Non-linear Rewrite Closure and Weak Normalization

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A rewrite closure is an extension of a term rewrite system with new rules, usually deduced by transitivity. Rewrite closures have the nice property that all rewrite derivations can be transformed into derivations of a simple form. This property has been useful for proving decidability results in term rewriting. Unfortunately, when the term rewrite system is not linear, the construction of a rewrite closure is quite challenging.

In this paper, we construct a rewrite closure for term rewrite systems that satisfy two properties: the right-hand side term in each rewrite rule contains no repeated variable (right-linear) and contains no variable at depth greater than one (right-shallow). The left-hand side term is unrestricted, and in particular, it may be non-linear.

As a consequence of the rewrite closure construction, we are able to prove decidability of the weak normalization problem for right-linear right-shallow term rewrite systems. Proving this result also requires tree automata theory. We use the fact that right-shallow right-linear term rewrite systems are regularity preserving. Moreover, their set of normal forms can be represented with a tree automaton with disequality constraints, and emptiness of this kind of automata, as well as its generalization to reduction automata, is decidable.

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1. INTRODUCTION

A term rewriting system is a collection of rules that describes how a term may be rewritten into another term. Depending on the application, these rules can be seen either as describing a computation, or as presenting an equational theory. When modeling computation, a term rewriting system (TRS) is used to rewrite a given starting term successively until a normal form is reached. A normal form is a term that can not be rewritten any more and it is usually considered as the result of the
computation. Properties such as reachability, termination (sometimes called strong normalization), weak normalization, and unique normalization become important in this context as they describe the result of the computation. When TRSs are used to study equational theories, properties that relate congruence with reachability, such as confluence, become important.

Term rewriting is a Turing-complete model of computation and hence almost any nontrivial property of general TRSs is undecidable. This motivates our interest in looking at restricted classes of TRSs for which interesting properties, such as confluence and (strong, weak, unique) normalization, may be decidable. When the rewriting rules do not contain any variables, then most of these properties are known to be decidable. So, the interesting question really is what restrictions do we impose on the occurrences of the variables in the rules so as to achieve decidability? Two restrictions are particularly important: linearity, i.e. no variable occurs more than once, and shallowness, i.e. no variable occurs at depth more than one.

This paper presents results for the class of right-shallow right-linear TRSs. In such a TRS, the right-hand side of each rule is restricted to be linear and shallow, while there is no restriction on the left-hand side of rules. In particular, variables can occur multiple times and at arbitrary positions on the left-hand sides of the rules. This class is quite expressive. Specifically, as far as equational theories are concerned, this class is as expressive as general TRSs: the following transformation shows how an arbitrary equation \( l = r \) can be equivalently written using two right-shallow right-linear equations.

\[
l = r \iff l = f(x_1, \ldots, x_m), r = f(x_1, \ldots, x_m),
\]

where \( f \) is a new function symbol and \( x_1, \ldots, x_m \) are the variables occurring simultaneously in \( l \) and \( r \). This implies that we cannot perform completion and obtain, in finite time, a convergent presentation for a right-shallow right-linear TRS. But, if the focus is on the reachability relation, then this paper shows how to obtain, in finite time, a rewrite closure presentation for a right-shallow right-linear TRS. Using the rewrite closure presentation, we obtain the second main result of the paper, which states that weak normalization — whether there is some terminating rewriting sequence starting from every term — is decidable for right-shallow right-linear TRSs.

1.0.0.1 Related Work. In recent years, significant progress has been made in determining decidability of several fundamental properties for restricted classes of TRSs. Restrictions (such as shallowness) can be imposed on both sides of rewrite rules (e.g., shallow TRSs), or just to the left-hand side (left-shallow TRSs), or just to the right-hand side (right-shallow TRSs).

We briefly mention some of the strongest known results. Reachability and joinability are decidable for right-shallow right-linear TRS [Nagaya and Toyama 2002], and even for some weaker restrictions [Takai et al. 2000]. Termination is decidable for right-shallow right-linear TRS [Godoy et al. 2007] and other variants of syntactic restrictions based on the form of the dependency pairs obtained from a TRS [Wang and Sakai 2006]. Confluence is decidable for shallow right-linear TRS [Godoy and Tiwari 2005], and for right-(ground or variable) TRS [Godoy and Tiwari 2004]. The weak normalization problem is decidable for left-shallow left-linear TRS [Nagaya
and Toyama 2002], linear right-shallow TRS and shallow right-linear TRS [Godoy and Tison 2007]. Thus, the second result in this paper extends the results in [Godoy and Tison 2007].

The results in this paper are obtained by using several known results on tree automata. One such result is that right-linear right-shallow TRSs are regularity preserving [Nagaya and Toyama 2002]. We will also use the fact that emptiness is decidable for certain classes of tree automata with disequality constraints [Comon and Jacquemard 2003] and restricted equality constraints (reduction automata) [Dauchet et al. 1995].

A rewrite closure of $R$ is a TRS that preserves the rewrite relation induced by $R$, but it has some additional nice properties. Rewrite closures have already been useful for proving decidability results, like in [Godoy and Tiwari 2004; Godoy and Tison 2007]. Unfortunately, when the TRS is not linear, the construction of a rewrite closure is quite challenging. Some of the techniques used in this paper to construct a rewrite closure for right-shallow right-linear TRSs are borrowed from previous works, most notably from [Godoy and Tiwari 2004; 2005], which were both targeted to showing decidability of confluence for more restricted classes of TRSs.

1.1 Summary of the Results

We briefly summarize the results in the paper and give a high-level overview of the proofs. In Section 3, using some fairly standard transformations, we first convert the given right-shallow right-linear TRS into a right-flat right-linear TRS over a signature that contains only constants and exactly one non-constant function symbol. This is done primarily to simplify the number of cases to consider when doing the proofs.

A rewrite closure presentation of $R$ has the property that whenever $s$ can reach $t$, it can also reach $t$ using a “valley” derivation; that is, all rewrite steps that are decreasing (with respect to some quasi-ordering, such as an ordering based on size) occur before any rewrite step that is increasing. This is easy to achieve for ground rules. Suppose we have the rules $r_1 : a \rightarrow f(b, c)$ and $r_2 : f(b, c) \rightarrow d$. The “peak” derivation, $a \rightarrow r_1 f(b, c) \rightarrow r_2 d$, is not a valley derivation. By adding the rule $r_3 : a \rightarrow d$, deduced by transitivity, the peak can be avoided and we can go from $a$ to $d$ using the valley derivation $a \rightarrow r_3 d$.

In general, whenever there is a peak, we can add a new rule deduced by transitivity, and thus construct a rewrite closure. However, this process need not finitely terminate because there may be infinitely many peaks to be considered. This may happen, for example, when there are variables in the rules.

When variables occur linearly in the rules and the TRS is right-flat, we can proceed as in the ground case and eliminate only those peaks that arise from “critical overlaps”. Suppose we have $r_1 : a \rightarrow f(b, c)$ and $r_2 : f(f(x, y), z) \rightarrow f(z, x)$. The peak, $f(a, d) \rightarrow r_1 f(f(b, c), d) \rightarrow r_2 f(d, b)$, can again be avoided by using the rule $r_3 : f(a, z) \rightarrow f(z, b)$, deduced by transitivity from $r_1$ and $r_2$ as before. The important point is that, due to linearity, we need not add any new rule for peaks produced below variables (“variable overlap”). This is because such peaks can be avoided by commuting the steps. For example, the peak derivation, $f(f(d, e), a) \rightarrow r_1 f(f(d, e), f(b, c)) \rightarrow r_2 f(f(b, c), d)$, can be transformed into the valley derivation, $f(f(d, e), a) \rightarrow r_2 f(a, d) \rightarrow r_1 f(f(b, c), d)$, by commuting the two
steps and we do not need any new rule.

