On the Stability of Generalized Second Price Auctions with Budgets

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Abstract. The Generalized Second Price (GSP) auction used typically to model sponsored search auctions does not include the notion of budget constraints, which is present in practice. Motivated by this, we introduce the different variants of GSP auctions that take budgets into account in natural ways. We examine their stability by focusing on the existence of Nash equilibria and envy-free assignments. We highlight the differences between these mechanisms and find that only some of them exhibit both notions of stability. This shows the importance of carefully picking the right mechanism to ensure stable outcomes in the presence of budgets.

1 Introduction

Advertising on Internet search engines has evolved into a phenomenal driving force both for the search engines and the advertising businesses. It is a modern and rapidly growing method that is now being implemented in various other popular sites beyond search engines, such as blogs, and social networking sites. Although some rightful concern has been raised regarding privacy issues and distinguishability from non-sponsored results, there are clear advantages to the advertisers who can efficiently reach their target audiences and observe the results of their ad campaign within days or even hours. At the same time, online ads account for a large share of the profits for search engines and other participating web-sites. Even the web-user experience can be enhanced, by the delivery of additional information relevant to their queries.

In a typical instance, a user queries a search engine for a particular keyword of commercial interest, and the search engine determines the ads to be displayed by means of an auction. The prevailing system uses a pay-per-click policy, i.e.,

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it only charges an advertiser when the user clicks on the corresponding link and is diverted to the advertiser' s web-site.

The mechanism used can be viewed as an auction for multiple homogenous indivisible items, the advertisement areas available, with single-demand buyers, since it is not desirable for the same advertisement to appear more than once. Such auctions can find applications in a wide variety of scenarios besides Internet advertising where buyers have a valuation per unit of a particular good but the setting is restricted to selling only single fixed-sized bundles. For example, consider selling different fixed-sized shipments of a food product when the seller cannot send more than one shipment to the same destination or frequency spectrum auctions of different fixed-sized bandwidths where regulations do not permit buying more than one continuous bandwidth. Noting that the mechanisms can be applied in much different scenarios, we will work in the context of Internet search advertising both because of its wide-spread application today and also because of the significant focus it receives in related literature.

In the early history of sponsored search auctions, the allocation of slots to advertisers was determined by a *first-price auction*, as in the systems originally used by Overture. Later on, Google was the first to switch to a *second-price auction*, an approach which demonstrated superior characteristics and was quickly adopted by the rest of the major search engines. The main idea is that the advertisers declare how much they are willing to pay for a click to their ad but they are charged instead a lesser amount equal to the next lower competing bid. Apart from its elegant simplicity, this scheme has been quite successful in terms of its generated revenue as well. In the literature, this system is commonly known as the *Generalized Second-Price* (GSP) auction and, by now, a large volume of work has emerged on the study of GSP auctions and related mechanisms, see e.g., the surveys [15] and [17].

However, an aspect that has been often ignored, especially in the early literature, is the presence of a budget constraint, requested from the advertisers to limit their exposure and expenditure. We believe this is a key parameter, essential in accurately understanding and evaluating the systems used in practice.

Our contribution. Our main conceptual contribution is a study of Generalized Second-Price auctions under the presence of budgets. First we showcase that ignoring these constraints might lead to unstable outcomes. We then introduce three simple and natural extensions of the GSP mechanism, that take budgets into account. As it is not straight-forward to define a single natural mechanism, we define these variants motivated by the key desirable properties of second-price auctions. For all mechanisms, we investigate the existence of Nash equilibria and envy-free assignments, which are the main notions of stability that have been considered in the literature. For the first mechanism, we show that a Nash equilibrium might not always exist, yet an envy-free assignment is always achievable. For the other two mechanisms, we show that they always possess envy-free Nash equilibria, in fact we show that any envy-free assignment can be realized as an equilibrium of the mechanisms under consideration. In our model, we consider the budget as part of a bidder's strategy, i.e., we have *private* budgets. An interesting and surprising outcome of our study is that in the case of public budgets Nash equilibria do not always exist, despite the existence of envy-free assignments. In contrast to mechanism design problems, where having public budgets usually eases the design of a truthful algorithm, here we realized that having public budgets may eliminate the existence of stable profiles. Of separate interest might be an algorithmic process than can construct an envy-free assignment in our model.

