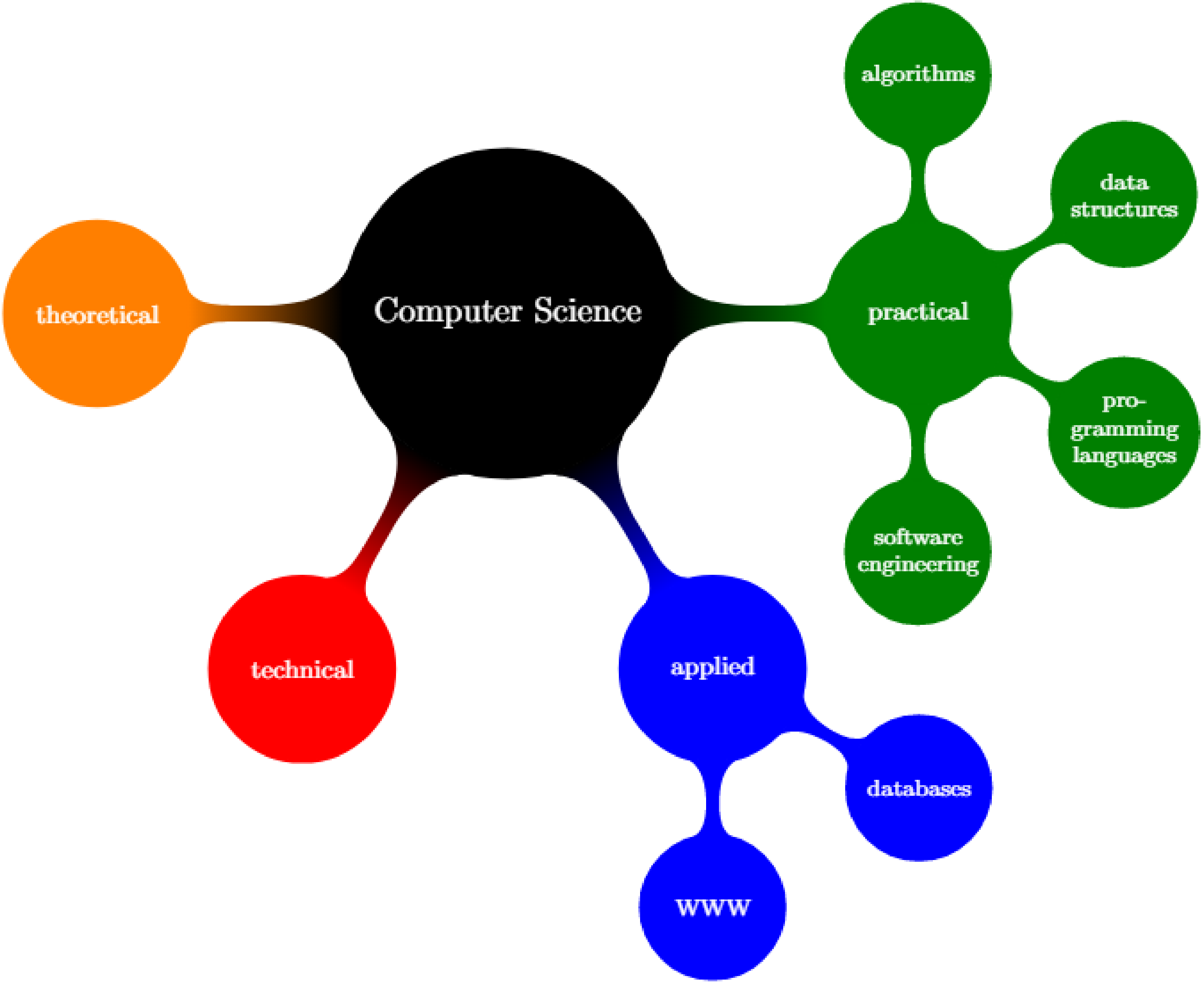
```
jelkner@rms:~$ tree webappdev/
webappdev/
├── css
│   ├── examples
│   └── notes
├── html
│   ├── examples
│   └── notes
├── unix
│   ├── examples
│   └── notes
│       ├── lesson1.txt
│       └── lesson2.txt
└── vim
    ├── examples
    └── notes

12 directories, 2 files
jelkner@rms:~$
```

Company structure



Mind maps


Computer Science
theoretical
technical
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databases
www
practical
algorithms
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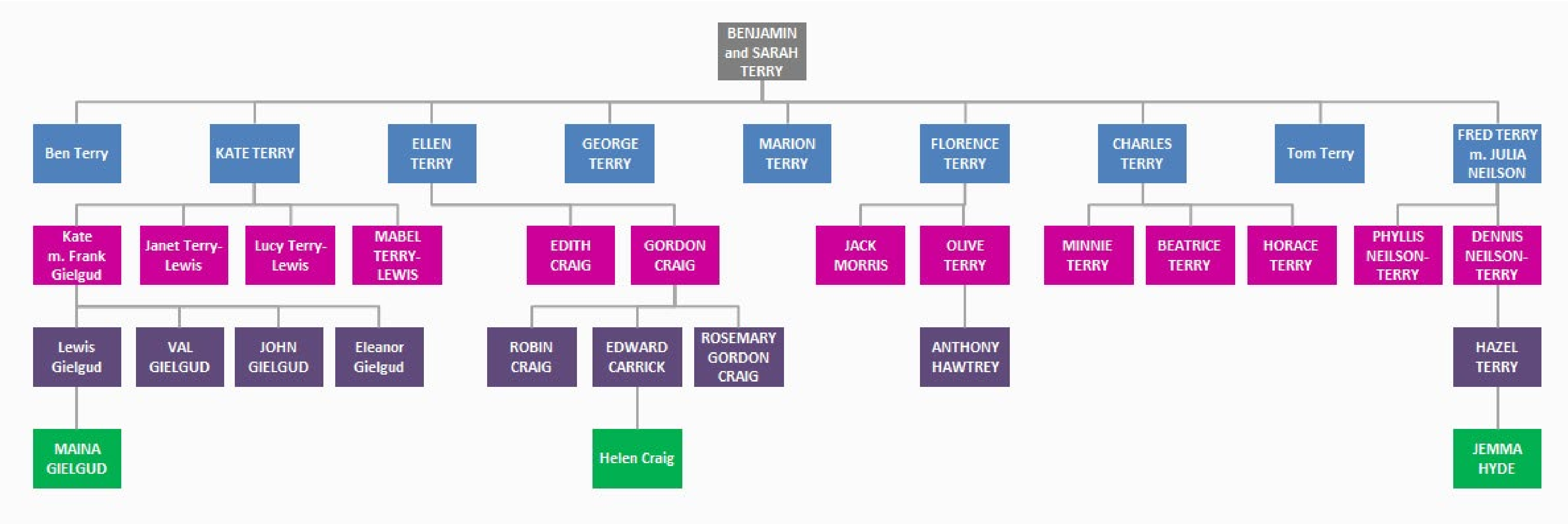
pro-gramming languages
software engineering

Genealogical trees



BENJAMIN and SARAH TERRY

Ben Terry
KATE TERRY
ELLEN TERRY
GEORGE TERRY
MARION TERRY
FLORENCE TERRY
CHARLES TERRY
Tom Terry
FRED TERRY m. JULIA NEILSON

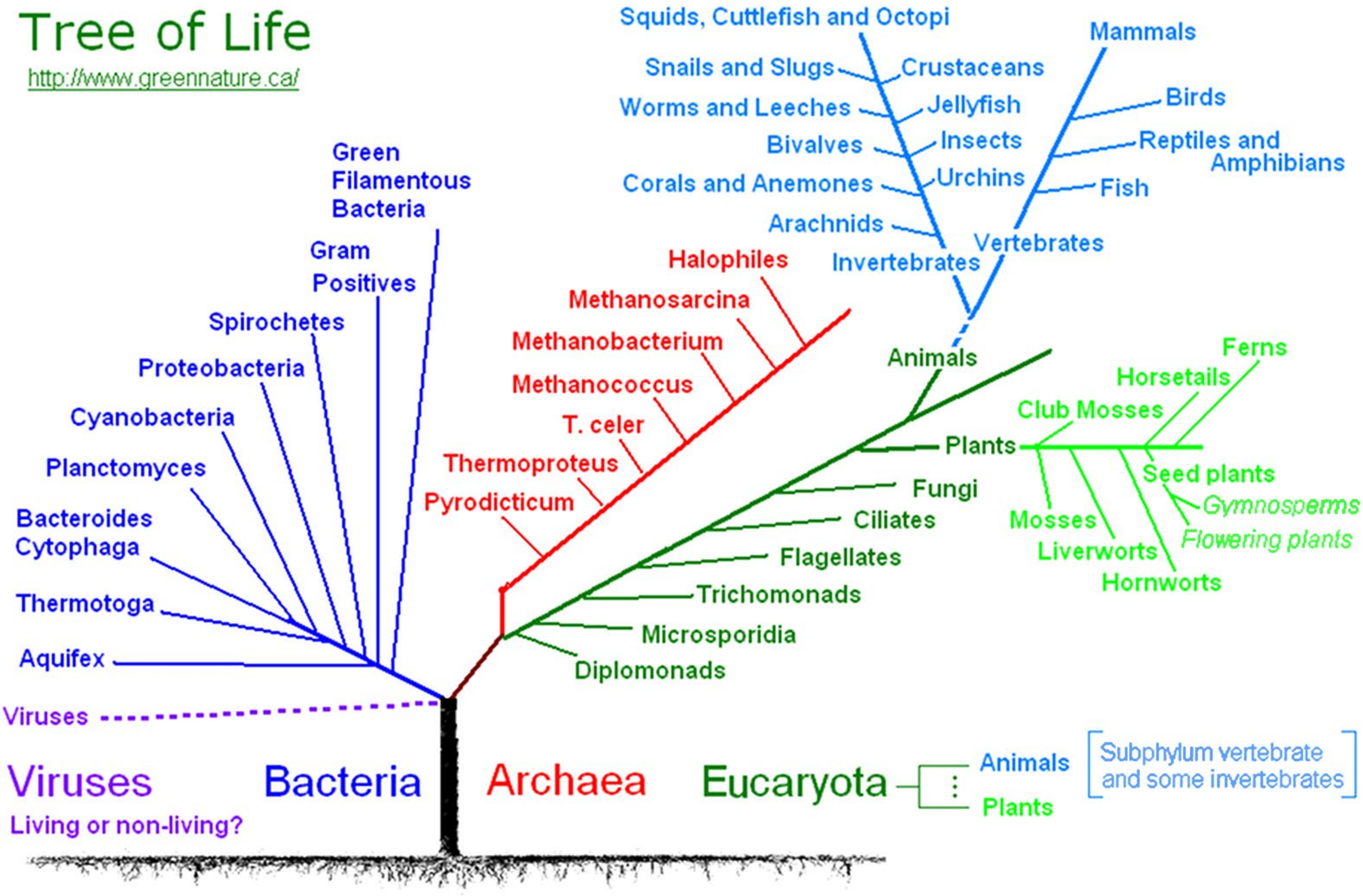
Kate m. Frank Gielgud
Janet Terry-Lewis
Lucy Terry-Lewis
MABEL TERRY-LEWIS
EDITH CRAIG
GORDON CRAIG
JACK MORRIS
OLIVE TERRY
MINNIE TERRY
BEATRICE TERRY
HORACE TERRY
PHYLLIS NEILSON-TERRY
DENNIS NEILSON-TERRY

Lewis Gielgud
VAL GIELGUD
JOHN GIELGUD
Eleanor Gielgud
ROBIN CRAIG
EDWARD CARRICK
ROSEMARY GORDON CRAIG
ANTHONY HAWTREY
HAZEL TERRY