When there is non-linearity in the rules, we cannot commute steps to eliminate “variable overlap” peaks. For example, suppose we have the rules $r_1 : a \rightarrow f(b, c)$ and $r_2 : f(x, x) \rightarrow e$ and the peak derivation $f(f(b, c), a) \rightarrow r_1 \rightarrow r_2 \rightarrow e$. We cannot directly apply $f(x, x) \rightarrow e$ to $f(f(b, c), a)$ and hence, we cannot avoid the peak by commutation. So, it may seem that we have to add the new rule $f(f(b, c), a) \rightarrow e$. In general, we may have to add a new rule $f(s, t) \rightarrow e$ for every pair $s, t$ of terms joinable by increasing rules. There could be infinitely many such pairs, and hence this is unacceptable. This problem was solved in [Godoy et al. 2004] by transforming rule $r_2$ into a constrained rule $r_3 : f(x_1, x_2) \rightarrow e(x_1 \downarrow x_2)$. Now, $x_1$ and $x_2$ do not need to be replaced by the same term; they only have to be joinable by increasing rules (this is the meaning of the constraint $x_1 \downarrow x_2$). Hence, rule $r_3$ can be used to reduce $f(f(b, c), a)$ to $e$ because $f(b, c)$ and $a$ are joinable using $r_1$. In general, constrained rules eliminate the need to add new rules to deal with peaks arising from variable overlaps.

Now, consider a more complicated situation where the duplicated variable on the left-hand side also occurs on the right-hand side. For example, a rule $r_0 : f(x, x) \rightarrow x$. We could proceed as before and generate a constrained rule of the form $r : f(x_1, x_2) \rightarrow x_3 \{x_1, x_2, x_3\}$, where the constraint $\{x_1, x_2, x_3\}$ means that the terms replacing $x_1$ and $x_2$ can reach the term replacing $x_3$ with increasing rules. Unfortunately, this is not enough. Suppose, along with $r_0$, we also have the rules:

\[
\begin{align*}
r_1 : a \rightarrow f(c, c') & \quad r_2 : b \rightarrow f(d, d') & \quad r_3 : c \rightarrow f(e, e') \\
r_4 : c' \rightarrow f(e, e') & \quad r_5 : d \rightarrow f(e, e') & \quad r_6 : d' \rightarrow f(e, e')
\end{align*}
\]

Now consider the “peak” derivation:

\[
\begin{align*}
& f(a, b) \\
\xrightarrow{r_1 \cdot r_2} & f(f(c, c'), f(d, d')) \\
\xrightarrow{r_3 \cdot r_4 \cdot r_5 \cdot r_6} & f(f(e, e'), f(e, e'), f(f(e, e'), f(e, e'))) \\
\xrightarrow{r_0} & f(f(e, e'), f(e, e')) \\
\xrightarrow{r_0} & f(e, e')
\end{align*}
\]

We could simplify this derivation by using the above constrained rule $r$, obtaining $f(a, b) \rightarrow_r f(e, e'), f(e, e') \rightarrow_r f(e, e')$. But the new derivation is not a valley. The problem is that, although the rule $r$ looks like a decreasing rule, it is used to give an increasing step: the term $f(f(e, e'), f(e, e'))$ used as $x_3$ is much bigger than the term $a$ (used as $x_1$) and the term $b$ (used as $x_2$). This problem was solved in [Godoy and Tiwari 2004] by extending the rewrite closure to generate rules ensuring that, when two terms $a$ and $b$ can both reach a term by increasing rules, then $a$ and $b$ can also reach a “small” term. The construction in [Godoy and Tiwari 2004] was rather complicated, but sufficient to deal with right-hand sides which are variables.

In this paper we use some of these same ideas, but we separate the part that ensures the existence of a “small” join term from the part that generates the rewrite closure. We thus achieve a finite rewrite closure presentation for $R$ in two phases. First, in Section 4, we conservatively extend the given TRS $R$ by some new constants.
and new rewrite rules to ensure that whenever two terms can both reach a term, they can also both reach a “small” term. Section 4 also shows that this extension of \( R \) preserves confluence, termination, weak normalization and unique normalization (the transformation in [Godoy and Tiwari 2004] only preserved confluence). Second, in Section 5 we construct a finite rewrite closure presentation of the extended TRS by using constrained rules. The correctness of the rewrite closure construction is shown using proof simplification arguments — similar to those used for showing correctness of critical-pair completion methods.

In the second part of the paper, we use the rewrite closure presentation to show decidability of weak normalization for right-shallow right-linear TRS. A term is normalizing if it can reach some normal form term. A TRS is weakly normalizing if all terms are normalizing. We first observe that we can decide if a given term \( s \) is normalizing using tree automata techniques: we compute a tree automaton representing the terms reachable from \( s \), which is possible thanks to the regularity preserving result [Nagaya and Toyama 2002]. This set is intersected with the set of normal forms, which can be represented by a tree automaton with disequality constraints [Comon and Jacquet-Margard 2003]. Emptiness of the last set proves that \( s \) is not normalizing. The crucial observation in Section 6 is that, using properties of the rewrite closure presentation, weak normalization property of a TRS can be reduced to the weak normalization of a finite set of terms. We establish this observation by showing that, if there is a large term that is not normalizing, then there will be a small term. The proof of this fact relies on the decidability for emptiness checking of reduction automata [Dauchet et al. 1995]. By using this result, we prove a computational bound for the size of the minimum non-normalizing term, whenever it exists. In Section 6 we also provide more intuition behind the election of this bound.

2. PRELIMINARIES

We use standard notation from the term rewriting literature [Baader and Nipkow 1998]. A signature \( \Sigma \) is a (finite) set of function symbols, which is partitioned as \( \bigcup_i \Sigma_i \) where \( f \in \Sigma_m \) if the arity of \( f \) is \( m \). In this case we may denote \( f \) as \( f^{(m)} \). Symbols in \( \Sigma_0 \), called constants, are denoted by \( a, b, c, d, e \), with possible subscripts. The elements of a set \( V \) of variable symbols are denoted by \( x, y, z \) with possible subscripts. The set \( T(\Sigma, V) \) of terms over \( \Sigma \) and \( V \), is the smallest set containing \( V \) and such that \( f(t_1, \ldots, t_m) \) is in \( T(\Sigma, V) \) whenever \( f \in \Sigma_m \), and \( t_1, \ldots, t_m \in T(\Sigma, V) \). \( \text{Vars}(t) \) denotes the set of variables occurring in \( t \). A position is a sequence of positive integers. If \( p \) is a position and \( t \) is a term, then by \( t|_p \) we denote the subterm of \( t \) at position \( p \). We have \( t|_\lambda = t \) (where \( \lambda \) denotes the empty sequence) and \( f(t_1, \ldots, t_m)|_{i, p} = t_i|_p \) if \( 1 \leq i \leq m \) (and is undefined if \( i > m \)). We also write \( t|_s \) to denote the term obtained by replacing in \( t \) the subterm at position \( p \) by the term \( s \). For example, if \( t \) is \( f(a, g(b, h(c)), d) \), then \( t|_{1, 2} = c \), and \( t|_{2, 2} = f(a, g(b, d), d) \). \( \text{Pos}(t) \) denotes the set of all positions in \( t \), and \( \text{VarPos}(t) \) denotes the set of all positions in \( t \) with occurrences of variables. The symbol occurring at the root of a term \( t \) is denoted as \( \text{root}(t) \). We write \( p_1 > p_2 \) (equivalently, \( p_2 < p_1 \)) and say \( p_1 \) is below \( p_2 \) (equivalently, \( p_2 \) is above \( p_1 \)) if \( p_2 \) is a proper prefix of \( p_1 \), that is, \( p_1 = p_2, p_2' \) for some non-empty \( p_2' \). Positions
$p$ and $q$ are disjoint if $p \not\supseteq q$ and $q \not\supseteq p$. The size of a term $s$, denoted $|s|$, is 1 if $s$ is a variable or a constant, and $1 + |s_1| + \ldots + |s_m|$ if $s = f(s_1, \ldots, s_m)$. The number of occurrences of a symbol $\alpha$ in $s$ is denoted $|s|_{\alpha}$. More formally, $|\alpha(s_1, \ldots, s_m)|_{\alpha}$ is $1 + |s_1|_{\alpha} + \ldots + |s_m|_{\alpha}$, and $|\beta(s_1, \ldots, s_m)|_{\alpha}$ for $\beta \neq \alpha$ is $|s_1|_{\alpha} + \ldots + |s_m|_{\alpha}$. The height of a term $s$, denoted $\text{height}(s)$, is 0 if $s$ is a variable or a constant, and $1 + \max(|\text{height}(s_i)|)$ if $s = f(s_1, \ldots, s_m)$ and $m > 0$. The depth of an occurrence at position $p$ of a term $t$ in a term $s = s[t]_p$ is $|p|$. Sometimes we will denote $t[s]_p$ by $t[s]$ when $p$ is clear from the context or not important.