1.1 Related work

Varian [19] and Edelman et al. [11] have been the two seminal works on equilibrium analysis of GSP auctions without budgets. They established the existence of a Nash equilibrium which also satisfies other desirable stability properties such as being welfare-maximizing and envy-free. A further analysis of envy-free Nash equilibria, by taking into account the quality factor of the advertisers was also provided in [16].

The notion of budget constraints has been introduced in various models and objectives, such as, among others, in [4], [6], [12], [7] and [13]. Recent work on truthful mechanism design, mainly inspired by the clinching auction of Ausubel [3], has led to the introduction of truthful Pareto-optimal mechanisms in the presence of budgets, see e.g. [10, 14, 9]. Ashlagi et al.introduced the model we'll be using in [2]. However, all these mechanisms employ techniques that are very different from the GSP scheme in order to achieve truthfulness. As a result, they lack the simplicity of second price auctions at the expense of achieving better properties.

The work of Arnon and Mansour [1] is conceptually closer to our approach. They studied second-price auctions with budget constraints but their model simplifies the items for sale to clicks, as opposed to the slots, allowing a player to potentially receive more or less clicks than a single slot could offer. This deviates from the one player per slot paradigm used in practice.

Finally, a different direction that has been pursued recently is the performance of mechanisms in terms of the generated social welfare. Price of Anarchy analysis for auctions was initiated in [8], see also [5], for sponsored search auctions without budgets. For certain settings with budget constraints, some results have been recently obtained in [18] (which however do not have any implications for our proposed mechanisms). Our work does not focus on Price of Anarchy, which we leave for future research, but on existence of stability concepts.

2 Preliminaries

2.1 Model

Our model is the same as in Ashlagi et al. [2], a natural extension to budget limited players of the model introduced by Varian [19] which is widely adopted in related literature.

We assume we have k slots, each with a fixed, distinct[4] and publicly known click-through rate (CTR), θ_j for slot j, representing the number of clicks received in a fixed time period (typically a day), independently of the advertisement displayed. Let us order the slots such that $\theta_1 > \theta_2 > \ldots > \theta_k$. Even though the click-through rates are probabilistic in nature, we will make the typical assumption that they are *deterministically* realized for simplification purposes; θ_i will really correspond to the expected click-through rate of slot i. Contrary to the numerical ordering, we will typically use the terminology "higher" and "lower" slots referring to slots of higher and lower CTR. Finally, for ease of illustration, we will ignore the bidder-dependent quality factor that is usually incorporated in calculating click-through rates in the separable model.

We have $n \geq k$ players (advertisers). Each player i has a private *valuation* v_i representing the perceived value per click. Each player also has a *budget constraint* B_i, indicating the total amount he is willing to spend in a fixed time period, not on a per click basis. We will also assume that these budget values are pairwise distinct. This assumption has been necessary in other works as well [2], and affects many properties in related mechanisms [10, 2]. In fact, as we will exhibit later on, envy-free assignments, which is one of the stability concepts we are interested in, are not guaranteed to exist when budgets are not distinct.Hence, similarly to previous works, we also choose to adopt the distinctness of budgets.

Each player i is interested in maximizing $\theta_{s(i)}(v_i - p(i))$, where $s(i)$ is the slot assigned to i and $p(i)$ the accompanying price per click requested by the respective mechanism. At the same time, he must also satisfy the budget constraint, $\theta_{s(i)}p(i) \leq B_i$. If this condition holds, we say the player can afford slot $s(i)$. We wish to enforce strict budget constraints so we define the utility of the players whose budget constraints are violated to be minus infinity as is typically done in the literature; any other negative value would also serve our purpose (i.e., budget violations are less desirable than not getting a slot). More formally,

$$u_i = \begin{cases} 0, & \text{if } i \text{ was not awarded a slot,} \\ \theta_{s(i)}(v_i - p(i)), & \text{if } \theta_{s(i)}p(i) \leq B_i, \\ -\infty, & \text{otherwise.} \end{cases}$$

2.2 Second-Price Auctions under Budgets

The players submit *value-bids* b_i, representing the maximum amount they are willing to pay per click. These bids do not necessarily form a truthful declaration of the players' values to the mechanism. Similarly, the players also submit a *budget-bid* g_i to declare their budget. We will use the term bid to refer to the combination of these two types or to one particular type when clear from context. In the case of ties, we assume there exists a fixed a priori defined ordering of the players based on which tie-breaking is resolved.