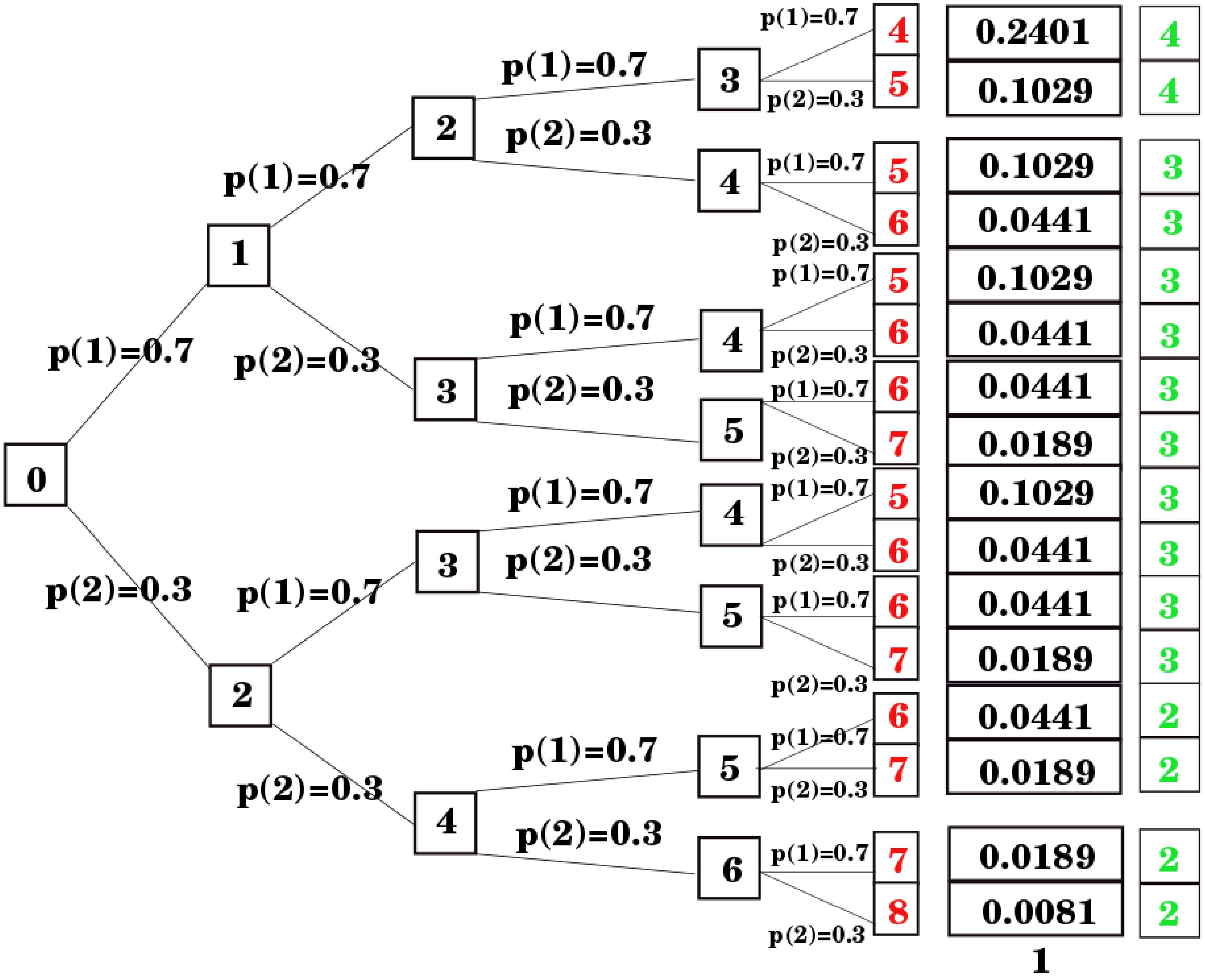
MAINA GIELGUD
Helen Craig
JEMMA HYDE

Tree of Life

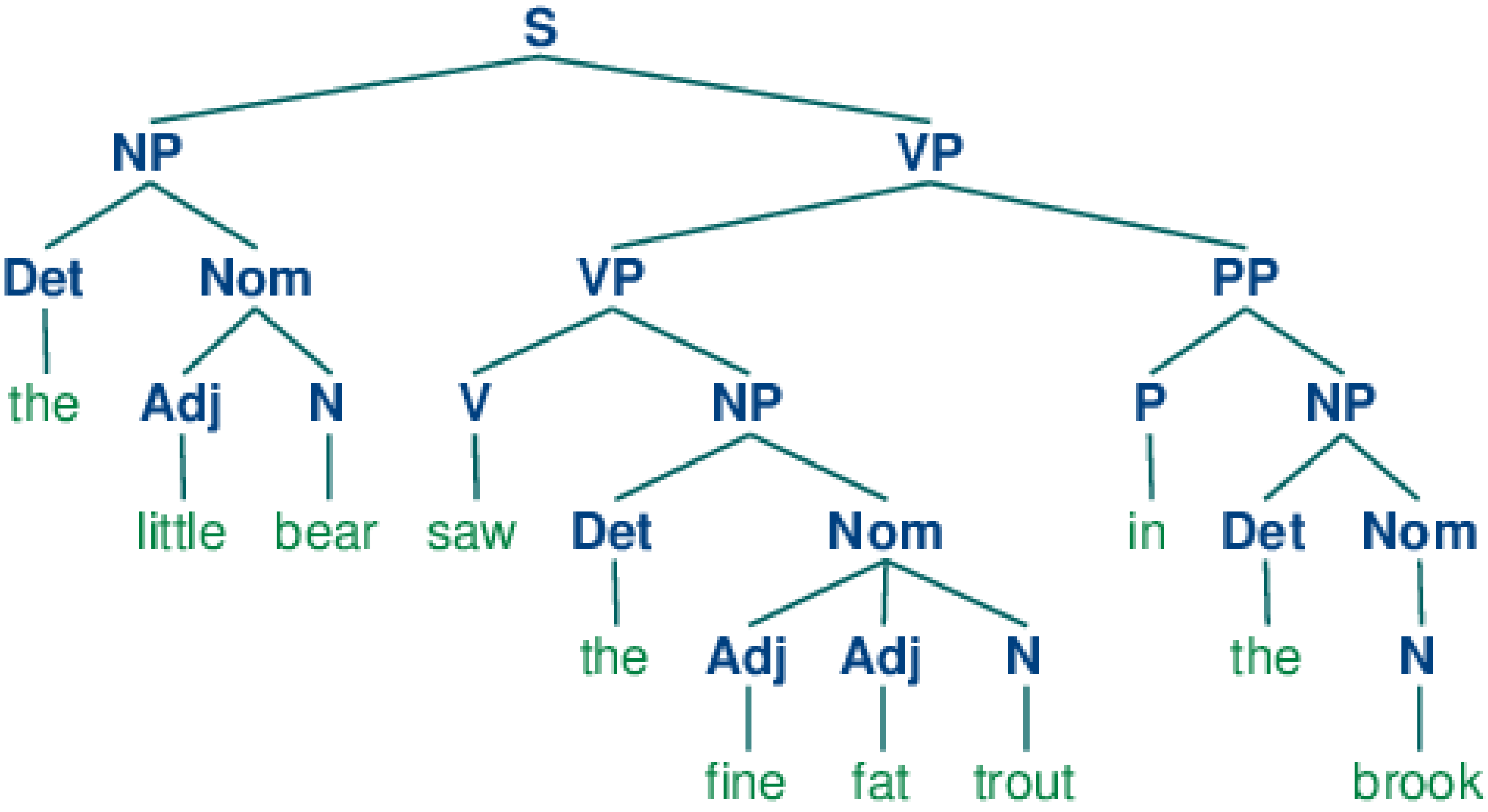
http://www.greennature.ca/



Probability trees



Parse trees


S
NP
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Nom
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PP
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Adj
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NP
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NP
little
bear
saw
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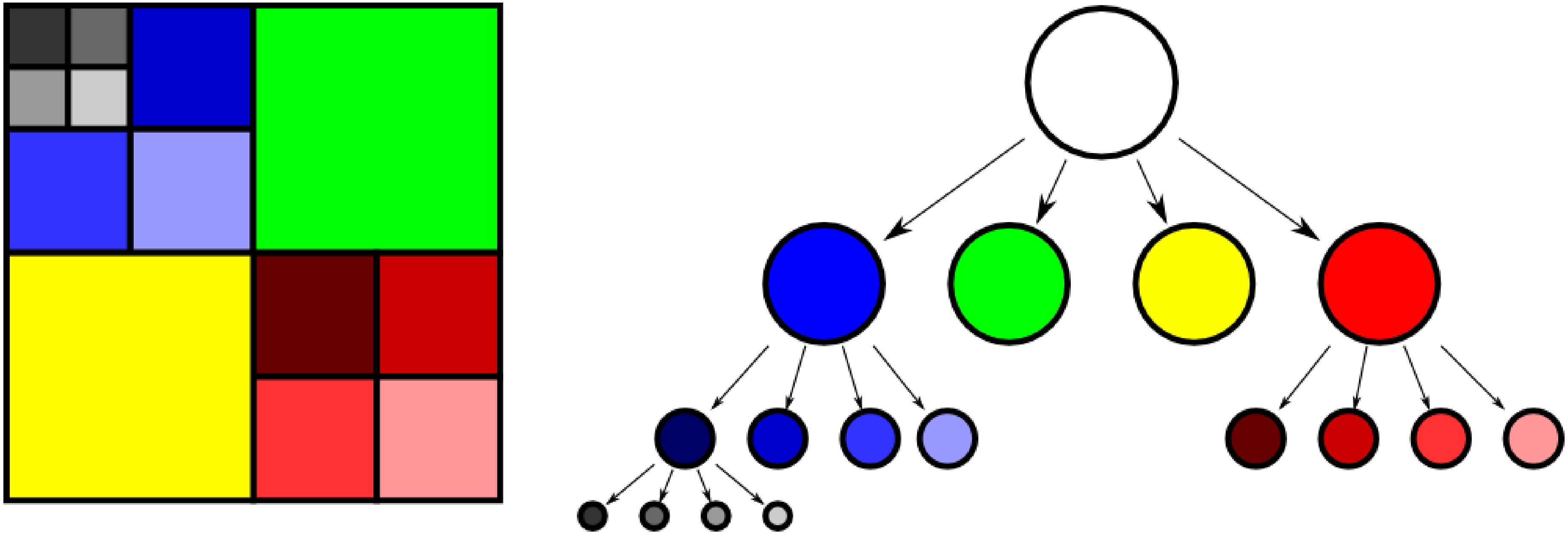
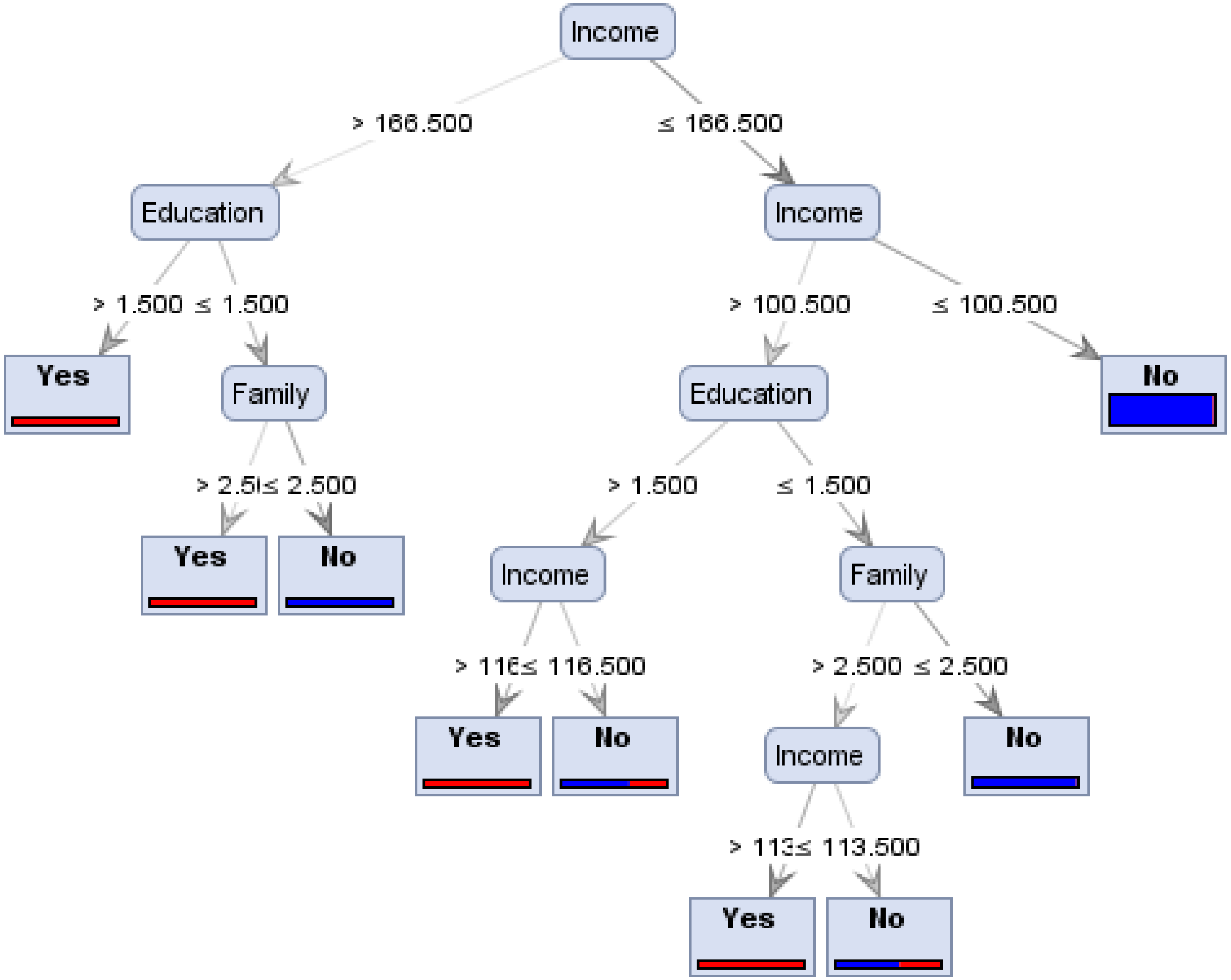
Adj
Adj
N
the
N
fine
fat
trout
brook

Image representation (quad-trees)