A substitution $\sigma$ is a mapping from variables to terms. It can be homomorphically extended to a function from terms to terms: $\sigma(t)$ denotes the result of simultaneously replacing in $t$ every $x \in \text{Dom}(\sigma)$ by $\sigma(x)$. For example, if $\sigma = \{x \mapsto f(b, y), y \mapsto a\}$, then $\sigma(g(x, y)) = g(f(b, y), a)$.

A rewrite rule is a pair of terms $(l, r)$, denoted by $l \to r$, with left-hand side $l$ and right-hand side $r$. A term rewrite system (TRS) $R$ is a finite set of rewrite rules. We say that $s$ rewrites to $t$ in one step at position $p$ (by $R$), denoted by $s \xrightarrow{R,p} t$, if $s_p = \sigma(l)$ and $t = \sigma(r)$, for some $l \to r \in R$ and substitution $\sigma$. We also denote such a rewrite step by $\xrightarrow{l\leftarrow r,\sigma,p}$ if we make explicit the used rule $l \to r$ and substitution $\sigma$. If $p = \lambda$, then the rewrite step $\xrightarrow{R,p}$ is said to be applied at the root. Otherwise, it is denoted by $s \xrightarrow{R,\lambda} t$. We make the usual assumption for the rules $l \to r$ of a TRS $R$, i.e. $l$ is not a variable, and all variables occurring in the term $r$ also occur in the term $l$.

If $\leftrightarrow$ is a binary relation on a set $S$, then $\leftrightarrow^+$ is its symmetric closure, $\leftarrow^+$ is its transitive closure, $\leftrightarrow^*$ is its reflexive-transitive closure.

A (rewrite) derivation (from $s$) is a sequence of rewrite steps (starting from $s$), that is, a sequence $s \xrightarrow{R} s_1 \xrightarrow{R} s_2 \xrightarrow{R} \ldots$. With $s \xrightarrow{R} t$ we denote that $t$ is $R$-reachable from $s$, or a concrete derivation from $s$ to $t$, depending on the context. For a set of terms $S$, with $\text{Reach}_R(S)$ we denote the set of all terms which are $R$-reachable from all terms in $S$, i.e. $\text{Reach}_R(S) = \{t : \forall s \in S : s \xrightarrow{R} t\}$. If $t$ is a term of $\text{Reach}_R(S)$, then we also say that $t$ is $R$-reachable from $S$. The length of a derivation is its number of rewrite steps. $R$ is terminating if no infinite derivation $s_1 \xrightarrow{R} s_2 \xrightarrow{R} \ldots$ exists. A term $s$ is $R$-irreducible (or, in $R$-normal form) if there is no term $t$ such that $s \xrightarrow{R} t$. A term $s$ is $R$-normalizing if $s$ $R$-reaches a $R$-normal form, and in this case we also say that $s$ has a $R$-normal form. Two terms $s$ and $t$ are $R$-joinable if there exists a term $u$ $R$-reachable from $s$ and $t$. A TRS $R$ is confluent if for every $s, u, v$ such that $s \xrightarrow{R} u$ and $s \xrightarrow{R} v$ there exists a term $t$ such that $u \xrightarrow{R} t$ and $v \xrightarrow{R} t$. A TRS $R$ satisfies the weak normalization property, or, equivalently, $R$ is weakly normalizing, if all terms are $R$-normalizing. It satisfies the unique normalization property, or, equivalently, $R$ is uniquely normalizing, if all terms are $R$-uniquely normalizing.

A term $t$ is called ground if it contains no variables. It is called shallow if all variable positions in $t$ are at depth 0 or 1. It is flat if its height is at most 1. It is linear if every variable occurs at most once.

A rule $l \to r$ is called ground (flat, shallow, linear) if both $l$ and $r$ are. A rule $l \to r$ is called left-ground (left-flat, left-shallow, left-linear) if $l$ is. A rule $l \to r$ is called right-ground (right-flat, right-shallow, right-linear) if $r$ is. A rule is called collapsing if $r$ is a variable.
A TRS \( R \) is called (left-, right-)ground (flat, shallow, linear) if all its rules are. A TRS \( R \) is called collapsing if it contains a collapsing rule.

An automaton with equality and disequality constraints (AWEDC) is a tree automaton \( \langle Q, \Sigma, Q_f, \Delta \rangle \) where \( \Sigma \) is a signature, \( Q \) is a finite set of states, \( Q_f \subseteq Q \) is the set of final states, and \( \Delta \) is a set of transitions \( f(q_1, \ldots, q_m) \xrightarrow{c} q \), where \( c \) is a conjunction of equality (\( p = p' \)) and disequality (\( p \neq p' \)) constraints. Here \( p, p' \) are positions. The language accepted by AWEDC is defined just as for tree automata, but a transition rule is applied to a term only when that term satisfies the constraint. Emptiness is undecidable for AWEDC, but it is decidable when all constraints are disequality constraints. A reduction automaton is an AWEDC with an ordering \( > \) on states \( Q \) such that whenever \( f(q_1, \ldots, q_m) \xrightarrow{c} q \) is a rule, then \( q_i \geq q \) for every \( i \), and moreover, whenever \( c \) contains an equality constraint, then \( q_i > q \) for every \( i \). Emptiness is decidable for reduction automata. For further details, see [Comon et al. 2007].

In the rest of the paper, we deal with a right-shallow right-linear TRS \( R \) and do the following usual assumption.

\[ (A0) \text{ For every rule } l \rightarrow r \in R, \text{ it is the case that } l \notin \mathcal{V} \text{ and } \text{Vars}(l) \supseteq \text{Vars}(r). \]

3. SIMPLIFYING ASSUMPTIONS ON THE SIGNATURE AND THE TRS

In this section we briefly present two assumptions on the signature \( \Sigma \) and the given right-shallow right-linear TRS \( R \), which we will make without loss of generality for the rest of the article.

\[ (A1) \text{ The signature } \Sigma \text{ is of the form } \Sigma_0 \cup \Sigma_m, \text{ and } \Sigma_m \text{ is of the form } \{ f \}. \text{ In other words, } \Sigma \text{ contains several constants and only one non-constant function symbol } f \text{ of a certain arity } m. \]

\[ (A2) \text{ The TRS } R \text{ is right-flat.} \]

The purpose of these assumptions is to simplify the proofs in the rest of the article. They are rather standard and have appeared in other works [Godoy et al. 2004; Godoy and Tiwari 2005; Godoy and Tison 2007; Godoy et al. 2007]. In these papers it is shown that the signature and the TRS can be transformed to satisfy Assumptions (A1) and (A2) while preserving confluence, termination, weak normalization and unique normalization.

In order to make the paper self-contained, we describe the two transformations on \( \Sigma \) and \( R \) to achieve (A1) and (A2).

We argue that Assumption (A1) is not restrictive by defining a transformation \( T \) from terms over \( \Sigma \) to terms over a new signature \( \Sigma' \) as follows. Let \( m \) be the maximum arity of a symbol in \( \Sigma \) plus 1. We choose a new function symbol \( f \) with arity \( m \) and define the new signature \( \Sigma' = \Sigma_0' \cup \Sigma_m' \) as \( \Sigma_0' = \Sigma \) and \( \Sigma_m' = \{ f \} \). Note that all function symbols in \( \Sigma \) are treated as constants in \( \Sigma' \). Now, we recursively define a transformation \( T : T(\Sigma, \mathcal{V}) \rightarrow T(\Sigma', \mathcal{V}) \) as \( T(c) = c \) and \( T(x) = x \) for constants \( c \in \Sigma_0 \) and variables \( x \in \mathcal{V} \), and \( T(g(t_1, \ldots, t_k)) = f(T(t_1), \ldots, T(t_k), g, \ldots, g) \) for terms headed with \( g \in \Sigma_k \), for \( k > 0 \). Given a TRS \( R \), by \( T(R) \) we denote \( \{ T(l) \rightarrow T(r) \mid l \rightarrow r \in R \} \).
Example 3.1. Consider $\Sigma := \{g^{(2)}_h, h^{(1)}_a, a^{(0)}\}$ and $R := \{g(h(a), x) \rightarrow x, g(x, y) \rightarrow g(y, x)\}$. In this case, $T(R) := \{f(f(a, h, h), x, g) \rightarrow x, f(x, y, g) \rightarrow f(y, x, g)\}$.