It is not trivial to introduce mechanisms that take budgets into account in a straightforward and natural way. To address this, we first ponder what con-

[4] This assumption is derived from the distinct space these slots occupy on a web-page.

stitutes a second-price mechanism by noting some key properties of generalized second-price auctions:

- The slot allocation should be performed by a simple and efficient process.
- The allocation should be in accordance with the bid ordering. If a player raises his bid he should be getting at least the slot he was getting before and should he lower his bid he should be getting at most the previous slot.
- Furthermore, if a bidder raises his bid, this should cause his total payment to potentially rise and respectively lowering his bid potentially lowers his payment.
- Finally, the price per click for each slot should be determined by either the next lower bid, the bid of the player awarded the next slot or the minimum bid required to obtain the slot. While these three concepts coincide in the regular GSP mechanism, this is not the case when one introduces budgets.

We first consider the GSP mechanism without budgets in our context, only to highlight that ignoring budgets can lead to unstable outcomes.

Definition 1 (Budget-Oblivious). *The* budget-oblivious *second-price auction, in short,* BOSP*, orders the value-bids in decreasing order and then assigns the slots in that order,* ignoring *the budget constraints. Naturally, the price for each slot is determined by the immediately lower value-bid.*

Since BOSP, as is shown in Section 3, does not have good stability properties, we turn our attention on mechanisms that respect the budget constraints by not assigning slots/prices to players that can afford them as declared by their budget-bids. The first interpretation of second-price pricing, charging the next lower bid, leads us to the following mechanism.

Definition 2 (Budget-Conscious by Price). *We define the* budget-conscious by price *second-price auction, in short,* BCSP(PRICE)*, as the mechanism which first orders the value-bids in decreasing order and assigns a price per click for each player equal to the immediately lower value-bid in the bid ordering. Then,* BCSP(PRICE) *assigns the players to slots in order of decreasing value-bids,* respecting *the budget constraints of each player as declared by their budget-bids, by assigning each player to the highest unassigned slot he can afford with his assigned price. If the player cannot afford any slot, he is left unassigned and he is not charged anything.*

Note that under this mechanism a player assigned to a slot might end up paying more per click than a player in a higher slot but we are guaranteed that all budget constraints of assigned players are satisfied. Also note that some slots might end up unassigned if no player can afford to occupy them.

The mechanism above is a natural way to guarantee budget compliance but raises a fairness issue, as players might be declaring value-bids as the maximum amount they are willing to pay and getting a slot that they cannot afford to pay if they were to pay their own bid. As will be evident in the later sections, this can lead to players intentionally raising their bid to just below their competitor's bid. The following mechanism addresses this.

Definition 3 (Budget-Conscious by Bid). *We define the* budget-conscious by bid *second-price auction, in short,* BCSP(BID)*, similarly to* BCSP(PRICE) *except the mechanism now requires the players to be able to afford their slot if they were to pay a price per click equal to their own value-bid. The players are ordered in decreasing order of value-bids and the price of each player is set to the next lower bid. Then, the players are assigned from the highest bidder to the lower, one by one, to the highest available unassigned slot that they can afford should they were to pay their own bid.*

The second way of interpreting second-price prices, charging prices equal to the value-bid of the player ending up occupying the next slot, first, has definitional issues since we cannot know if our player can afford a slot without knowing who gets the next one and secondly, might charge a player more than his bid. For these reasons, we do not investigate mechanisms of this type in this work.

Finally, we introduce a mechanism that considers what the players are willing and afford to pay for a slot by considering the minimum as implied from their value and budget bids. This mechanism essentially captures pricing by charging the minimum amount required to obtain a slot.