Decision trees

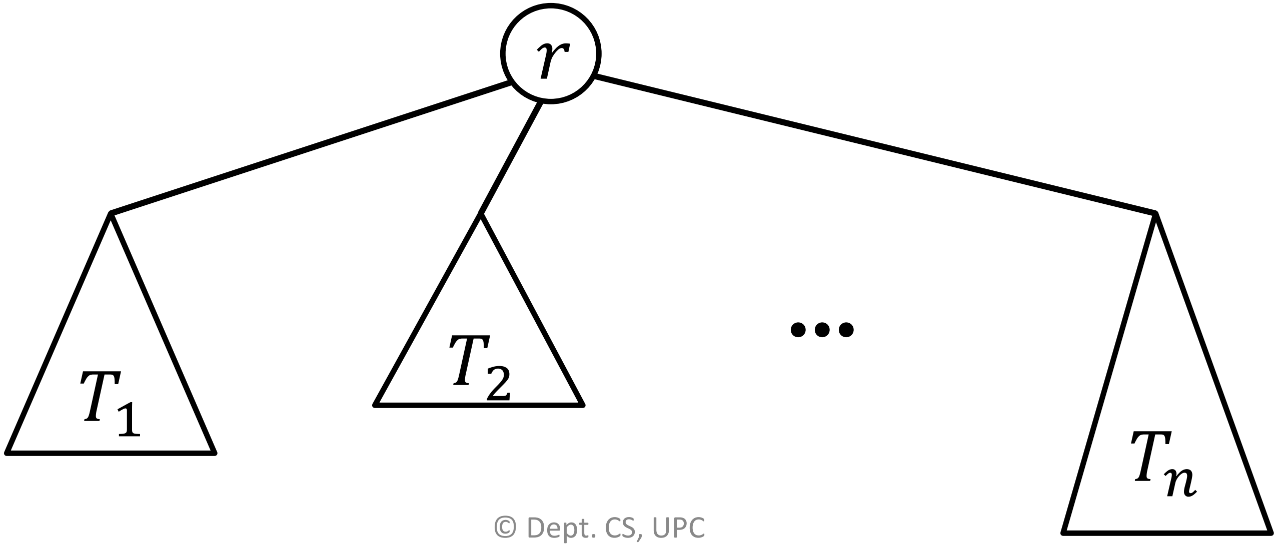


source: http://www.simafore.com/blog/bid/94454/A-simple-explanation-of-how-entropy-fuels-a-decision-tree-model

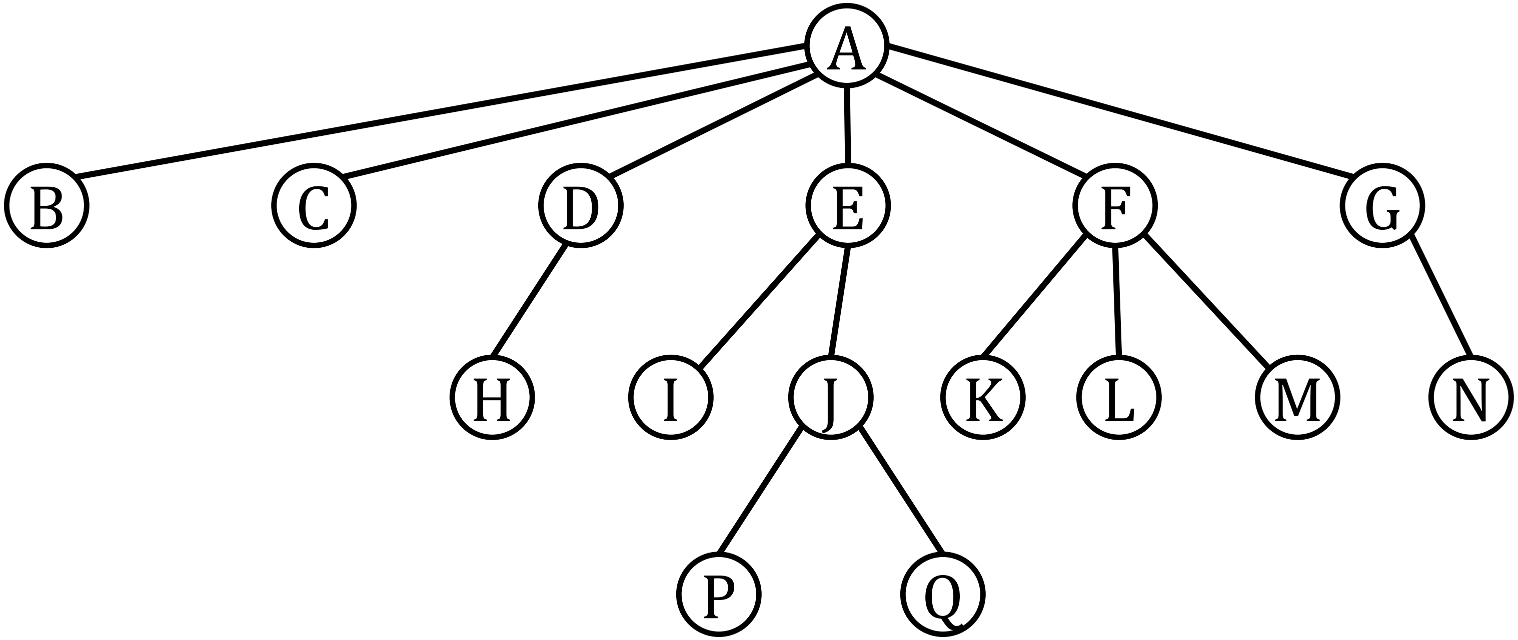
Tree: definition

- Graph theory: a tree is an undirected graph in which any two vertices are connected by exactly one path.

- Recursive definition (CS). A non-empty tree T consists of:
  - a root node r
  - a list of non-empty trees $T_1, T_2, \dots, T_n$ that hierarchically depend on r. The list can be possibly empty ($n \geq 0$).



Tree: nomenclature



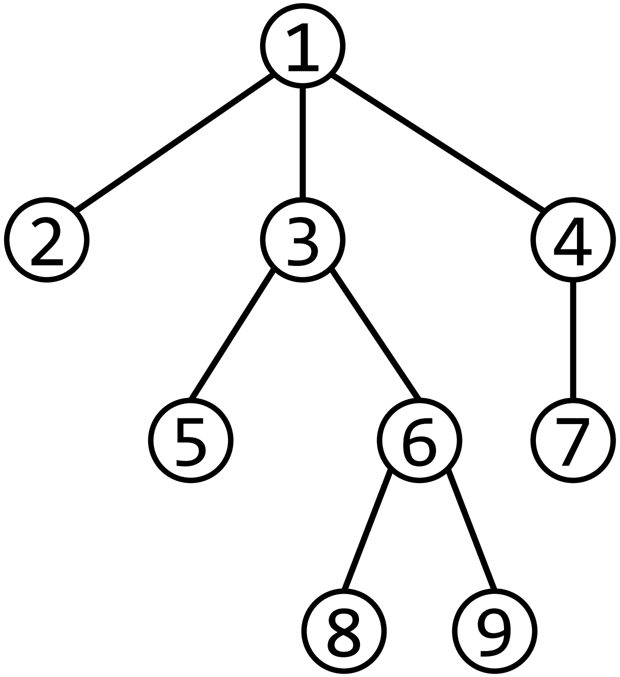
- A is the **root** node.
- Nodes with no children are **leaves** (e.g., B and P).
- Nodes with the same parent are **siblings** (e.g., K, L and M).
- The **depth** of a node is the length of the path from the root to the node. Examples: depth(A)=0, depth(L)=2, depth(Q)=3.

Tree: representation

There is a plethora of data structures that can be used to represent a tree, e.g., a hierarchical list.

$\texttt{[root, child}_1\texttt{, child}_2\texttt{, …, child}_n\texttt{]}$

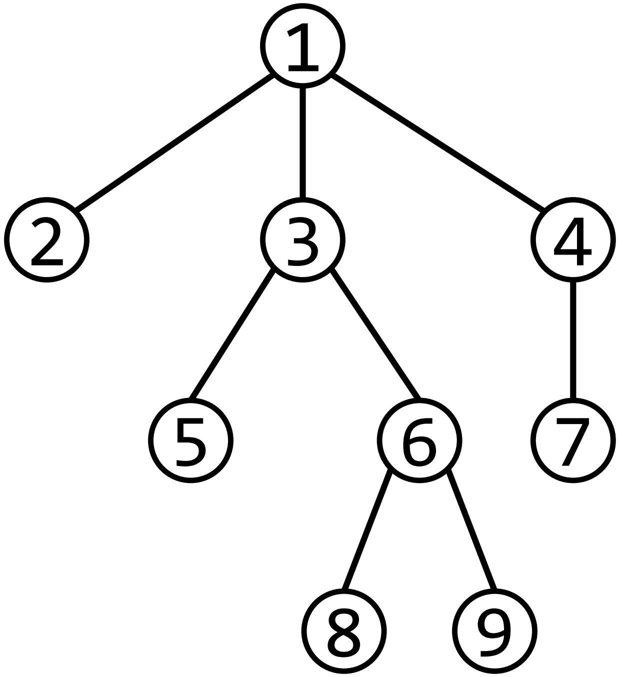
another tree



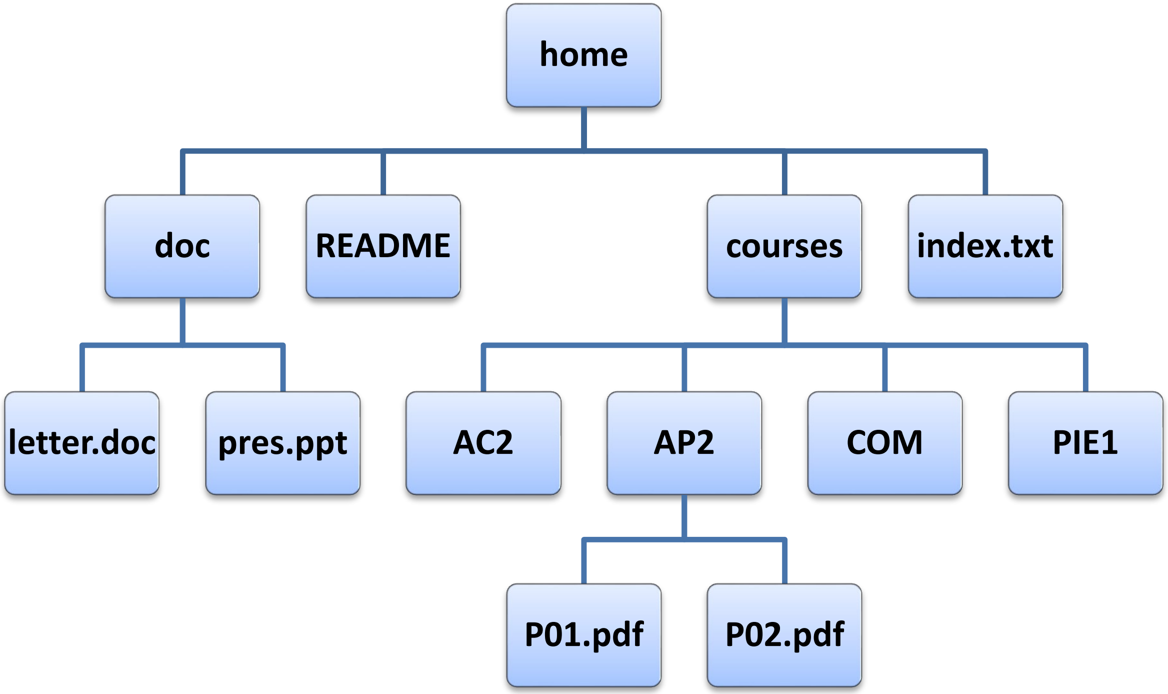
```
tree = [1, 2,
              [3, 5, [6, 8, 9]],
              [4, 7]
        ]
```

Tree: Abstract Data Type

```
from dataclasses import dataclass
from typing import TypeVar, Generic

T = TypeVar('T')

@dataclass
class Tree(Generic[T]):
    """Class to represent a generic tree"""
    data: T
    children: list[Tree[T]]


def size(t: Tree) -> int:
    return 1 + sum(c.size() for c in t.children)



def num_levels(t: Tree) -> int:
    # implement it!
```



Write a tree



```
home
  doc
    letter.doc
    pres.ppt
  README
  courses
    AC2
    AP2
      P01.pdf
      P02.pdf
    COM
    PIE1
  index.txt
```

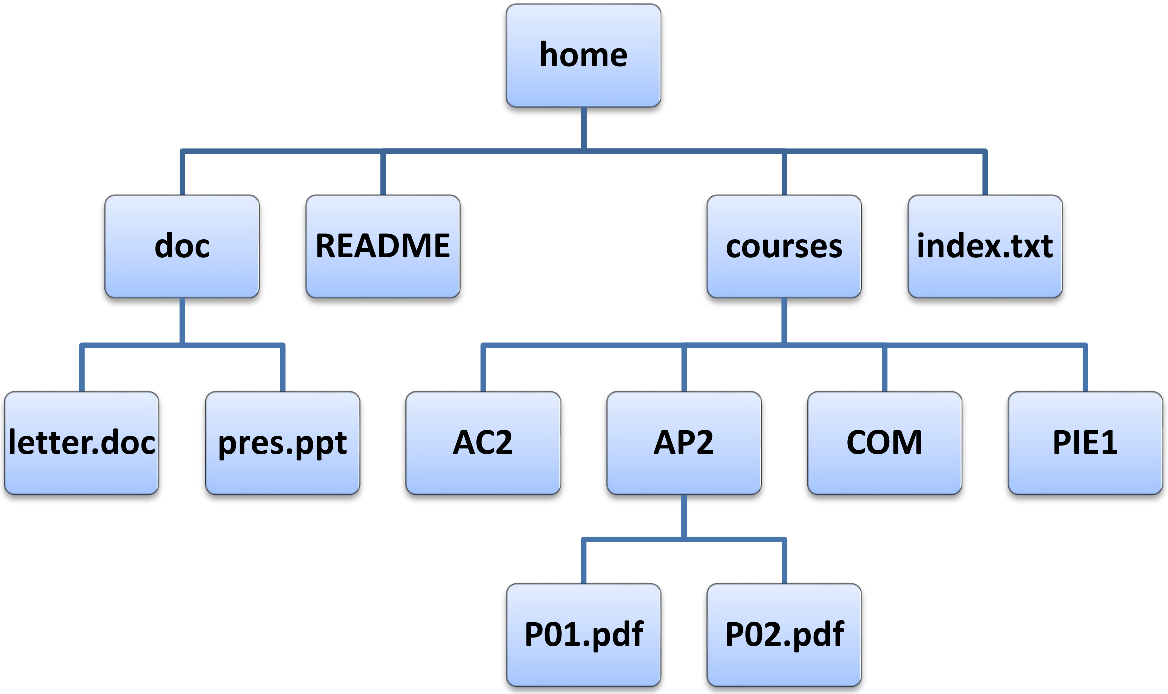
```
def write(t: Tree[T], depth: int = 0) -> None:
    """Writes a tree indented according to the depth"""
```

Write a tree

```
def write(t: Tree[T], depth: int = 0) -> None:
    """Writes a tree indented according to the depth"""

    # print the root
    print(' '*2*depth, t.data, sep='')

    # print the children with depth + 1
    for c in t.children:
        write(c, depth + 1)
```

This function executes a ***preorder*** traversal of the tree: each node is processed *before* the children.