We satisfy Assumption (A2) by applying the following transformation whenever it is possible:

If there is a non-constant ground term $t$ that is a subterm at depth 1 of a right-hand side of a rule in $R$, then create a new constant $c$, replace all occurrences of $t$ at depth 1 in the right-hand sides of the rules of $R$ by $c$, and add the rule $c \rightarrow t$ to $R$.

Clearly, after we have exhaustively applied the above transformation, the final TRS is necessarily right-flat. The transformation is always a conservative extension of the original TRS: the old and the new TRS induce the same rewrite relation on terms not containing $c$.

4. Adding Constants Representing Minimal Joins

We now assume that $R$ is a TRS over the signature $\Sigma$ and it satisfies Assumptions (A0), (A1), and (A2). The goal of this section is to transform $R$ further so that it has the following two additional properties. Let $B$ be the subset of $R$ having a constant as left-hand side.

(A3). Let $c$ be a constant of $\Sigma$, and let $t$ be a term $R$-reachable from $c$. Then, $t$ is $B$-reachable from $c$.

(A4). Let $t_1, \ldots, t_n$ be terms in $T(\Sigma, V)$ and $s$ a term $B$-reachable from $\{t_1, \ldots, t_n\}$. Then, there exists a term $s'$ in $T(\Sigma, V)$ $B$-reachable from $\{t_1, \ldots, t_n\}$ such that $s' \succeq_B s$ and $\text{Pos}(s') = \bigcup_{i \in \{1, \ldots, n\}} \text{Pos}(t_i)$.

Example 4.1. Consider the TRS $R := \{c \rightarrow f(a), d \rightarrow f(a), f(a) \rightarrow b\}$ over the signature $\Sigma := \{a^{(0)}, b^{(0)}, c^{(0)}, d^{(0)}, f^{(1)}\}$. The TRS $R$ satisfies Assumptions (A0), (A1), and (A2). Clearly, $B := \{c \rightarrow f(a), d \rightarrow f(a)\}$. However, $R$ does not satisfy Assumption (A3): $b$ is $R$-reachable from $c$, but it is not $B$-reachable from $c$. Also, $R$ does not satisfy Assumption (A4): $f(a)$ is $B$-reachable from both $c$ and $d$, but there is no depth 0 term that is $B$-reachable from both $c$ and $d$.

We achieve the desired properties by extending the signature $\Sigma$ and the TRS $R$ by, respectively, new constants and new rewrite rules over these new constants. The main technical result of this section is that the proposed transformation preserves several properties (such as weak normalization and confluence) of $R$. Additionally, the result of the transformation will also satisfy Assumptions (A3) and (A4), which are key ingredients required to compute a rewrite closure for $R$ (in the next section).

Given the TRS $R$ over the signature $\Sigma$, we define the new signature $\Sigma'$ as follows.

Definition 4.2. Let $f$ be the only non-constant function symbol (of arity $m$) in
The new signature $\Sigma$ is defined as follows.

$$
\Sigma := \Sigma_m \cup \Sigma_0
$$

$$
\Sigma_m := \{f\}
$$

$$
\Sigma_0 := \{c_S | S \subseteq \Sigma \land \text{Reach}_R(S) \neq \emptyset \land
\left(|S| = 1 \lor \forall S' \subseteq \Sigma : (\text{Reach}_R(S') = \text{Reach}_R(S) \land
\left(|S'| = 1 \lor S' \supset S\right))\right)\}
$$

For each term $t \in T(\Sigma, \mathcal{V})$, $t^{(1)}$ represents its transformation by replacing every occurrence of each constant $d$ by $c(d)$, and $R^{(1)}$ is obtained from $R$ by the same transformation. Formally, $d^{(1)} = c(d)$ for $d \in \Sigma_0$, $x^{(1)} = x$ for $x \in \mathcal{V}$, $(f(t_1, \ldots, t_m))^{(1)} = f(t_1^{(1)}, \ldots, t_m^{(1)})$, and $R^{(1)} = \{t^{(1)} \rightarrow r^{(1)} | t \rightarrow r \in R\}$.

Informally, $c_S$ belongs to $\Sigma$ if either $S$ is singleton, or the set $S$ of constants is $R$-joinable ($\text{Reach}_R(S) \neq \emptyset$) and neither a larger nor a singleton set $S'$ satisfies $\text{Reach}_R(S') = \text{Reach}_R(S)$. The goal of the new $c_S$ is to make all constants in $S$, which are $R$-joinable to a term whose size may be “large”, to be joinable to $c_S$, which is “small”.

**Example 4.3.** Let $R := \{a \rightarrow f(a, b), b \rightarrow f(a, b), f(x, x) \rightarrow f(a, a)\}$. Note that $f(a, b)$ can $R$-reach $f(a, a)$. If the smallest term reachable from both $a$ and $b$ is bigger than both $a$ and $b$ (such as, $f(a, b)$ above), then it will be impossible to show that $f(a, b)$ can $R$-reach $f(a, a)$ without first using “increasing” steps to show that $a$ and $b$ are joinable. However, if we extend $R$ forcing $a$ and $b$ to be joinable to a new constant term, such as $c(a, b)$, then we get a rewrite derivation of the desired form.

The following two lemmas are straightforward consequences of the definition 4.2.

**Lemma 4.4.** Let $S$ be a subset of $\Sigma$ satisfying $\text{Reach}_R(S) \neq \emptyset$. Then, there exists $c_S \in \Sigma$ such that $\text{Reach}_R(S') = \text{Reach}_R(S)$.

**Proof.** Let $S$ be a subset of $\Sigma$. If there exists a singleton subset $S'$ of $\Sigma$ satisfying $\text{Reach}_R(S') = \text{Reach}_R(S)$, then $c_S$ belongs to $\Sigma$ and the statement holds. Otherwise, let $S'$ be a maximal (in the inclusion relation) subset of $\Sigma$ satisfying $\text{Reach}_R(S') = \text{Reach}_R(S)$. Again, we conclude that $c_S$ belongs to $\Sigma$ and the statement holds.

**Lemma 4.5.** Let $S$ be a non-singleton subset of $\Sigma$ such that $c_S \in \Sigma$. Then, there is no $c_S$ in $\Sigma$ such that $S' \neq S$ and $\text{Reach}_R(S') = \text{Reach}_R(S)$.

**Proof.** We prove it by contradiction. Let $S$ be a non-singleton subset of $\Sigma$ such that $c_S \in \Sigma$, and let $S'$ be a subset of $\Sigma$ different from $S$ such that $\text{Reach}_R(S') = \text{Reach}_R(S)$. The set $S'$ cannot be a singleton set: otherwise, $S$ contradicts the definition of $\Sigma$. The set $S'$ cannot be a subset of $S$: otherwise, by the definition of $\Sigma$, $c_S$ cannot be in $\Sigma$. Thus, $S'$ contains a constant $d \in \Sigma - S$. Since $\text{Reach}_R(S') = \text{Reach}_R(S)$, it holds that $\text{Reach}_R(S') \supset \text{Reach}_R(S)$. Therefore, $\text{Reach}_R(S \cup \{d\}) = \text{Reach}_R(S)$. But $S \cup \{d\}$ strictly includes $S$, and this is in contradiction with the fact that $c_S \in \Sigma$. 

\[\text{Journal of the ACM, Vol. V, No. N, Month \text{YYYY}.} \]
In the following, we use a notation where a set of terms occurs as a “subterm” in a term. For example, in the term $t[S_1]_{p_1} \ldots [S_n]_{p_n}$, the sets $S_1, \ldots, S_n$ occur at positions $p_1, \ldots, p_n$, respectively. We identify such a term with a set of terms as follows: $t[S_1]_{p_1} \ldots [S_n]_{p_n} := \{t[t_1]_{p_1} \ldots [t_n]_{p_n} \mid t_1 \in S_1, \ldots, t_n \in S_n\}$.