Definition 4 (Best Offer Budget-Conscious). *We define the* best offer budget-conscious *second-price auction, in short,* BCSP(BEST OFFER)*, as the mechanism which intuitively awards each slot to the player that can offer the most "money" but charges them the next lower amount offered. More formally, each slot s, one by one from higher to lower, is awarded to the unassigned player with the largest $min\{b_i, g_i/\theta_s\}$ and he is charged a price per click equal to the second largest such value among unassigned players. We note that under this mechanism, the price charged for each slot is the minimum bid required to secure the slot. Alternatively, one can think of the slot rewarded to the player with the highest $min\{\theta_s b_i, g_i\}$, and paying the second highest such amount, representing the total offer of the player and the total price charged.*

It should be pointed out that a slot in BCSP(BEST OFFER), in turn of decreasing CTR, is offered to the player who is willing to pay the most, given that does not violate his declared budget; he is then charged what the second such player would pay (not counting players who already got a slot). Whereas in the previous two budget-conscious mechanisms, each player, in turn of its bid, chooses the best object he can afford and pays the bid of the next player in line. The above two approaches are obviously equivalent in any mechanism that does not refuse giving a slot to a player who cannot afford it. However, once we introduce into the mechanism the additional requirement of refusing to give objects to anybody who cannot afford it, then the above distinction becomes necessary.

In all mechanisms, we assume that players not awarded a slot are not charged a payment and that the price of the lowest bidding player is zero, should he be awarded a slot. Finally, given a finite set of players, we note that the allocation of slots and pricing can be determined efficiently in all defined mechanisms.

2.3 Stable assignments

It is easy to see that none of the mechanisms defined above are *incentive-compatible.* There are cases where a player might receive a higher utility by "lying" about his value and getting a lower slot at a beneficial price, even if other players are truthfully bidding their values. Naturally, we turn our attention to notions of stability, a requirement to analyze significant properties of these auctions and in general a desired property for the advertisers as well. As usual, we will focus on the notion of *Nash equilibrium.*

Definition 5 (Nash Equilibrium). *A profile of bids, $\langle b_i, g_i\rangle$ for each player i, forms a Nash equilibrium if no player has an incentive to deviate to a different strategy $\langle b'_i, g'_i\rangle$, for any $\langle b'_i, g'_i\rangle$.*

In related work [19, 11], the notion of *symmetric* or *envy-free* equilibrium was defined. Under the generalized second-price auction without budget constraints, this class of envy-free equilibria is a subset of Nash equilibria.

Definition 6 (Envy-Free Assignment). *We define an envy-free assignment as a slot allocation $s(\cdot)$, where no slot is left unassigned, along with a set of prices per click $p(\cdot)$, assigning slot $s(i)$ to player i and charging him $p(i)$ per click, such that for all players i we have*

$$\forall i' \text{ with } 1 \le s(i') \le k, u_i \ge \begin{cases} \max\{\theta_{s(i')}(v_i - p(i')), 0\}, & \text{if } \theta_{s(i')}p(i') \le B_i, \\ 0, & \text{otherwise,} \end{cases}$$

where u_i is the utility of player i as defined earlier.

Note that an envy-free assignment also guarantees *rationality*: $p(i) \le v_i$ for all players i. We say that an envy-free assignment is *realizable* under a certain mechanism, if a set of bids exists such that the allocation and pricing generated by the mechanism under this set of bids matches the allocation and pricing of the envy-free assignment.

Under BOSP, where slot allocation depends only on the value-bids and not on the budgets or budget-bids, the constraints on the bids that realize an envy-free assignment are stricter than those of a Nash equilibrium. The same holds for BCSP(BID), as all players can pay their own bid for their slot and intuitively cannot be forced out of position by someone else' s bid[5]. Similarly, under BCSP(BEST OFFER), a player cannot get a higher slot without paying more than the current player occupying the slot; again a realizable envy-free assignment effectively produces a Nash equilibrium. Hence, under the mechanisms BOSP, BCSP(BID), BCSP(BEST OFFER), the realizable envy-free assignments form a subset of the set of Nash equilibria.

Under BCSP(PRICE) however, the slot allocation is dependent on budgets and intuitively, one could alter the allocation to his benefit by forcing other

[5] In more detail, the instability arises when some player can alter his bid to raise someone else' s price, forcing the mechanism to evict him from his slot based on budget constraints and subsequently benefiting the first player.

players out of budget, hence there might exist bids that realize an envy-free assignment but do not form a Nash equilibrium.