Write a tree (postorder traversal)



```
    letter.doc
    pres.ppt
  doc
  README
    AC2
      P01.pdf
      P02.pdf
    AP2
    COM
    PIE1
  courses
  index.txt
home
```

Postorder traversal: each node is processed after the children.

Write a tree (postordre traversal)

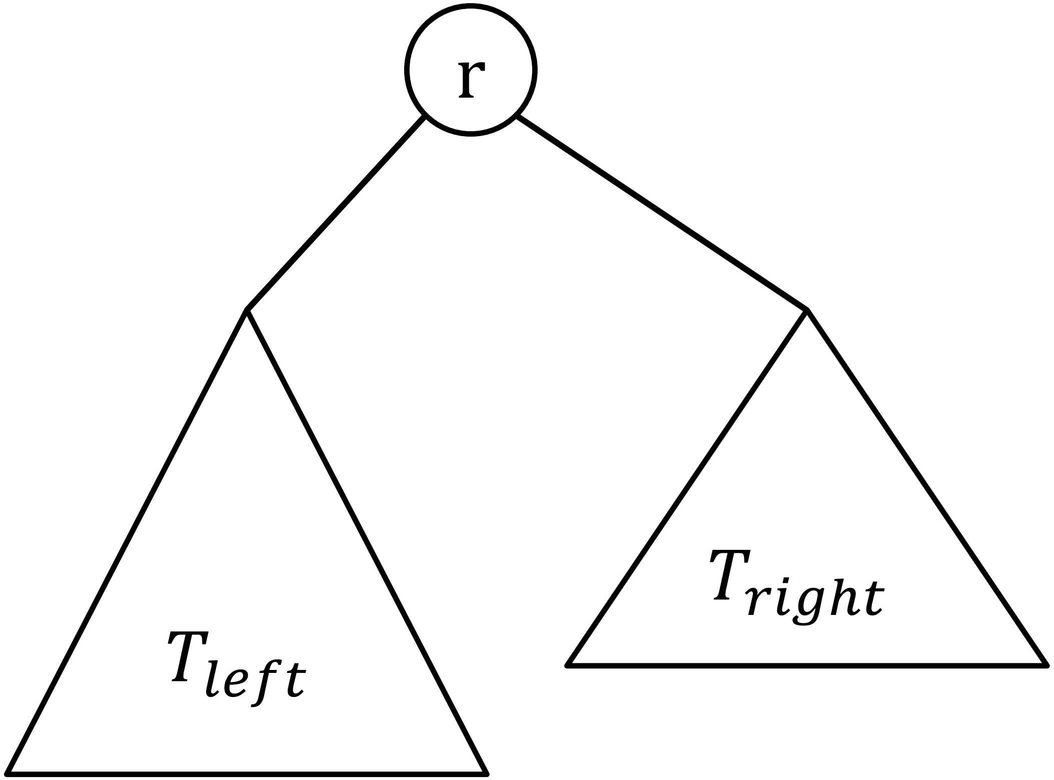
```
def write_postorder(t: Tree[T], depth: int = 0) -> None:
    """Writes a tree (in postorder) indented according
       to the depth"""

    # print the children with depth + 1
    for c in t.children:
        write_postorder(c, depth + 1)

    # print the root
    print(' '*2*depth, t.data, sep='')
```

This function executes a ***postorder*** traversal of the tree: each node is processed *after* the children.

Binary tree: definition

A binary tree is a finite set of nodes that either

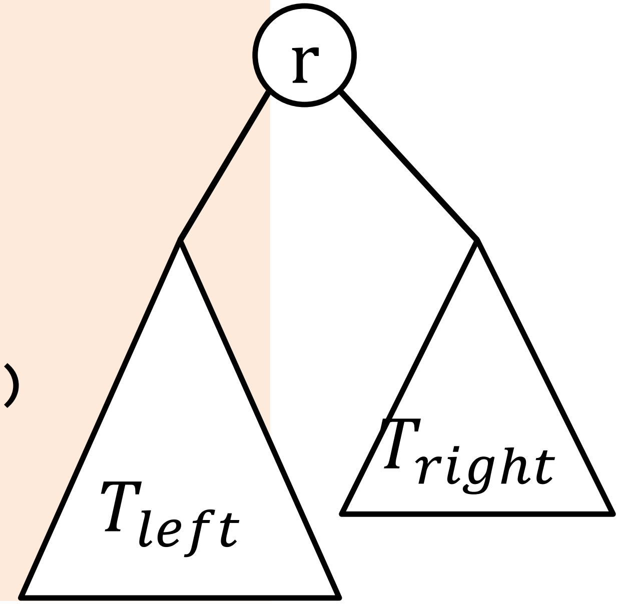
- is empty, or
- is comprised of three disjoint sets of nodes: a root node and two binary trees called its left and right subtrees



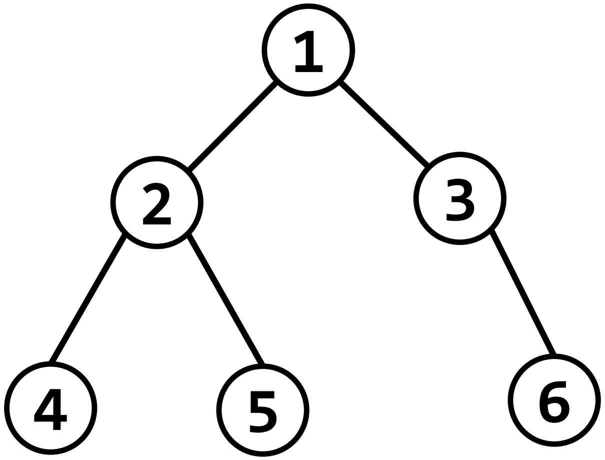
Binary tree: representation

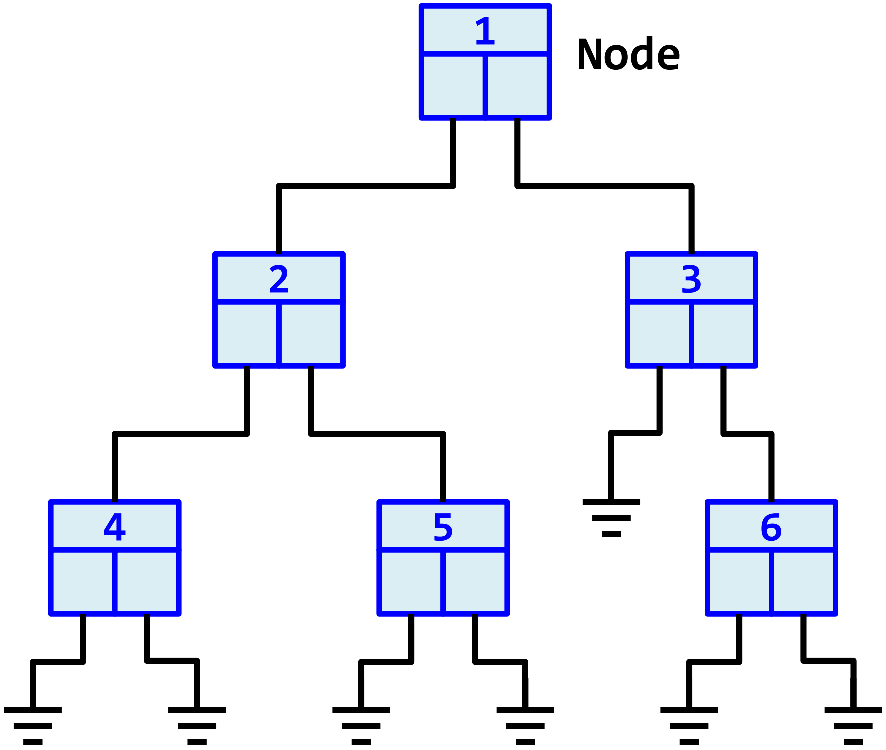
Data structures to represent binary trees are typically based on the definition of a node.

```
from dataclasses import dataclass, field
from typing import TypeVar, Generic, Optional, Iterator

T = TypeVar('T')

@dataclass
class Node(Generic[T]):
    """Node of a bin tree"""
    data: T
    left: 'BinTree[T]' = field(default = None)
    right: 'BinTree[T]' = field(default = None)

BinTree = Optional[Node[T]]
NodeIter = Iterator[Node[T]]
```

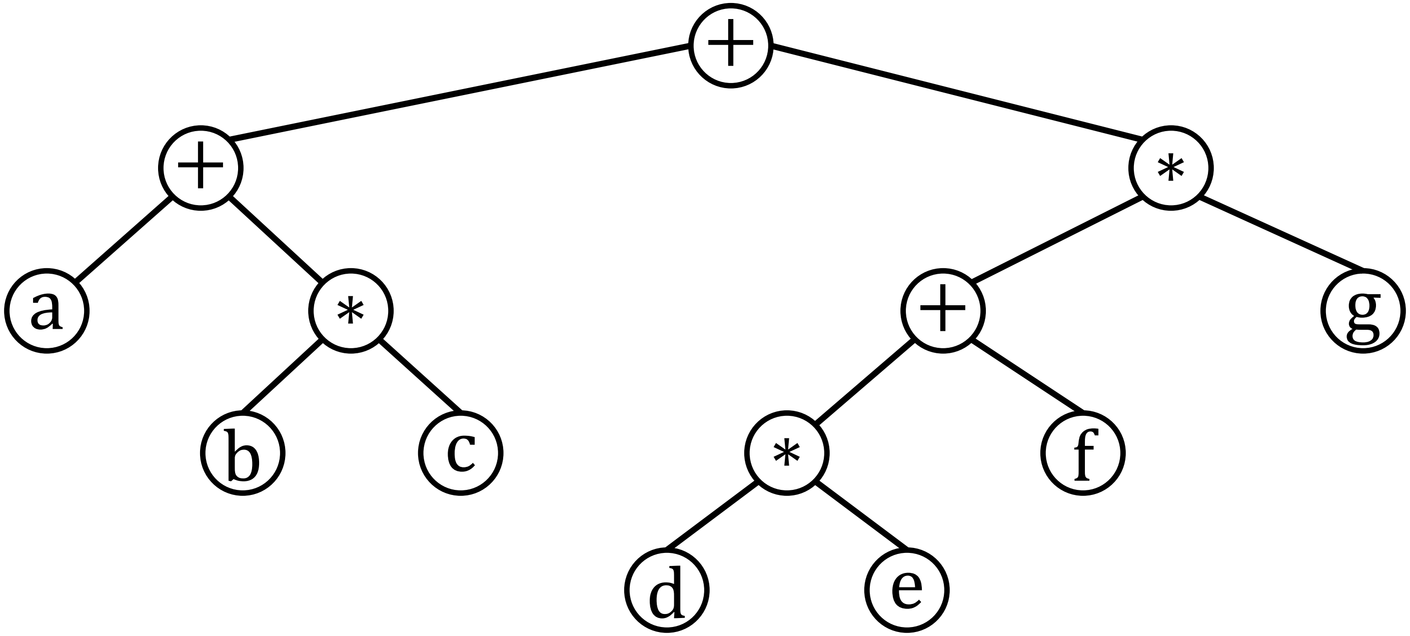


Binary tree: representation





Example: expression trees



Expression tree for: **a + b*c + (d*e + f) * g**

Postfix representation: **a b c * + d e * f + g * +**

How can the postfix representation be obtained?