**Definition 4.6.** The TRS $\overline{R}$ is defined over $\Sigma$ as follows.

\[
\overline{R} = R^1 \cup R^c \cup R^l
\]

\[
R^c = \{c_S \rightarrow c_{S'} \mid c_S, c_{S'} \in \Sigma_0 \land S \neq S' \land \text{Reach}_R(S') \subseteq \text{Reach}_R(S)\}
\]

\[
R^l = \{c_S \rightarrow f(c_{S_1}, \ldots, c_{S_m}) \mid c_S, c_{S_1}, \ldots, c_{S_m} \in \Sigma_0 \land f(\text{Reach}_R(S_1), \ldots, \text{Reach}_R(S_m)) \subseteq \text{Reach}_R(S)\}
\]

Right-linear right-flat TRS are regularity preserving [Nagaya and Toyama 2002]. Thus the sets Reach$_R(S)$, where $S \subseteq \Sigma$, are computable as regular sets. Therefore, since emptiness and set inclusion are decidable for regular languages [Comon et al. 2007], and since the conditions in the definition of $\Sigma$ and $\overline{R}$ are based on emptiness and inclusions, it follows that $\Sigma$ and $\overline{R}$ are computable.

**Lemma 4.7.** $\Sigma$ and $\overline{R}$ are computable from $\Sigma$ and $R$.

The following example illustrates the definition of $\overline{R}$.

**Example 4.8.** Let $R := \{a \rightarrow f(a,b), b \rightarrow f(a,b), f(x,x) \rightarrow f(a,a)\}$. Note that Reach$_R(a,b)$ is non-empty (it contains $f(a,b)$). It is also different from Reach$_R\{a\}$ and Reach$_R\{b\}$ since it contains neither $a$ nor $b$. There is no larger subset $S'$ of $\Sigma$ that can reach the exact same set of terms. Hence, we add $c_{(a,b)}$ to $\overline{R}$. Intuitively, $c_{(a,b)}$ will be the “small” term that is reachable from both $a$ and $b$. Thus, $\Sigma := \{c_{(a)}, c_{(b)}, c_{(a,b)}, f\}$ and $\overline{R} := R^1 \cup R^c \cup R^l$, where $R^1 := \{c_{(a)} \rightarrow f(c_{(a)}, c_{(b)}), c_{(b)} \rightarrow f(c_{(a)}, c_{(b)}), f(x,x) \rightarrow f(c_{(a)}, c_{(a)})\}$, $R^c := \{c_{(a)} \rightarrow c_{(a,b)}, c_{(b)} \rightarrow c_{(a,b)}\}$, and $R^l := \{c_{(a)} \rightarrow f(d_1, d_2), c_{(b)} \rightarrow f(d_1, d_2), c_{(a,b)} \rightarrow f(d_1, d_2) \mid d_1 \in \{c_{(a)}, c_{(b)}\}, d_2 \in \{c_{(a)}, c_{(b)}\}\}$.

Note that, when a rule $l \rightarrow r$ of the form $d \rightarrow e$ or of the form $d \rightarrow f(d_1, \ldots, d_m)$ belongs to $R$, then $l^1 \rightarrow r^1$ belongs to $\overline{R}$ because it belongs to $R^1$, but also because it belongs to $R^c$ (whenever $d \neq e$) or $R^l$.

In the rest of this section, we prove that $\overline{R}$ satisfies Properties (A3) and (A4), and that it preserves the following possible properties from $R$: confluence, termination, weak normalization and unique normalization. These proofs are interesting and require some additional definitions and lemmas.

An important observation that will enable proving properties of $\overline{R}$ is that $\overline{R}$ is a conservative extension of $R^1$. To prove this and establish all the links between $R$ and $\overline{R}$, we define a notion of marking of positions in terms, which will be extended later to markings of derivations.

**Definition 4.9.** With $\mathcal{P}_f$ we denote the set containing all finite subsets of $T(\Sigma, \mathcal{V})$. Let $t$ be a term in $T(\Sigma, \mathcal{V})$. A marking $M$ of $t$ is a function $M : \text{Pos}(t) \rightarrow \mathcal{P}_f$. A marking is valid if the following conditions hold:

1. For every $p \in \text{Pos}(t)$ such that root$(t)_p$ is some constant $c_S$, $M(p) = S$.
2. For every $p \in \text{Pos}(t)$ such that root$(t)_p$ is some variable $x$, $M(p) = \{x\}$.
(3) For every \( p \in \text{Pos}(t) \) such that \( \text{root}(t|_p) = f \) it holds that:

\[
f(\text{Reach}_R(M(p,1)), \ldots, \text{Reach}_R(M(p,m))) \subseteq \text{Reach}_R(M(p))
\]

We can generalize the property in Definition 4.9 to arbitrary depths.

**Lemma 4.10.** Let \( t \) be a term in \( T(\Sigma, V) \) and let \( M \) be a valid marking of \( t \). Let \( \{p_1, \ldots, p_k\} \) be a set of disjoint positions of \( t \) such that \( P = \{p_1, \ldots, p_k\} \) is complete, that is, for any position \( p.q \) there exists \( i \) satisfying \( p_i \leq q \) or \( q \leq p_i \). Then, \( (t|_p)[\text{Reach}_R(M(p,1))]|_{p_1} \ldots [\text{Reach}_R(M(p,p_k))]|_{p_k} \subseteq \text{Reach}_R(M(p)) \).

**Proof.** It can be easily proved by induction on the size of \( t|_p \). For the particular case \( P = \{\lambda\} \) the result trivially follows. Thus, suppose \( P \neq \{\lambda\} \). Then, necessarily height\( (t|_p) \neq 0 \). Then, \( t|_p \) is of the form \( f(t_1, \ldots, t_m) \), and by the definition of valid marking, \( f(\text{Reach}_R(M(p,1)), \ldots, \text{Reach}_R(M(p,m))) \subseteq \text{Reach}_R(M(p)) \) holds. For any position \( i \in \{1, \ldots, m\} \) we can consider the set \( P_i = \{q \mid q \in P \} \). Note that \( \{p_i.q \mid q \in P_i\} \) is a set of disjoint positions of \( t \) such that \( P_i \) is complete. Since induction hypothesis applies for all such \( t_i \) and \( P_i \), we conclude that \( (t|_p)[\text{Reach}_R(M(p,p_1))]|_{p_1} \ldots [\text{Reach}_R(M(p,p_k))]|_{p_k} \subseteq f(\text{Reach}_R(M(p,1)), \ldots, \text{Reach}_R(M(p,m))) \), and hence

\[
(t|_p)[\text{Reach}_R(M(p,p_1))]|_{p_1} \ldots [\text{Reach}_R(M(p,p_k))]|_{p_k} \subseteq \text{Reach}_R(M(p)).
\]

\[ \square \]

If \( M \) is a valid marking and \( p \) a position, then \( \text{Reach}_R(M(p)) \) is always non-empty.

**Corollary 4.11.** Let \( t \) be a term in \( T(\Sigma, V) \), let \( M \) be a valid marking of \( t \), and let \( p \) be any position of \( t \). Let \( p_1, \ldots, p_k \) be the leaf positions of \( t \) below \( p \). Then, \( \text{Reach}_R(M(p)) \supseteq (t|_p)[\text{Reach}_R(M(p,p_1))]|_{p_1} \ldots [\text{Reach}_R(M(p,p_k))]|_{p_k} \neq \emptyset \).