For the other direction, it is trivial to find an example where the outcome of a Nash equilibrium is not an envy-free assignment in all mechanisms building on the intuition that someone might be envious of someone else' s higher slot but they are not able to get it at that price.

3 The Budget-Oblivious Second-Price Auction

BOSP lacks the notions of stability defined earlier.

Theorem 1. *There are settings where no Nash equilibrium exists under* BOSP*.*

A simple counterexample is a game with two slots with rates 1 and 0.4 and three players with value/budget 50/50, 16/5 and 8/2 and the proof is presented in the full version.

Since under BOSP, realizable envy-free assignments are a subset of Nash equilibria it follows that:

Corollary 1. *There are settings where no realizable envy-free assignment exists under* BOSP*.*

Let us point out here that in Lemma 3, stated below, we show that envy-free assignments always exist (under the assumption of distinctness of budgets). By the above corollary, such assignments are not realizable under BOSP.

4 The Budget-Conscious by Price Second-Price Auction

We now turn our attention to BCSP(PRICE). We first show

Theorem 2. *There are settings in which no Nash equilibrium exists for* BCSP-(PRICE)*.*

To prove Theorem 2, we will first show that Nash equilibria do not always exist in the special case where players are budget-bidding their true constraints and then extend the result to the general case.

Lemma 1. *There are settings where players are budget-bidding their true constraints, and no Nash equilibrium exists under* BCSP(PRICE)*.*

Proof. Consider two slots with $\theta_1 = 1, \theta_2 = 0.4$ and 3 players with attributes as shown in Figure 1. We assume that the tie-breaking ordering favors player 3 and then player 2. In order to show that a Nash equilibrium does not exist we have to consider all orderings of bids and for each such case all possible slot assignments. Intuitively, we will showcase two types of instability. If a bid is low so that the player paying it has the budget constraint satisfied then it can be easily overbid or if a bid is high then underbidding below it will force that player out of budget for the slot.

	Player 1	Player 2	Player 3
Value	50	16	8
Budget	50	5	2

Fig. 1. Example of non-existence of Nash equilibrium under BCSP(PRICE), having two slots with $\theta_1 = 1, \theta_2 = 0.4$.

We start by considering the case where $b_1 > b_2 > b_3$ and slot 1 is assigned to player 1 and slot 2 to player 2. This means that these players can afford these slots, therefore we must have $b_2 \leq 50$ and $b_3 \leq 12.5$. If $b_2 > 5$ then player 1 can bid $b_2 - \varepsilon > b_3$, for some small ε, and still get slot 1 at a lower price, since player 2 cannot afford it. If $b_2 \leq 5$ then player 3 can bid $b_2 + \varepsilon < b_1$ and gain strictly positive utility.

If player 1 is assigned to slot 2 because he cannot afford it and player 2 gets slot 1, we must have $b_2 > 50$ and $b_3 \leq 5$. If player 1 bids $b_2 - \varepsilon > 5 > b_3$, then player 2's bid will be the highest bid but he will not be able to afford slot 1 anymore, which will end up at player 1 for a lower price per click than before.

The rest of the cases follow similarly and we present them in the full version of this paper. □

Lemma 2. *Under* BCSP(PRICE)*, if a Nash equilibrium exists then a Nash equilibrium also exists where the players are budget-bidding their true constraints.*

To prove Lemma 2, we show that when a player changes his budget-bid to his true budget the same slot allocation can be achieved. This is combined with an adequate shifting of value-bids that maintains the same prices and that concludes the proof of Lemma 2 and of Theorem 2.

Despite the non-existence of Nash equilibria, all envy-free assignments are realizable under BCSP(PRICE) (and there exists at least one such assignment).

Theorem 3. *There exists an envy-free assignment which is realizable under* BCSP(PRICE)*.*

The proof of the Theorem is based on Lemma 3 and Lemma 4 below.

Lemma 3. *Under budget constraints, there always exists an envy-free assignment.*

Proof outline: We present a proof outline. We first note that an envy-free assignment can always be obtained from the mechanism in [2]. We present here an alternate way to obtain an envy-free assignment. Our procedure produces a different assignment than that of [2] in some settings, has a shorter proof and we believe it has some value on its own.