Example: expression trees

Expressions are represented by strings in postfix notation in which **`'a'…'z'`** represent operands and **`'+'`** and **`'*'`** represent operators.

```
Exprtree: TypeAlias = BinTree[str]

def build_expr(expr: str) -> Exprtree:
    """Builds an expression tree from a correct
       expression represented in postfix notation"""

def infix_expr(t: Exprtree) -> str:
    """Generates a string with the expression in
       infix notation"""

def eval_expr(t: Exprtree, v: dict[str, int]) -> int:
    """Evaluates an expression taking v as the value of the
       variables (e.g., v['a'] contains the value of a)"""
```

Example: expression trees

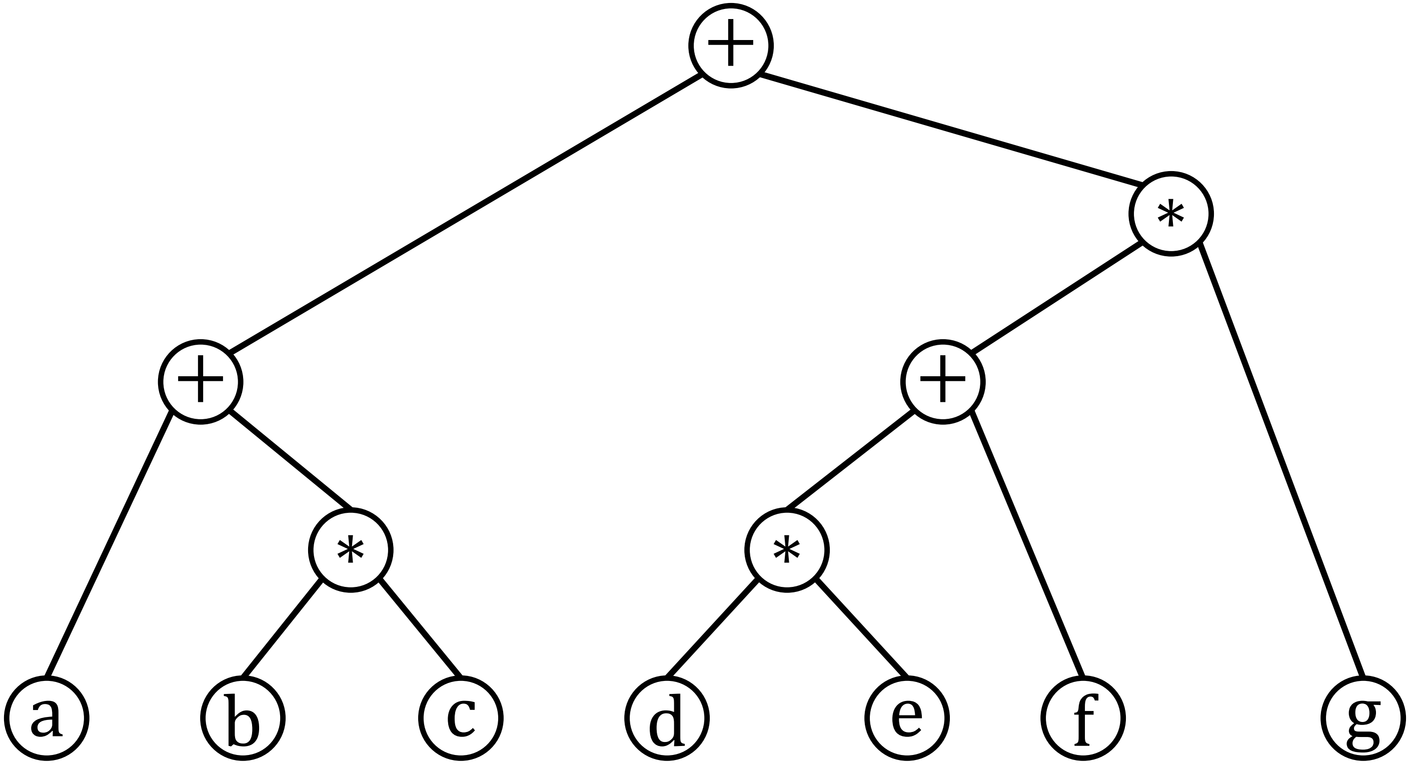
```
def main():
    t = build_expr('a b c * + d e * f + g * +')
    print(infix_expr(t))
    print(eval_expr(t, {'a':3, 'b':1, 'c':0, 'd':5,
                        'e':2, 'f':1, 'g':6}))
```