**Proof.** Note that \( \{p_1, \ldots, p_k\} \) is a set of disjoint positions of \( t \) and that \( P = \{p_1, \ldots, p_k\} \) is a complete set of positions. By Lemma 4.10, \( (t|_p)[\text{Reach}_R(M(p,p_1))]|_{p_1} \ldots [\text{Reach}_R(M(p,p_k))]|_{p_k} \subseteq \text{Reach}_R(M(p)) \). By the definition of valid marking, each \( M(p_i) \) is either of the form \( \{x\} \), if \( t|_{p_i} \) is a certain variable \( x \), or of the form \( S \), if \( t|_{p_i} \) is a certain constant \( c_S \in \Sigma \). In the first case, \( \text{Reach}_R(M(p,p_i)) \) is \( \{x\} \). In the second case, by the definition of \( \Sigma \), \( \text{Reach}_R(M(p,p_i)) \) is not empty. Thus, in any case \( \text{Reach}_R(M(p,p_i)) \) is not empty. Since this is valid for any \( i \) in \( \{1, \ldots, k\} \), it holds that \( (t|_p)[\text{Reach}_R(M(p,p_1))]|_{p_1} \ldots [\text{Reach}_R(M(p,p_k))]|_{p_k} \neq \emptyset \).

**Lemma 4.12.** Let \( t \) be a term in \( T(\Sigma, V) \) and let \( M \) be a valid marking of \( t \). Let \( q_1, \ldots, q_n \) be positions of \( t \) satisfying \( t|_{q_1} = t|_{q_2} = \ldots = t|_{q_n} \).

Then, \( \text{Reach}_R(M(q_1)) \cap \ldots \cap \text{Reach}_R(M(q_n)) \neq \emptyset \).

**Proof.** Let \( s \) be \( t|_{q_i} \). Let \( P = \{p_1, \ldots, p_k\} \) be the set of leaf positions of \( s \). Note that, by the definition of valid marking, for each \( j \in \{1, \ldots, k\} \) and independently on \( i \), it holds that \( M(q_j, p_j) \) is \( \{x\} \) if \( s|_{p_j} \) is a certain variable \( x \), and \( S \) if \( s|_{p_j} \) is a certain constant \( c_S \in \Sigma \). Therefore, \( M(q_1, p_j) = M(q_2, p_j) = \ldots = M(q_n, p_j) \). Thus, by Corollary 4.11, the set \( s[\text{Reach}_R(M(q_1, p_j))]|_{p_1} \ldots [\text{Reach}_R(M(q_1, p_j))]|_{p_k} \) is not empty and included in all \( \text{Reach}_R(M(q_1)) \ldots \text{Reach}_R(M(q_n)) \), and this concludes the proof. \( \square \)
We can push valid markings through \( R \)-rewrite steps. Note that we use the fact that \( R \) is right-shallow and right-linear below.

**Definition 4.13.** Let \( t, t' \) be terms in \( T(\Sigma, \mathcal{V}) \) and \( l \rightarrow r \in R \) such that \( t \rightarrow_{t \rightarrow r, p} t' \). Let \( M \) be a valid marking of \( t \).

Then, the derived marking \( M' : \mathsf{Pos}(t') \rightarrow \mathcal{P}_f \) of \( t' \) from \( t \rightarrow_{t \rightarrow r, p} t' \) is defined as follows for every position \( p' \) in \( \mathsf{Pos}(t') \).

(a) If \( p' \) is disjoint with \( p \) (case a.1) or such that \( p' < p \) (case a.2) then we define \( M'(p') = M(p') \).

(b) If \( p' \) is \( p \cdot p_1 \) for some \( p_1 \) satisfying \( p_1 \in \mathsf{Pos}(r) \) and \( r|_{p_1} \) is a constant \( c_S \), then we define \( M'(p \cdot p_1) = S \).

(c) If \( p' = p \cdot p_2 \), for some \( p_1 \) and \( p_2 \) such that \( p_1 \in \mathsf{Pos}(r) \) and \( r|_{p_1} \) is a variable, then let \( q_1, \ldots, q_k \) be the positions in \( l \) where this variable occurs. We define \( M'(p \cdot p_2) = \bigcup_{i \in \{1, \ldots, k\}} M(p \cdot q_i \cdot p_2) \).

(d) If \( p' = p \) and \( \text{root}(r) \) is \( f \) then we define \( M'(p) = M(p) \). (Note that the situations where \( p' = p \) and \( \text{root}(r) \) is not \( f \) are captured by cases (b) and (c).)

The following technical lemma helps in proving the validness of the derived marking from a valid marking.

**Lemma 4.14.** Let \( t, t' \) be terms in \( T(\Sigma, \mathcal{V}) \) such that \( t \rightarrow_{t \rightarrow r, p} t' \). Let \( M \) be a valid marking of \( t \).

Then, the derived marking \( M' : \mathsf{Pos}(t') \rightarrow \mathcal{P}_f \) of \( t' \) from \( t \rightarrow_{t \rightarrow r, p} t' \) satisfies \( \mathsf{Reach}_R(M'(p)) \subseteq \mathsf{Reach}_R(M(p)) \).

**Proof.** The only difficult situation appears when \( M(p) \neq M'(p) \). Thus, we assume that the applied rule has a variable or a constant as right-hand side.

---

If the applied rule is of the form \( c_S \rightarrow c_S' \), then, by the definition of \( R \), \( \mathsf{Reach}_R(S') \subseteq \mathsf{Reach}_R(S) \). By the definition of derived marking, \( M(p) = S \) and \( M'(p) = S' \). Thus \( \mathsf{Reach}_R(M'(p)) \subseteq \mathsf{Reach}_R(M(p)) \) holds.

---

If the applied rule is of the form (i) \( l_1^1 \rightarrow c(d) \) for \( l_1 \rightarrow x \in R \), or of the form (ii) \( l_1^1 \rightarrow x \) for \( l_1 \rightarrow x \in R \), we consider any \( u \) in \( \mathsf{Reach}_R(M'(p)) \) and prove that \( u \) belongs to \( \mathsf{Reach}_R(M(p)) \). Let \( P = \{p_1, \ldots, p_n\} \) be the set of leaf positions of \( l \) (which coincides with the set of leaf positions of \( l_1 \)). Note that \( p \cdot p_1, \ldots, p \cdot p_n \) are not necessarily the leaf positions of \( t \) below \( p \). For each \( p_i \) with \( i \in \{1, \ldots, n\} \), by Corollary 4.11, it holds that \( \mathsf{Reach}_R(M(p \cdot p_i)) \) is not empty. Moreover, for every \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \) satisfying \( l|_{p \cdot p_{i_1}} = l|_{p \cdot p_{i_2}} = \ldots = l|_{p \cdot p_{i_k}} \in \mathcal{V} \), it holds that \( l|_{p \cdot p_{i_1}} = l|_{p \cdot p_{i_2}} = \ldots = l|_{p \cdot p_{i_k}} \), and by Lemma 4.12, \( \mathsf{Reach}_R(M(p \cdot p_{i_1})) \cap \ldots \cap \mathsf{Reach}_R(M(p \cdot p_{i_k})) = \emptyset \). Therefore, we can choose terms \( t_1, \ldots, t_n \) satisfying \( t_1 \in \mathsf{Reach}_R(M(p \cdot p_1)), \ldots, t_n \in \mathsf{Reach}_R(M(p \cdot p_n)) \) and such that \( l|_{p \cdot p_1} = l|_{p \cdot p_2} = \ldots = l|_{p \cdot p_n} \), and \( t_i \) implies \( t_i = t_j \). Moreover, in case (ii), if some \( l|_{p \cdot i} = x \) then we can choose \( t_i \) to be \( x \). Note that, by the definition of derived marking, \( M'(p) = \bigcup_{q \in \mathsf{Pos}(l) : l|_q = x} M(p, q) \), and hence, \( \mathsf{Reach}_R(M'(p)) = \bigcap_{q \in \mathsf{Pos}(l) : l|_q = x} \mathsf{Reach}_R(M(p, q)) \). With this election, \( l|_{t_1 \cdot p_1}, \ldots, l|_{t_n \cdot p_n} \) \( R \)-reaches \( u \) in case (ii), and in case (i). But in case (i) \( l|_{t_1 \cdot p_1}, \ldots, l|_{t_n \cdot p_n} \) \( \rightarrow_{R} u \) also holds, because in this case \( M(p) \) equals \( \{d\} \) and \( u \) was chosen from \( \mathsf{Reach}_R(M'(p')) \). Since \( l|_{t_1 \cdot p_1}, \ldots, l|_{t_n \cdot p_n} \in l[\mathsf{Reach}_R(M(p \cdot p_1)) \mid p_1 \ldots
\[\text{Reach}_R(M(p,p_0))]_{p_n}; \text{ and } P \text{ is a set of disjoint and complete positions, by Lemma 4.10 } \{\text{Reach}_R(M(p,p_1))_{p_1}, \ldots, \text{Reach}_R(M(p,p_n))_{p_n}\} \subseteq \text{Reach}_R(M(p)) \text{ holds. We conclude that } u \text{ belongs to } \text{Reach}_R(M(p)). \]

\[\Box\]

**Lemma 4.15.** Let \(t, t'\) be terms in \(T(\Sigma, \mathcal{V})\) such that \(t \rightarrow_{r,p} t'\). Let \(M\) be a valid marking of \(t\).