We run the following procedure that completes with all slots assigned in an envy-free assignment. The slots are initially free and have an assigned infinite price. We pick a free slot and lower its price until some player can obtain non-negative utility, at which point we award him the slot at that price and stop lowering it.

We repeat similarly with other slots but allow players that are assigned to request the particular slot if its price makes it more beneficial for them. This leads to a reassignment of the player. We then effectively maintain the envy-free conditions during the whole process.

We have to be more careful when a player can receive equal utility from his assigned slot and the slot whose price is being lowered as we cannot further lower either price without potentially inducing envy or infinite switch loops among the players. We deal with this by simultaneously lowering both prices in a uniform manner and by a careful analysis on potential outcomes and progress measures to guarantee the termination and maintenance of envy-free conditions. For more details, we refer the reader to the full version of the paper. □

Lemma 4. *Any envy-free assignment is realizable under* BCSP(PRICE)*.*

We define value bids such that the mechanism produces the same allocation and prices as in the envy-free assignment. The budget-bids can be set to match the true budgets of the players.

5 Budget-Conscious by Bid and Best Offer Second-Price Auctions

Recall that under BCSP(BID) and BCSP(BEST OFFER), realizable envy-free assignments are a subset of Nash equilibria. We are going to show that for these two mechanisms, Nash equilibria exist, by establishing that envy-free assignments are realizable in both BCSP(BID) and BCSP(BEST OFFER). Given an envy-free assignment, we pick the bids such that the mechanism assigns the players in the desired slots and achieves the same prices. The proofs of the following theorems are presented in the full version.

Theorem 4. *Under* BCSP(BID)*, there is always a Nash equilibrium that produces an envy-free assignment.*

Theorem 5. *Under* BCSP(BEST OFFER)*, there is always a Nash equilibrium that produces an envy-free assignment.*

Note that in both theorems we need to set the budget-bids of the players different from their true budgets. It turns out this is necessary, a perhaps surprising result as having budgets publicly known was necessary for truthful mechanisms in related work [10, 2].

Theorem 6. *There are settings with public budgets, where a Nash equilibrium does not exist under both* BCSP(BID) *and* BCSP(BEST OFFER)*.*

As envy-free assignments realizable under BCSP(BID) or BCSP(BEST OFFER) form a subset of Nash equilibira of these two mechanisms, respectively, the Theorem above implies that there are settings where neither of these mechanisms can realize any envy-free assignment (guaranteed to exist by Lemma 3), unless players are allowed to bid non-true budgets.

6 Discussion

The consideration of several variants of mechanisms, introduced not out of idle curiosity, but as representations of all the natural answers to natural questions raised by the introduction of budgets, and the examination of their properties and differences is what we consider as our primary contribution in this work. We believe our work can serve as a starting point for studying further the properties of GSP auctions under budget constraints. Although we studied these auctions in the context of sponsored search, second-price auctions are widely used in many different settings both in off-line and on-line scenarios. As such, our results are applicable in a much wider context.

In the area of sponsored search, a further step is required towards the more accurate modeling of the deployed systems. In practice, a player can transition between slots during a time period, as players are moderated according to their budget depletion rate. Similarly to the majority of related work on keyword auctions with budgets, we chose to study the static setting first both as a stepping stone and in its own interest for settings outside of keyword auctions. An analysis on the effects of budget constraints in a dynamic setting that would extend upon our results, would contribute towards a more accurate modeling of sponsored search auctions. Another interesting direction for future research is to evaluate the performance of these mechanisms in terms of the generated welfare. This type of Price of Anarchy analysis for non-truthful auctions was initiated in [8], and for certain settings with budget constraints, some results have been recently obtained in [18].