Output:

```
((a+(b*c))+(((d*e)+f)*g))
69
```

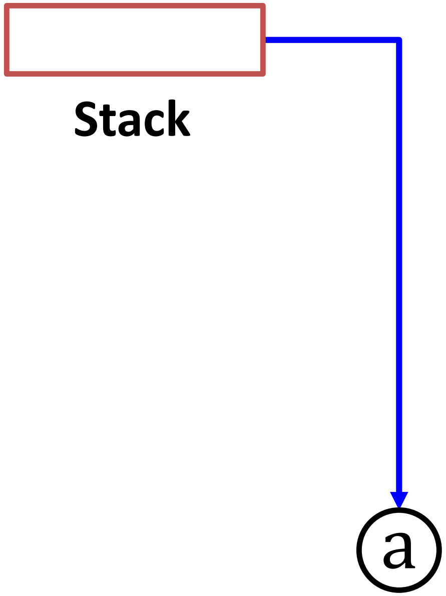
How to build an expression tree

a b c * + d e * f + g * +



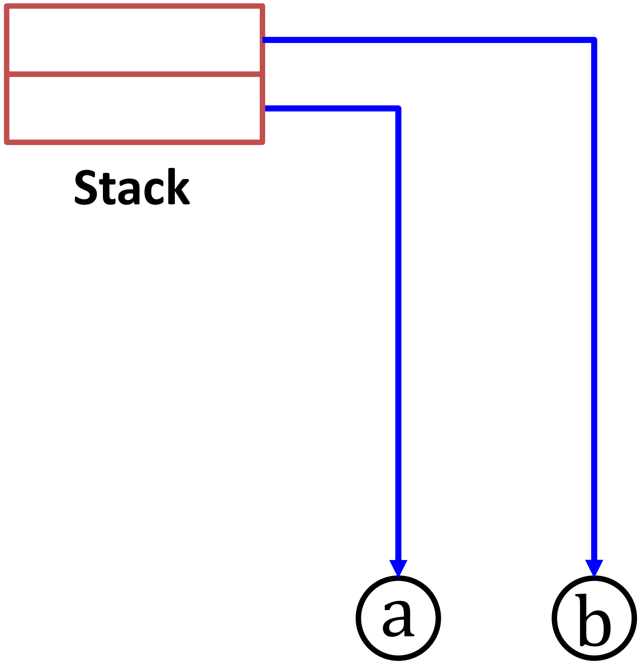
How to build an expression tree

a b c * + d e * f + g * +



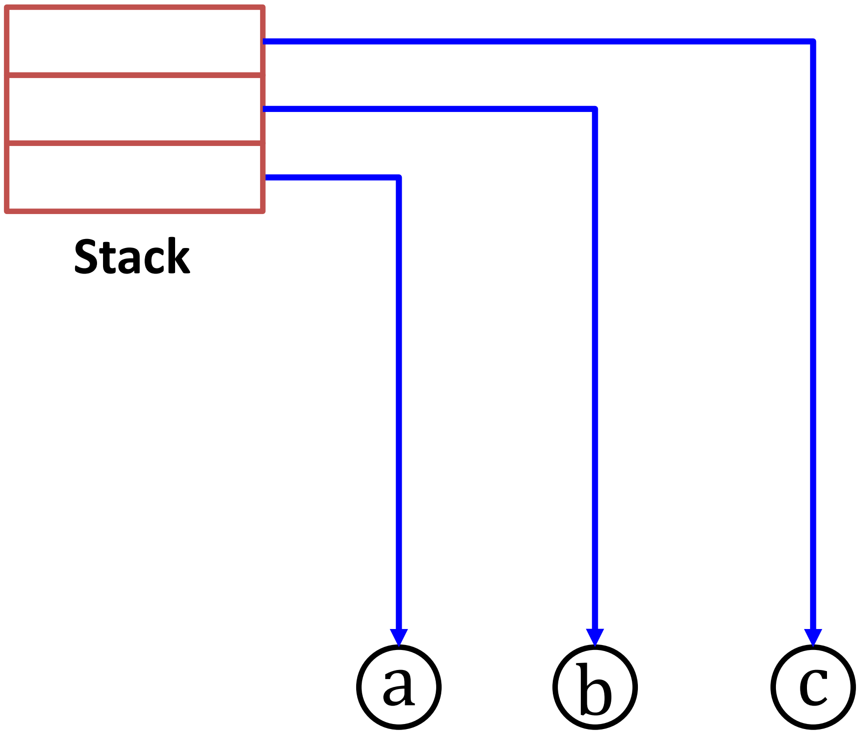
How to build an expression tree

a b c * + d e * f + g * +



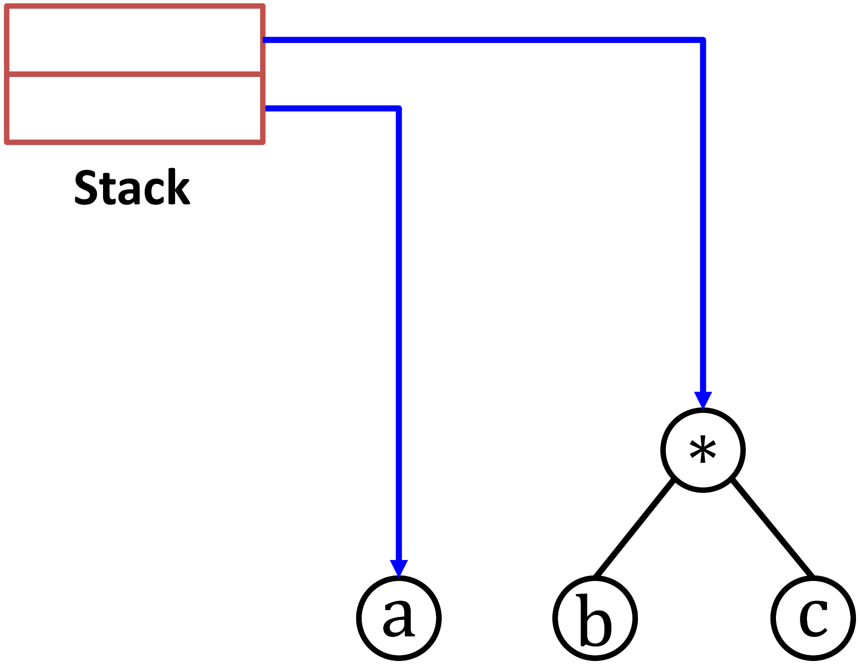
How to build an expression tree

a b c * + d e * f + g * +



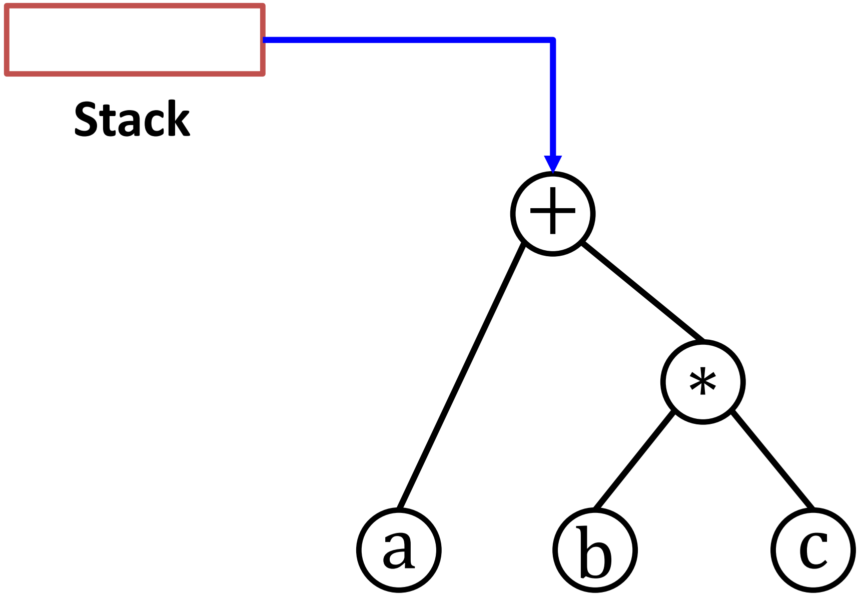
How to build an expression tree

a b c * **+ d e * f + g * +**



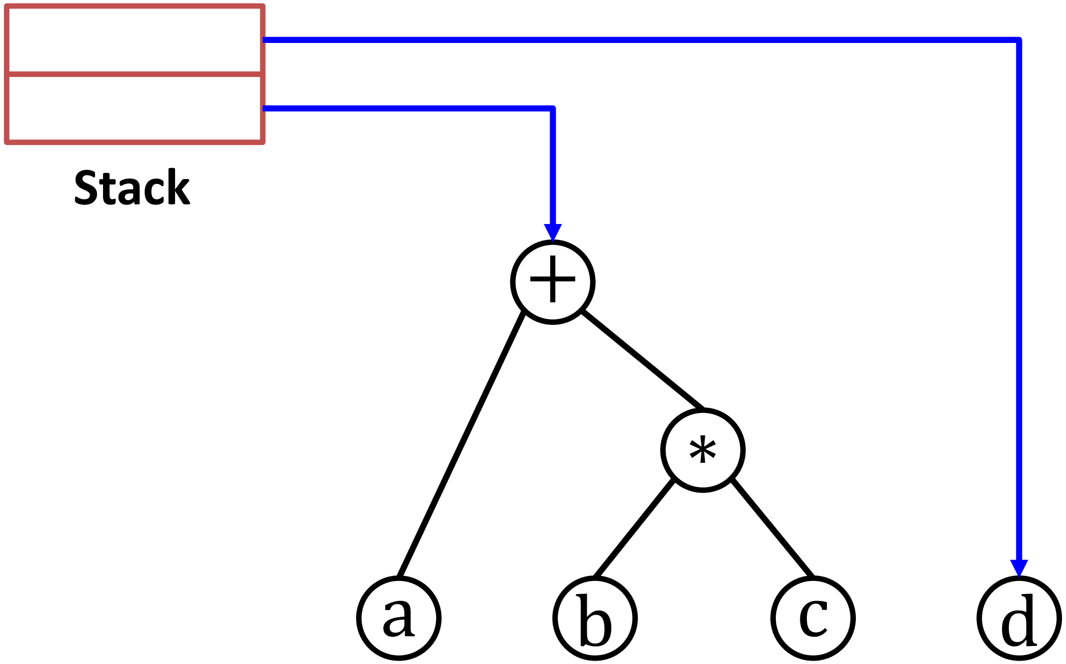
How to build an expression tree

a b c * + **d e * f + g * +**



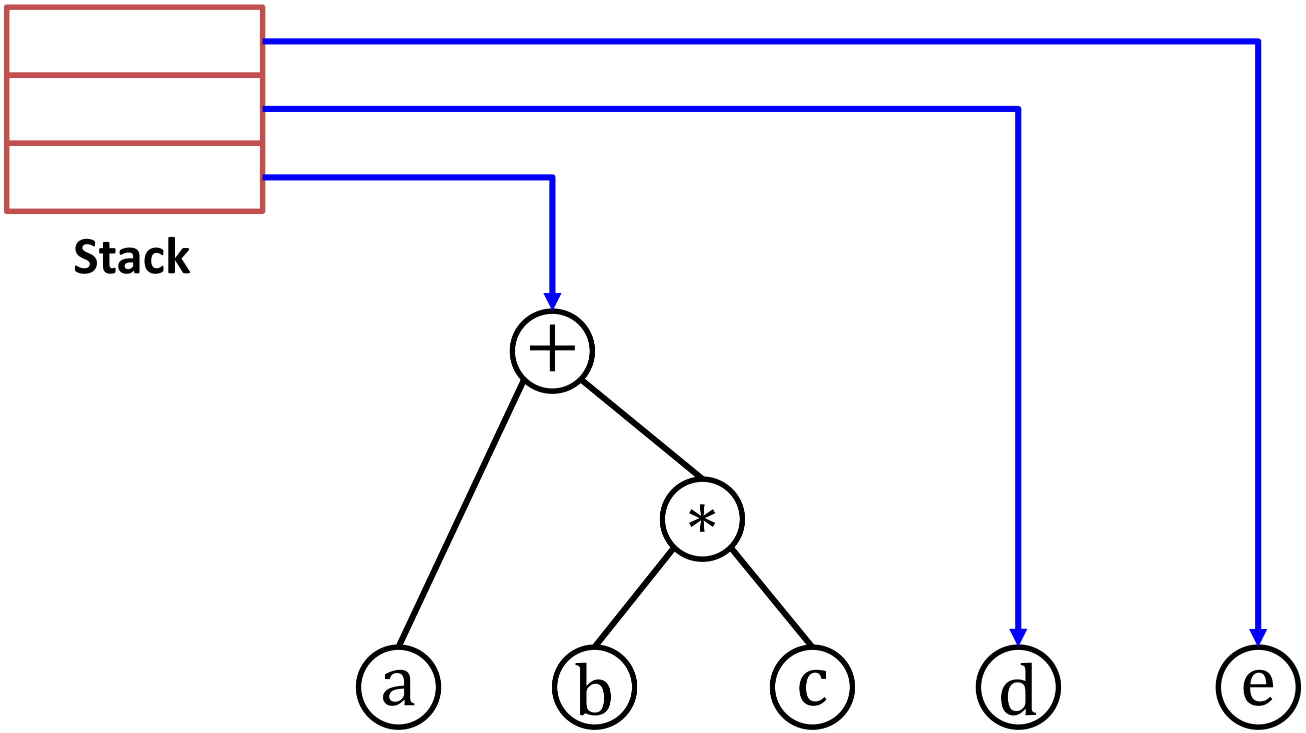
How to build an expression tree

a b c * + d **e * f + g * +**



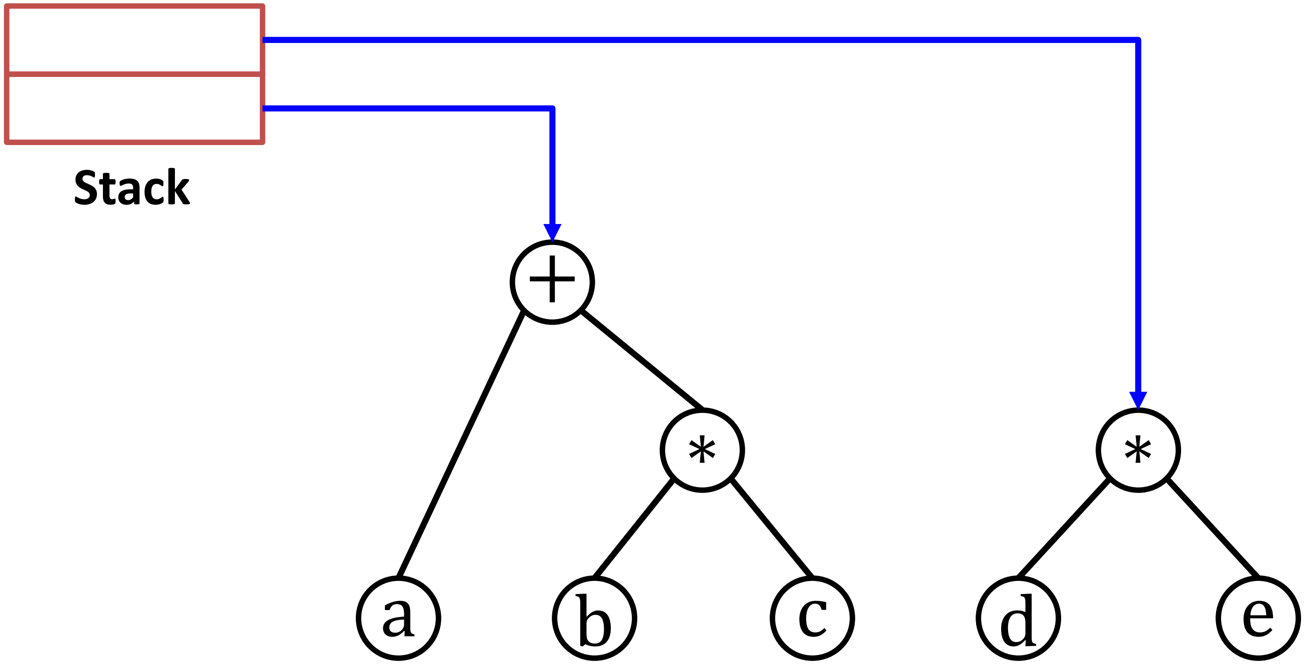
How to build an expression tree

a b c * + d e *** f + g * +**



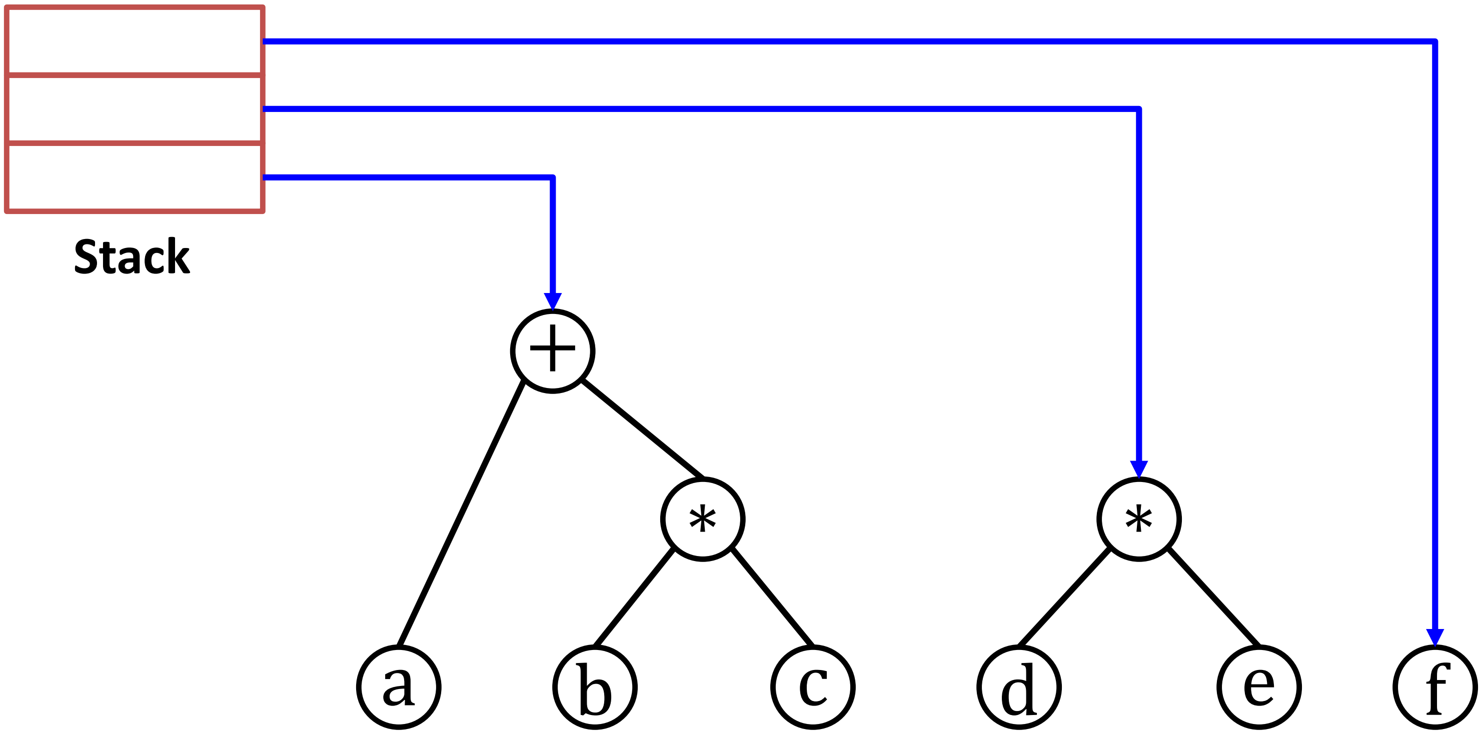
How to build an expression tree

a b c * + d e * **f + g * +**



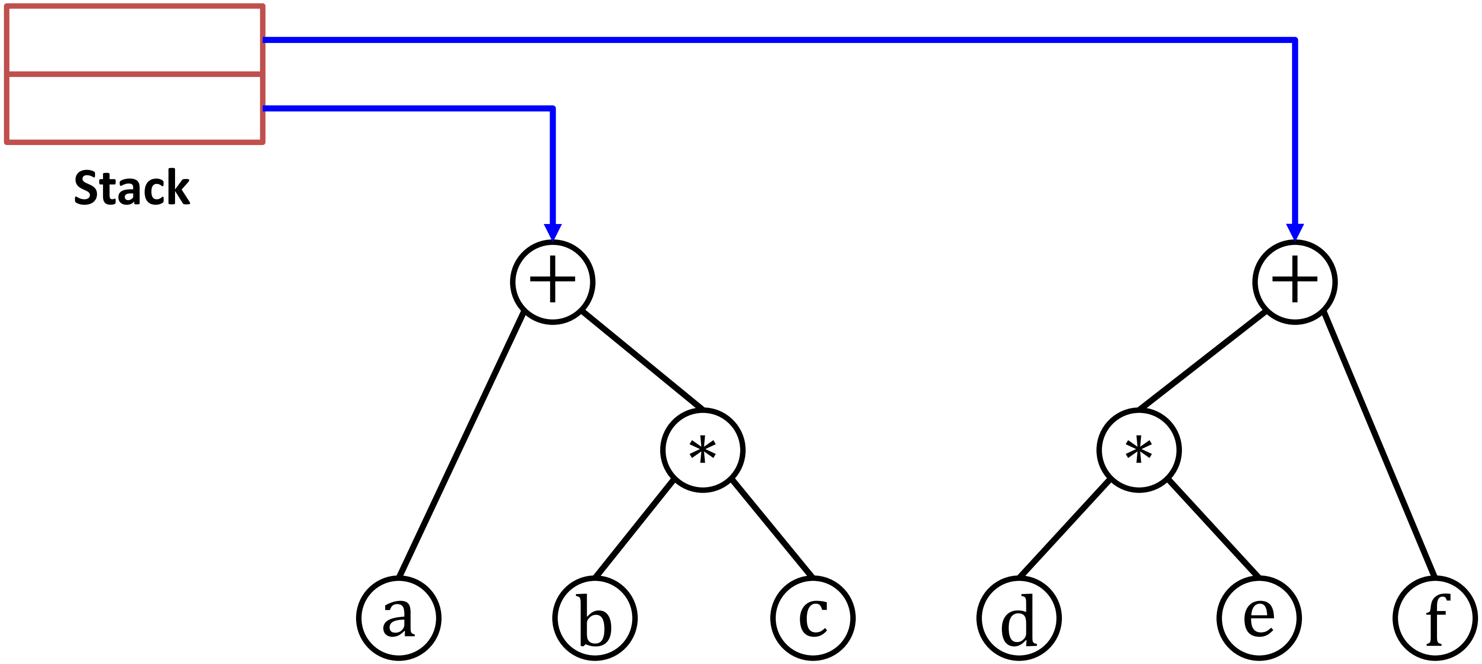
How to build an expression tree

a b c * + d e * f **+ g * +**



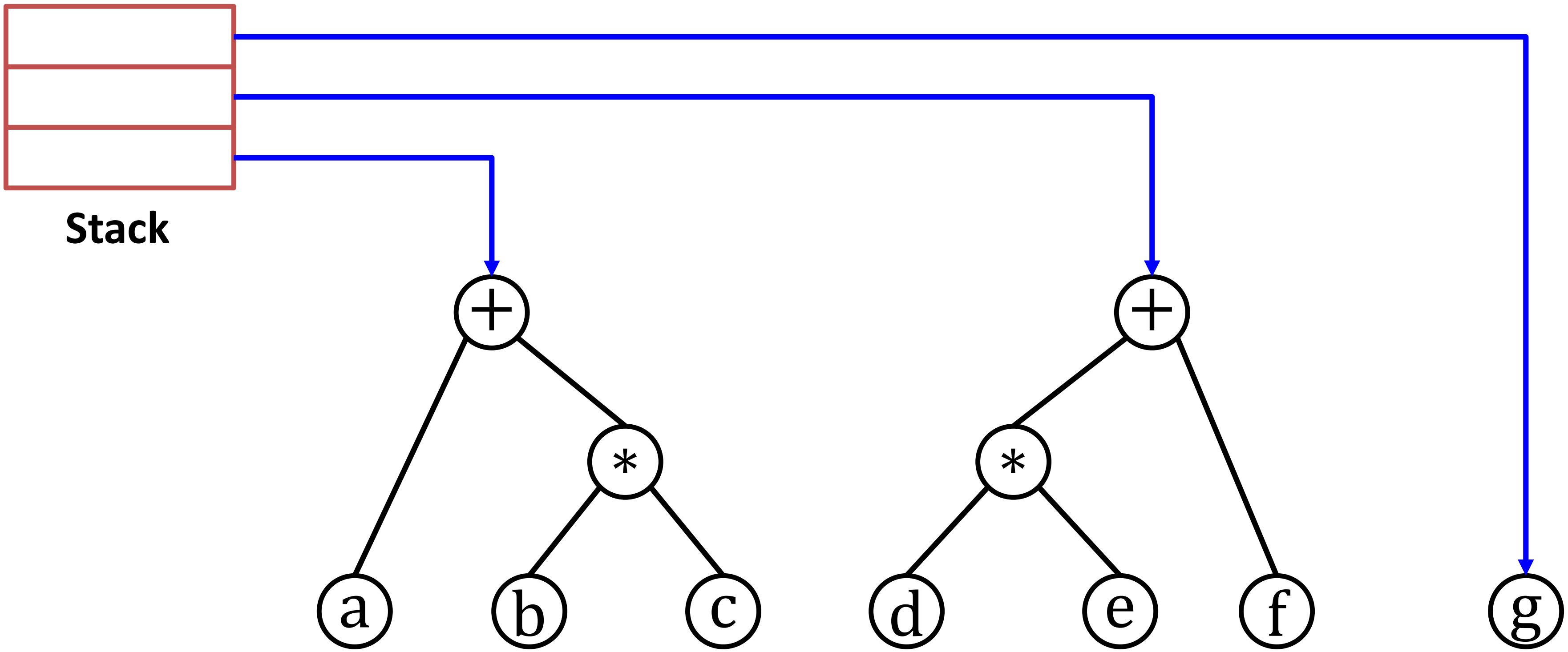
How to build an expression tree

a b c * + d e * f + **g * +**



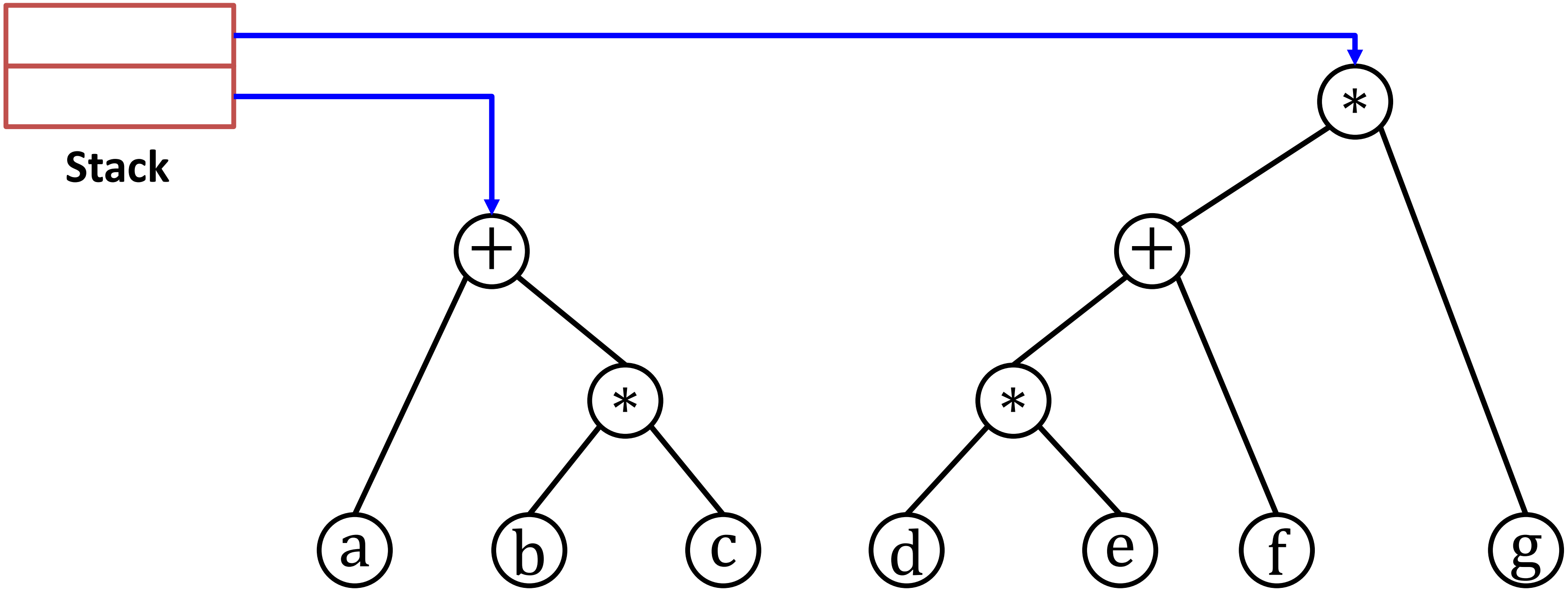
How to build an expression tree

a b c * + d e * f + g *** +**



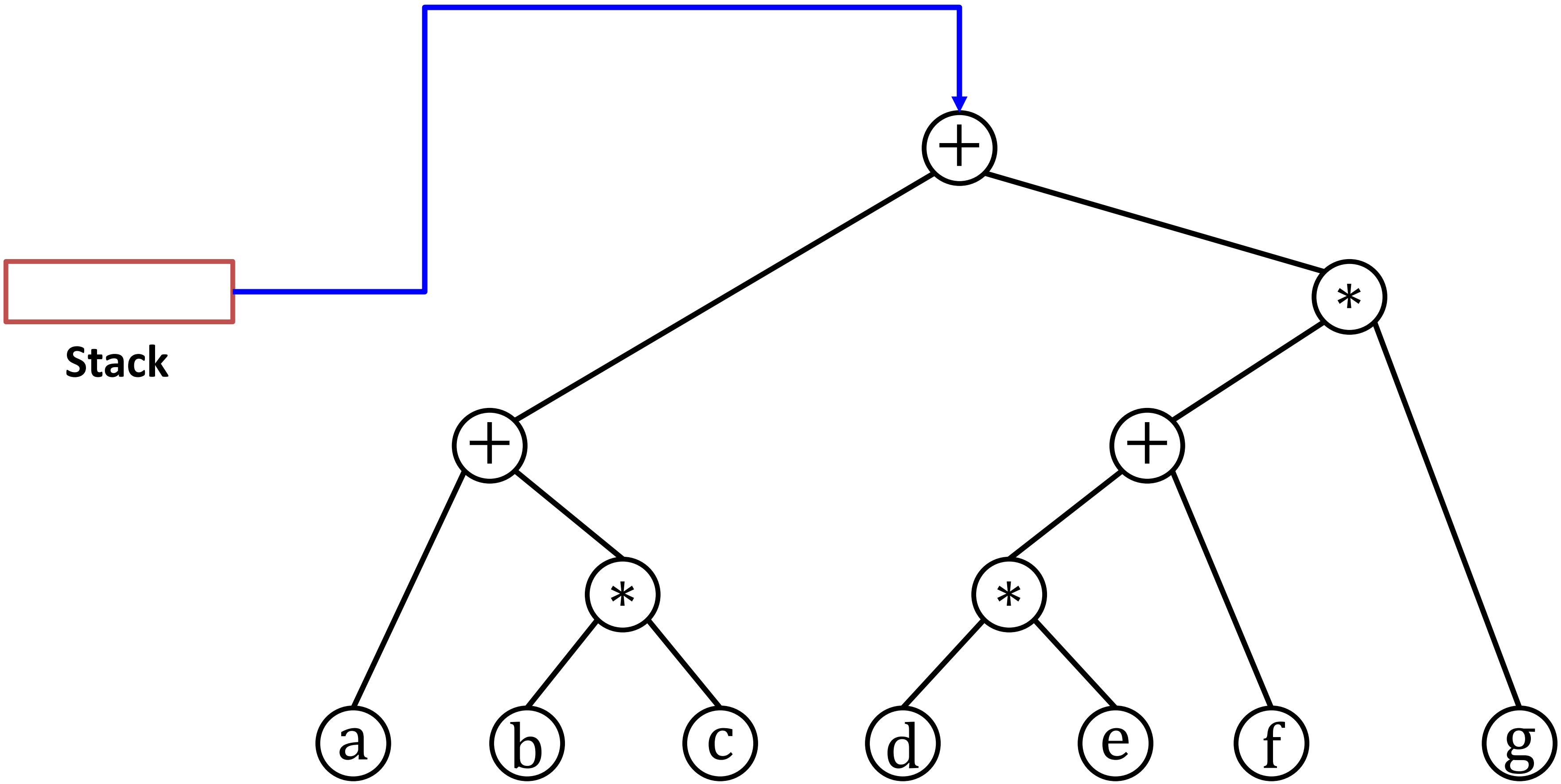