Then, the corresponding derived marking \(M' : \text{Pos}(t') \rightarrow \mathcal{P}_f\) is valid.

**Proof.** We first prove that \(M'\) satisfies item (1) of the definition of valid marking. Let \(p' \in \text{Pos}(t')\) be such that \(\text{root}(t'|p')\) is some constant \(c_S\). We distinguish cases according to the definition of derived marking:

(a.1) If \(p'\) is disjoint with \(p\), then \(t'|p'\) is also \(c_S\), and hence, \(M'(p') = M(p') = S\).

(a.2,d) The case where \(p'\) is a prefix of \(p\), as well as the case where \(\text{root}(t'|p')\) is \(f\), are not possible when \(t'|p'\) is \(c_S\).

(b) If \(p' = p, p_1\) for some \(p_1\) satisfying \(p_1 \in \text{Pos}(r)\) and \(r|_{p_1}\) is a constant \(c_S\), then, by the definition of derived marking \(M'(p,p_1) = S\) holds.

(c) If \(p' = p, p_1, p_2\), for some \(p_1\) and \(p_2\) such that \(p_1 \in \text{Pos}(r)\) and \(r|_{p_1}\) is a variable, let \(q_1, \ldots, q_k\) be the positions in \(l\) where this variable occurs. Note that all \(t|_{p,q_1, p_2}\) are \(c_S\). Therefore, all \(M(p,q_1, p_2)\) are \(S\), and hence, \(M'(p,p_1, p_2) = \bigcup_{i \in \{1, \ldots, k\}} M(p,q_i, p_2) = S\).

We now prove that \(M'\) satisfies item (2) of the definition of valid marking. Let \(p' \in \text{Pos}(t')\) be such that \(\text{root}(t'|p')\) is some variable \(x\). We distinguish cases according to the definition of derived marking:

(a.2,b,d) The case where \(p'\) is a prefix of \(p\), as well as the case where \(\text{root}(t'|p')\) is \(f\), as well as the case where \(p' = p, p_1\) for some \(p_1\) satisfying \(p_1 \in \text{Pos}(r)\) and \(r|_{p_1}\) is a constant \(c_S\), are not possible when \(t'|p'\) is \(x\).

(a.1) If \(p'\) is disjoint with \(p\), then \(t'|p'\) is also \(x\), and hence, \(M'(p') = M(p') = \{x\}\).

(c) If \(p' = p, p_1, p_2\), for some \(p_1\) and \(p_2\) such that \(p_1 \in \text{Pos}(r)\) and \(r|_{p_1}\) is a variable, let \(q_1, \ldots, q_k\) be the positions in \(l\) where this variable occurs. Note that all \(t|_{p,q_1, p_2}\) are \(x\). Therefore, since \(M\) is valid, all \(M(p,q_1, p_2)\) are \(\{x\}\), and hence, \(M'(p,p_1, p_2) = \bigcup_{i \in \{1, \ldots, k\}} M(p,q_i, p_2) = \{x\}\).

Finally, we prove that \(M'\) satisfies item (3) of the definition of valid marking. Let \(p' \in \text{Pos}(t')\) be such that \(\text{root}(t'|p')\) is \(f\). We prove \(f(\text{Reach}_R(M'(p',1)), \ldots, \text{Reach}_R(M'(p',m))) \subseteq \text{Reach}_R(M'(p'))\) distinguishing cases according to the definition of derived marking:

(a.1) If \(p'\) is disjoint with \(p\), then \(M'(p') = M(p')\) and \(M'(p', i) = M(p', i)\) for all \(i \in \{1, \ldots, m\}\), and hence, \(f(\text{Reach}_R(M'(p',1)), \ldots, \text{Reach}_R(M'(p',m))) = \text{Reach}_R(M'(p'))\) follows as in the previous case.
(a.2.2) If \( p' \neq j \) for some \( j \in \{1, \ldots, m\} \), then \( M'(p') = M(p) \). Moreover, for all \( i \neq j \) with \( i \in \{1, \ldots, m\} \) it holds that \( M'(p',i) = M(p,i) \). Furthermore, by Lemma 4.14, \( \text{Reach}_R(M'(p',j)) = \text{Reach}_R(M'(p)) \subseteq \text{Reach}_R(M(p)) = \text{Reach}_R(M'(p,j)). \) Thus, \( f(\text{Reach}_R(M'(p',1)), \ldots, \text{Reach}_R(M'(p',m))) \subseteq f(\text{Reach}_R(M'(p',1)), \ldots, \text{Reach}_R(M'(p',m))) \subseteq \text{Reach}_R(M'(p')). \)

(b) The case where \( p' \) is \( p,p_1 \) for some \( p_1 \) satisfying \( p_1 \in \text{Pos}(r) \) and \( r|_{p_1} \) is a constant \( c_S \) is not possible when \( \text{root}(t'_{p'}) \) is \( f \).

c) If \( p' \) is \( p,p_1,p_2 \), for some \( p_1 \) and \( p_2 \) such that \( p_1 \in \text{Pos}(r) \) and \( r|_{p_1} \) is a variable, let \( Q \) be the positions in \( l \) where this variable occurs, i.e. \( Q = \{ q \in \text{Pos}(l) : l|_q = r|_{p_1} \} \). By the definition of derived marking, \( M'(p,p_1,p_2) = \bigcup_{q \in Q} M(p,q,p_2) \) holds, but also \( M'(p,p_1,p_2,i) = \bigcup_{q \in Q} M(p,q,p_2,i) \) for every \( i \in \{1, \ldots, m\} \) (recall that \( \text{root}(t'_{p'}) = f \)). For every \( q \in Q \) it holds that \( l|_{p,q,p_2} = t'_{p,p_1,p_2} \) and, since \( M \) is valid, it also holds that \( f(\text{Reach}_R(M(p,q,p_2)), \ldots, \text{Reach}_R(M(p,q,p_2,m))) \subseteq \text{Reach}_R(M(p,q,p_2)). \) Therefore:

\[
\begin{align*}
&f(\text{Reach}_R(M'(p,p_1,p_2,1)), \ldots, \text{Reach}_R(M'(p,p_1,p_2,m))) \\
&= f(\text{Reach}_R(\bigcup_{q \in Q} M(p,q,p_2,1)), \ldots, \text{Reach}_R(\bigcup_{q \in Q} M(p,q,p_2,m))) \\
&= f(\bigcap_{q \in Q} \text{Reach}_R(M(p,q,p_2,1)), \ldots, \bigcap_{q \in Q} \text{Reach}_R(M(p,q,p_2,m))) \\
&\subseteq \bigcap_{q \in Q} \text{Reach}_R(M(p,q,p_2)) \\
&= \text{Reach}_R(\bigcup_{q \in Q} M(p,q,p_2)) \\
&= \text{Reach}_R(M'(p,p_1,p_2)).
\end{align*}
\]

(d) If \( p' \) is \( p \) and \( \text{root}(r) \) is \( f \), then we have \( M'(p) = M(p) \).