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References

1. Arnon, A., Mansour, Y.: Repeated budgeted second price ad auction. In: Proceedings of the 4th Symposium on Algorithmic Game Theory. pp. 7–18. Springer-Verlag (2011)
2. Ashlagi, I., Braverman, M., Hassidim, A., Lavi, R., Tennenholtz, M.: Position auctions with budgets: existence and uniqueness. B.E. journal of Theoretical Economics Advances (to appear, 2013)
3. Ausubel, L.M.: An efficient ascending-bid auction for multiple objects. The American Economic Review 94(5), 1452–1475 (2004)
4. Borgs, C., Chayes, J., Immorlica, N., Mahdian, M., Saberi, A.: Multi-unit auctions with budget-constrained bidders. In: ACM Conference on Electronic Commerce (EC). pp. 44–51 (2005)

5. Caragiannis, I., Kaklamanis, K., Kanellopoulos, P., Kyropoulou, M., Lucier, B., Paes Leme, R., Tardos, E.: On the efficiency of equilibria in generalized second price auctions. arxiv:1201.6429 (2012)
6. Chakrabarty, D., Zhou, Y., Lukose, R.: Budget constrained bidding in keyword auctions and online knapsack problems. In: Workshop on Internet and Network Economics (WINE). pp. 566–576 (2008)
7. Charles, D., Chakrabarty, D., Chickering, M., Devanur, N.R., Wang, L.: Budget smoothing for internet ad auctions: a game theoretic approach. In: Proceedings of the fourteenth ACM conference on Electronic commerce. pp. 163–180. EC '13, ACM, New York, NY, USA (2013), http://doi.acm.org/10.1145/2482540.2482583
8. Christodoulou, G., Kovács, A., Schapira, M.: Bayesian combinatorial auctions. In: ICALP (1). pp. 820–832 (2008)
9. Colini-Baldeschi, R., Henzinger, M., Leonardi, S., Starnberger, M.: On multiple keyword sponsored search auctions with budgets. In: Czumaj, A., Mehlhorn, K., Pitts, A.M., Wattenhofer, R. (eds.) Automata, Languages, and Programming - 39th International Colloquium, ICALP 2012. Lecture Notes in Computer Science, vol. 7392, pp. 1–12. Springer (2012)
10. Dobzinski, S., Lavi, R., Nisan, N.: Multi-unit auctions with budget limits. In: Proceedings of the 2008 49th Annual IEEE Symposium on Foundations of Computer Science. pp. 260–269. FOCS '08, IEEE Computer Society, Washington, DC, USA (2008), http://dx.doi.org/10.1109/FOCS.2008.39
11. Edelman, B., Ostrovsky, M., Schwarz, M.: Internet advertising and the generalized second-price auction: Selling billions of dollars worth of keywords. The American Economic Review 97(1), 242–259 (2007)
12. Feldman, J., Muthukrishnan, S., Pal, M., Stein, C.: Budget optimization in search-based advertising auctions. In: ACM Conference on Electronic Commerce (EC). pp. 40–49 (2007)
13. Fiat, A., Leonardi, S., Saia, J., Sankowski, P.: Single valued combinatorial auctions with budgets. In: Proceedings of the 12th ACM conference on Electronic commerce. pp. 223–232. EC '11, ACM, New York, NY, USA (2011), http://doi.acm.org/10.1145/1993574.1993609
14. Goel, G., Mirrokni, V.S., Leme, R.P.: Polyhedral clinching auctions and the adwords polytope. In: ACM Symposium on Theory of Computing (STOC). pp. 107–122 (2012)
15. Lahaie, S., Pennock, D., Saberi, A., Vohra, R.: Sponsored search auctions. In: Nisan, N., Roughgarden, T., Tardos, E., Vazirani, V. (eds.) Algorithmic Game Theory, chap. 28, pp. 699–716. Cambridge University Press (2007)
16. Lahaie, S., Pennock, D.: Revenue analysis of a family of ranking rules for keyword auctions. In: Proc. ACM Conference on Electronic Commerce (EC). pp. 50–56. San Diego, California, USA (June 2007)
17. Maillé, P., Markakis, E., Naldi, M., Stamoulis, G.D., Tuffin, B.: Sponsored search auctions: an overview of research with emphasis on game theoretic aspects. Electronic Commerce Research 12(3), 265–300 (2012)
18. Syrgkanis, V., Tardos, É.: Composable and efficient mechanisms. In: ACM Symposium on Theory of Computing (STOC 2013). pp. 211–220 (2013)
19. Varian, H.: Position auctions. International Journal of Industrial Organization 25, 1163–1178 (2005)