How to build an expression tree

a b c * + d e * f + g * **+**



How to build an expression tree

a b c * + d e * f + g * +



Example: expression trees

```
def build_expr(expr: str) -> Exprtree:
    """Builds an expression tree from a correct
       expression represented in postfix notation"""
    # Create a list of all characters (without spaces)
    expr_char = [x for x in expr if not x.isspace()]
    stack: list[Node[str]] = []
    for c in expr_char:
        if c.isalpha():
            # We have an operand. Create a leaf node
            stack.append(Node(c))
        else:
            # We have an operator (+ or *)
            right = stack.pop()
            left = stack.pop()
            stack.append(Node(c, left, right))
    # The stack has only one element: the root of the expression
    return stack.pop()
```

Example: expression trees

```
def infix_expr(t: Exprtree) -> str:
    """Generates a string with the expression in
       infix notation"""

    if not t.left:  # it is a leaf node (operand)
        return t.data

    # We have an operator. Add enclosing parenthesis (for safety)
    return '(' + infix_expr(t.left) + t.data +
           infix_expr(t.right) + ')'
```

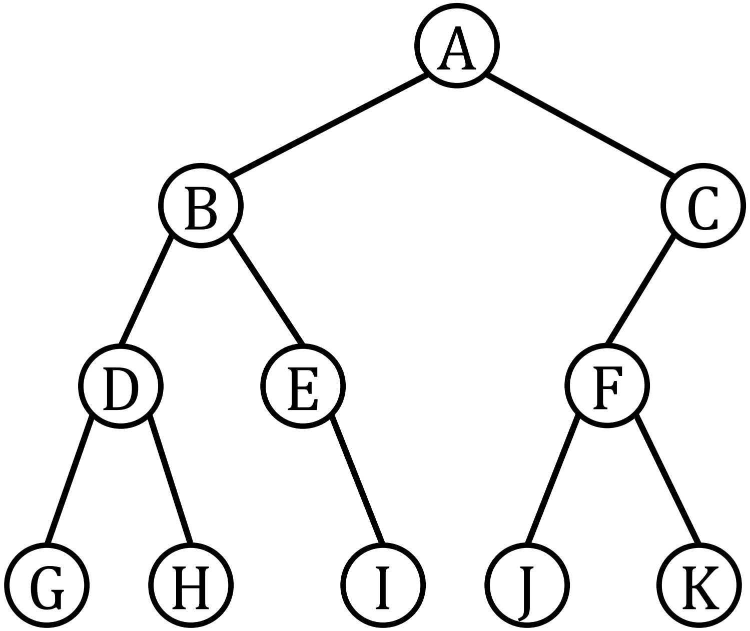
Inorder traversal: node is visited *between* the left and right children.

