The EM algorithm

- Start with guess for values of your model parameters
- **Step E**
  - Compute distribution of the missing/latent data given the observed data and your current guess of the model parameters.
  - Use the missing/latent data distribution to compute the expectation of the likelihood function with respect to the unobserved variables.
- **Step M**
  - Use the expected likelihood function with no unobserved variables to maximize the function as you would in the fully observed case, to get a new estimate of your model parameters.
- Repeat steps E-M until convergence (no further changes).
The EM algorithm - Example

- Three coins with probability of heads $(\lambda, p_1, p_2)$.
- Hidden variable coin$_0$ $(\lambda)$: $Y = \{H, T\}$
- $Y = H \Rightarrow$ flip coin$_1$ ($p_1$) three times
- $Y = T \Rightarrow$ flip coin$_2$ ($p_2$) three times
- Observed sequence: $X = \{\text{HHT, HTT, TTT, HHH}\}$
The EM algorithm - Example

- **Start with a guess model** \( \mu = (\lambda, p_1, p_2) \)

- **Step E - Expectation**

  Use current model parameters \( \mu \) to compute probability distribution of hidden data given the observations:

  \[
  P_\mu(H \mid x_i) = \frac{P_\mu(x_i, H)}{P_\mu(x_i)}; \quad P_\mu(T \mid x_i) = \frac{P_\mu(x_i, T)}{P_\mu(x_i)} \quad \forall x_i \in X
  \]

  where \( P(x_i, H), P(x_i, T), \) and \( P_\mu(x_i) \) are computed from current model:

  \[
  P_\mu(HHT, H) = \lambda p_1^2 (1 - p_2) \\
  P_\mu(HTT, T) = (1 - \lambda) p_2 (1 - p_2)^2
  \]

  \[
  \ldots \text{etc.} \ldots
  \]

  \[
  P_\mu(x_i) = P_\mu(x_i, H) + P_\mu(x_i, T) \quad \forall x_i \in X
  \]

  Compute expected number of occurrences for hidden variable values:

  \[
  E[Y = H] = \sum_i P(H \mid x_i) \\
  E[Y = T] = \sum_i P(T \mid x_i)
  \]
The EM algorithm - Example

- **Step M - Maximization**
  
  Use expectations computed above to compute new MLE estimates of model parameters given observations
  \( X = \{ \text{HHT, HTT, TTT, HHH, HTT} \} \)

  \[ \lambda' = \frac{E[Y=H]}{N} \]

  \[ p_1' = \frac{2 \cdot P(\text{HHT}, \text{H}) + 1 \cdot P(\text{HTT}, \text{H}) + 0 \cdot P(\text{TTT}, \text{H}) + 3 \cdot P(\text{HHH}, \text{H})}{E[Y=H]} \]

  \[ p_2' = \frac{2 \cdot P(\text{HHT}, \text{T}) + 1 \cdot P(\text{HTT}, \text{T}) + 0 \cdot P(\text{TTT}, \text{T}) + 3 \cdot P(\text{HHH}, \text{T})}{E[Y=\text{T}]} \]