Exercise: redesign infix_expr to minimize the number of parenthesis.

Example: expression trees

```
def eval_expr(t: Exprtree, v: dict[str, int]) -> int:
    """Evaluates an expression taking v as the value of the
       variables (e.g., v['a'] contains the value of a)"""

    if not t.left:  # it is a leaf node: return the value
        return v[t.data]

    # We have an operator: evaluate subtrees and operate
    left = eval_expr(t.left, v)
    right = eval_expr(t.right, v)
    return left + right if t.data == '+' else left * right
```

Tree traversals



Let us consider generators to visit the nodes of the tree in some specific order.

```
t: BinTree[str] = … # some tree constructor

Lpreorder  = [n.data for n in preorder(t)]
Lpostorder = [n.data for n in postorder(t)]
Linorder   = [n.data for n in inorder(t)]
Llevels    = [n.data for n in level_order(t)]
```

```
Lpreorder:  ['A', 'B', 'D', 'G', 'H', 'E', 'I', 'C', 'F', 'J', 'K']

Lpostorder: ['G', 'H', 'D', 'I', 'E', 'B', 'J', 'K', 'F', 'C', 'A']

Linorder:   ['G', 'D', 'H', 'B', 'E', 'I', 'A', 'J', 'F', 'K', 'C']

Llevels:    ['A', 'B', 'C', 'D', 'E', 'F', 'G', 'H', 'I', 'J', 'K']
```

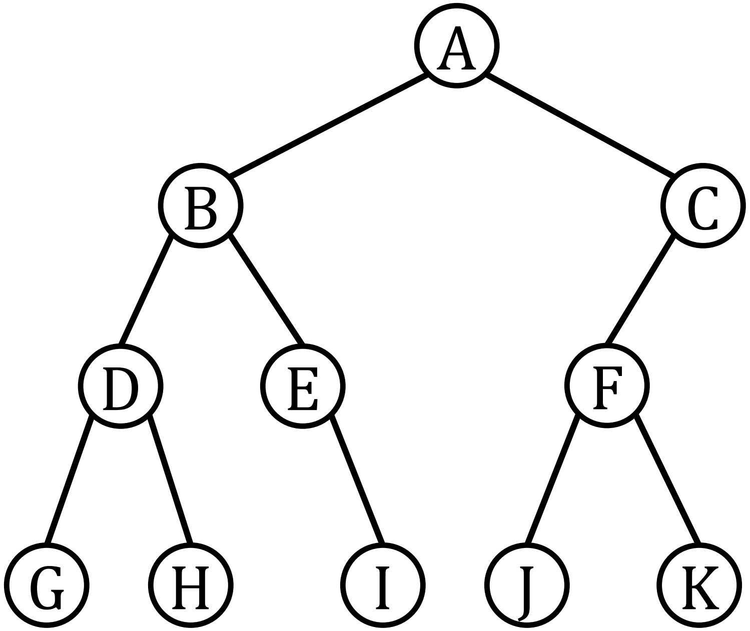
Tree traversals

```
# Remember:
#     BinTree = Optional[Node[T]]
#     NodeIter = Iterator[Node[T]]

def preorder(t: BinTree) -> NodeIter:
    """Iterator to visit the nodes in preorder"""
    if t:
        yield t.data
        yield from preorder(t.left)
        yield from preorder(t.right)


def postorder(t: BinTree) -> NodeIter:
    """Iterator to visit the nodes in postorder"""
    if t:
        yield from postorder(t.left)
        yield from postorder(t.right)
        yield t.data
```

Tree traversals

```
def inorder(t: BinTree) -> NodeIter:
    """Iterator to visit the nodes in inorder"""
    if t:
        yield from inorder(t.left)
        yield t.data
        yield from inorder(t.right)


def level_order(t: BinTree) -> NodeIter:
    """Iterator to visit the nodes by levels"""
    if not t:
        return
    q: deque[Node] = deque([t])
    while q:
        n = q.popleft()
        yield n
        if n.left:
            q.append(n.left)
        if n.right:
            q.append(n.right)
```

Tree visitors



A visitor is a function that is applied to all nodes of a tree.

Similar to the **map** function applied to iterables (e.g., lists)

```
def visit_preorder(t: BinTree[T], f: Callable[[T], T]) -> None:
    """Visits all the nodes of the tree in preorder and applies
       f() to the data. The result is reassigned to the data"""
```

Type: **`Callable[[T1,…Tn], Tr]`**.
A function with parameters `[T1,…,Tn]` and result `Tr`.

Tree visitors

```
def visit_preorder(t: BinTree[T], f: Callable[[T], T]) -> None:
    """Applies f to all data in preorder"""
    if t:
        t.data = f(t.data)
        visit_preorder(t.left, f)
        visit_preorder(t.right, f)

# Example
def square(x: int) -> int:
    return x*x

t: Bintree[int] = …  # some tree constructor
visit_preorder(t, square) # squares all data in the tree

# equivalent with lambda: visit_preorder(t, lambda x: x*x)
```

EXERCISES

Expression tree

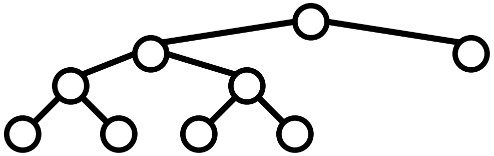
- Modify `infixExpr` for a nicer printing:
  - Minimize number of parenthesis.
  - Add spaces around + (but not around *).

- Extend the functions to support other operands, including the unary – (e.g., –a/b).

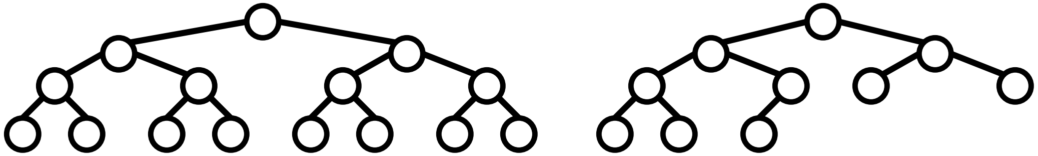
Binary tree types

Design the function "`def check_type(t: BinTree) -> bool:`" for each type tree.

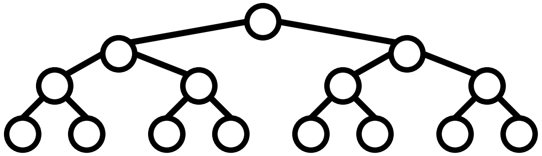
- **Full Binary Tree**: each node has 0 or 2 children.



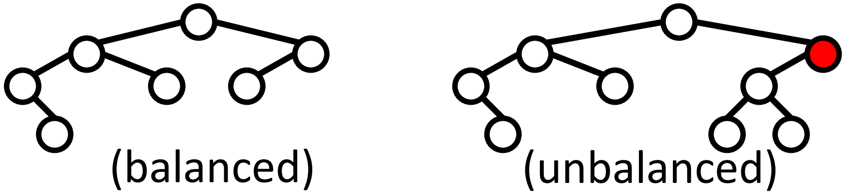
- **Complete Binary Tree:** all levels are filled entirely with nodes, except the lowest level. In the lowest level, all nodes reside on the left side.



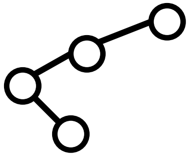
- **Perfect Binary Tree:** all the internal nodes have exactly two children and all leaves are at the same level.



- **Balanced Binary Tree:** the tree height is $O(\log n)$, where n is the number of nodes. The height of the left and right subtrees of each node should vary by at most one.



- **Degenerated Binary Tree:** every internal node has a single child.



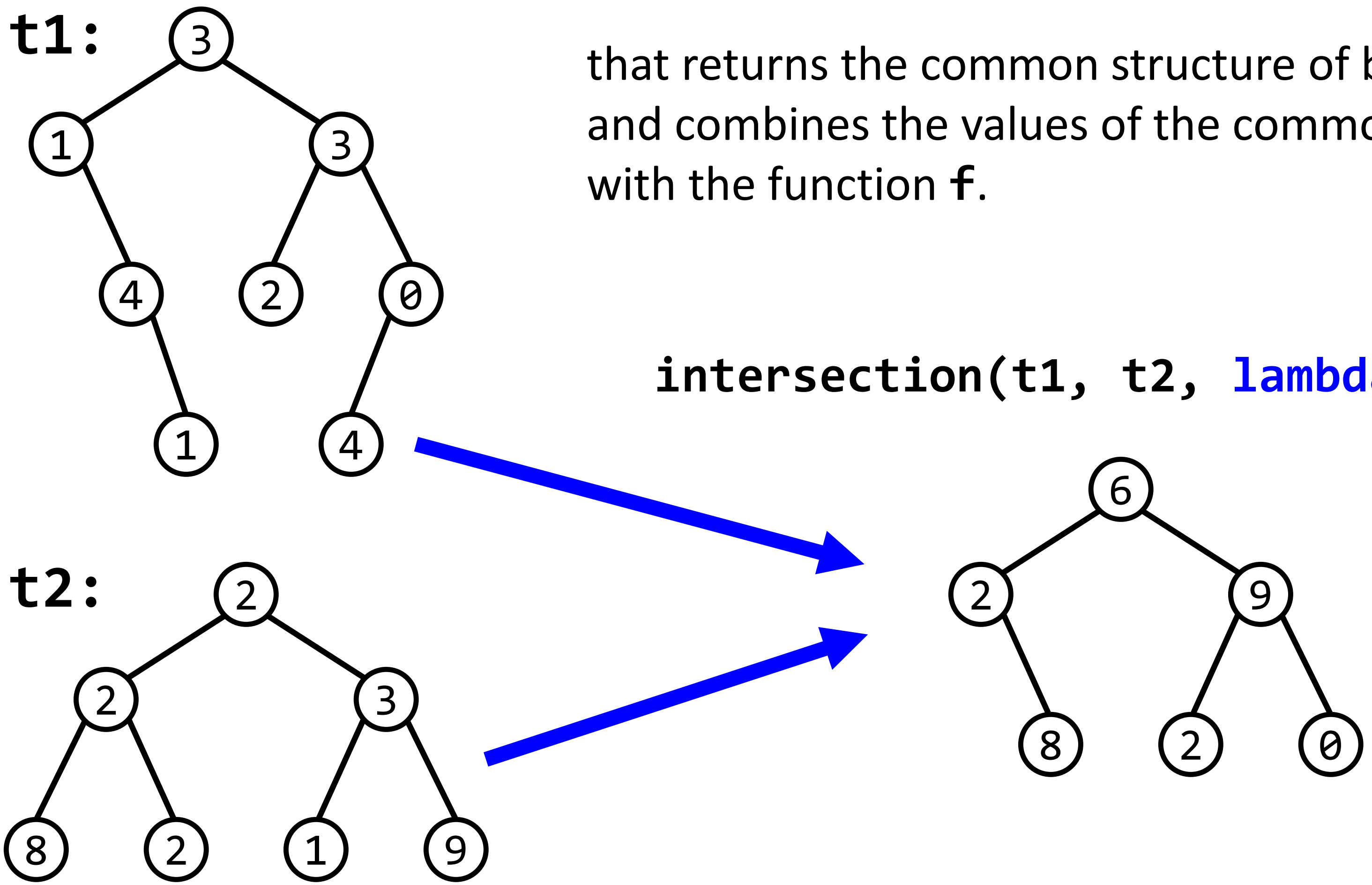
Intersection of binary trees

Design the function

```
def intersection(t1: BinTree[T], t2: BinTree[T],
                 f: Callable[[T, T], T]) -> BinTree[T]:
```

that returns the common structure of both trees and combines the values of the common nodes with the function **`f`**.

```
intersection(t1, t2, lambda x, y: x*y)
```



Traversals: Full Binary Trees

- A Full Binary Tree is a binary tree where each node has 0 or 2 children.
- Draw the full binary trees corresponding to the following tree traversals:
  - Preorder: 2 7 3 6 1 4 5; Postorder: 3 6 7 4 5 1 2
  - Preorder: 3 1 7 4 9 5 2 6 8; Postorder: 1 9 5 4 6 8 2 7 3
- Given the pre- and post-order traversals of a binary tree (not necessarily full), can we uniquely determine the tree?
  - If yes, prove it.
  - If not, show a counterexample.

Traversals: Binary Trees

- Draw the binary trees corresponding the following traversals:
  - Preorder: 3 6 1 8 5 2 4 7 9; Inorder: 1 6 3 5 2 8 7 4 9
  - Level-order: 4 8 3 1 2 7 5 6 9; Inorder: 1 8 5 2 4 6 7 9 3
  - Postorder: 4 3 2 5 9 6 8 7 1; Inorder: 4 3 9 2 5 1 7 8 6

- Describe an algorithm that builds a binary tree from the preorder and inorder traversals.

Drawing binary trees

We want to draw the skeleton of a binary tree as it is shown in the figure. For that, we need to assign (x, y) coordinates to each tree node. The layout must fit in a pre-defined bounding box of size $W \times H$, with the origin located in the top-left corner. Design the function:

```
T = TypeVar('T')
Coordinate = tuple[float, float]
Coordinates = dict[Bintree, Coordinate]

def draw(t: Bintree, w: float, h: float) -> Coordinates:
```

that returns a dictionary with the coordinates of all tree nodes in such a way that the lines that connect the nodes do not cross.