In this case the used rule \( l \rightarrow r \) is either of the form (i) \( c_S \rightarrow f(c_S, \ldots, c_{S_m}) \) for \( \bar{x} \neq f(\text{Reach}_R(S_1), \ldots, \text{Reach}_R(S_m)) \subseteq \text{Reach}_R(S) \), or of the form (ii) \( \bar{x} \rightarrow f(\bar{a}_1, \ldots, \bar{a}_m) \) for \( l_1 \rightarrow f(\alpha_1, \ldots, \alpha_m) \).

—In case (i) it holds that \( M'(p) = M(p) = S \) and \( M'(p,1) = S_1, \ldots, M'(p,m) = S_m \), and hence, by the definition of \( R' \), \( f(\text{Reach}_R(M'(p,1)), \ldots, \text{Reach}_R(M'(p,m))) \subseteq \text{Reach}_R(M'(p)) \) follows.

—In case (ii) we choose any term \( u = f(u_1, \ldots, u_m) \in f(\text{Reach}_R(M'(p,1)), \ldots, \text{Reach}_R(M'(p,m))) \) and prove that \( f(u_1, \ldots, u_m) \in \text{Reach}_R(M'(p)) \), which in this case coincides with \( \text{Reach}_R(M(p)) \) since \( M'(p) = M(p) \). We consider all the leaf positions \( p_1, \ldots, p_n \) of \( l \) (which coincide with the leaf positions of \( l_1 \)), and choose terms \( t_1, \ldots, t_n \) satisfying \( t_1 \in \text{Reach}_R(M(p,p_1)), \ldots, t_n \in \text{Reach}_R(M(p,p_n)) \) and such that \( l|_{p_1} = l|_{p_2} \in V \) implies \( t_i = t_j \). This election is possible by the same argument given in the proof of Lemma 4.14, which is based on Corollary 4.11 and Lemma 4.12. Moreover, if \( l|_{p_i} \) is some \( c_k \in V \), then we choose \( t_i \) to be \( u_k \), note that, by the definition of derived marking, \( M'(p,k) = \bigcup_{q \in \text{Pos}(l): l|_q = c_k} M(p,q) \), and hence, \( \text{Reach}_R(M'(p,k)) = \bigcup_{q \in \text{Pos}(l): l|_q = c_k} \text{Reach}_R(M(p,q)) \). With this election, \( l[t_1|_{p_1}, \ldots, t_n|_{p_n}] \) \( R \)-reaches a term in one step that \( R \)-reaches \( u \) in several rewrite steps. Moreover,
\[ \{t_1, \ldots, t_n\} \in \mathcal{U}_{\text{Reach}}(M(p, p_1), \ldots, \mathcal{U}_{\text{Reach}}(M(p, p_n)), p_n, \text{ and since } \{p_1, \ldots, p_n\} \text{ is a set of disjoint and complete positions, by Lemma 4.10, } \\
\mathcal{U}_{\text{Reach}}(M(p, p_1)), \ldots, \mathcal{U}_{\text{Reach}}(M(p, p_n)) \subseteq \mathcal{U}_{\text{Reach}}(M(p)). \text{ Thus, } \\
u \in \mathcal{U}_{\text{Reach}}(M(p)) = \mathcal{U}_{\text{Reach}}(M'(p)) \text{ and we are done.} \]

\[\square\]

**Definition 4.16.** Let \( s \) be a term of the original signature \( T(\Sigma, \mathcal{V}) \), and let \( s^{(i)} = s_0 \xrightarrow{\mathcal{R}} s_1 \xrightarrow{\mathcal{R}} s_2 \xrightarrow{\mathcal{R}} \ldots \xrightarrow{\mathcal{R}} s_n = t \) be a derivation with \( \mathcal{R} \). The **marking of this derivation** is a list of functions \( M_0 : \text{Pos}(s_0) \rightarrow 2^{\text{Subterm}(s) \cup \Sigma_0}, \ldots, M_n : \text{Pos}(s_n) \rightarrow 2^{\text{Subterm}(s) \cup \Sigma_0} \) such that \( M_0(p) = \{s|_p\} \) for each \( p \in \text{Pos}(s_0) \) (Note that \( \text{Pos}(s_0) = \text{Pos}(s) \)), and each \( M_{i+1} \) is the derived marking from \( M_i \) and the \((i+1)\)th rewrite step.

**Lemma 4.17.** Let \( s \) be a term in \( T(\Sigma, \mathcal{V}) \). Let \( s^{(i)} = s_0 \xrightarrow{\mathcal{R}} s_1 \xrightarrow{\mathcal{R}} s_2 \xrightarrow{\mathcal{R}} \ldots \xrightarrow{\mathcal{R}} s_n \) be a derivation. Let \( M_0, M_1, M_2, \ldots, M_n \) be the marking of this derivation.

Then, for all \( i \) in \( \{1, \ldots, n\} \), \( M_i \) is a valid marking and \( \mathcal{U}_{\text{Reach}}(M_i(\lambda)) \subseteq \mathcal{U}_{\text{Reach}}(s) \).

**Proof.** The fact that \( M_0 \) is valid follows by definition of valid marking. Note that, for leaf positions \( p \in \text{Pos}(s_0) \) it holds that \( s_0|_p \) is either a variable \( x \) or a constant \( c_d \). In the first case \( s_0|_p \) is \( x \) and in the second it is \( d \). Thus, the definition \( M_0(p) = \{s|_p\} \) satisfies conditions (1) and (2) of the definition of valid marking. Moreover, any term \( t = f(t_1, \ldots, t_m) \in T(\Sigma, \mathcal{V}) \) satisfies \( \mathcal{U}_{\text{Reach}}(t) \supseteq f(\mathcal{U}_{\text{Reach}}(t_1), \ldots, \mathcal{U}_{\text{Reach}}(t_m)) \). Thus, condition (3) of the definition of valid marking is also satisfied for \( M_0 \).

The fact that the rest of \( M_i \) are valid markings follows by Lemma 4.15 and induction hypothesis.

The fact that \( \mathcal{U}_{\text{Reach}}(M_0(\lambda)) \subseteq \mathcal{U}_{\text{Reach}}(s) \) follows by the definition \( M_0(\lambda) = \{s|_\lambda\} = \{s\} \).

The fact that \( \mathcal{U}_{\text{Reach}}(M_i(\lambda)) \subseteq \mathcal{U}_{\text{Reach}}(s) \) for the rest of \( i \)'s follows by Lemma 4.14 and induction hypothesis. Note that, when the rewrite step \( s_i \xrightarrow{\mathcal{R}} s_{i+1} \) is done at a position \( p \) different from \( \lambda \), it holds that \( M_{i+1}(\lambda) = M_i(\lambda) \), and hence, \( \mathcal{U}_{\text{Reach}}(M_{i+1}(\lambda)) = \mathcal{U}_{\text{Reach}}(M_i(\lambda)) \). Otherwise, if \( p \) is \( \lambda \) then, by Lemma 4.14, it holds that \( \mathcal{U}_{\text{Reach}}(M_{i+1}(\lambda)) \subseteq \mathcal{U}_{\text{Reach}}(M_i(\lambda)) \).

**Lemma 4.18.** Let \( t \) be a term in \( T(\Sigma) \) such that there is a valid marking \( M : \text{Pos}(t) \rightarrow 2^\Sigma \).

Then, there exists a valid marking \( M' : \text{Pos}(t) \rightarrow 2^\Sigma \) such that, for all \( p \in \text{Pos}(t) \), it holds that \( c_{M'(p)} \) belongs to \( \Sigma \), and \( \mathcal{U}_{\text{Reach}}(M'(p)) = \mathcal{U}_{\text{Reach}}(M(p)) \).

**Proof.** Recall that, by Lemma 4.4, for every subset \( S \) of \( \Sigma \) satisfying \( \mathcal{U}_{\text{Reach}}(S) \neq \emptyset \), there exists a subset \( S' \) of \( \Sigma \) satisfying that \( c_S \) belongs to \( \Sigma \) and \( \mathcal{U}_{\text{Reach}}(S') = \mathcal{U}_{\text{Reach}}(S) \). Thus, we can define \( M' \) from \( M \) by defining each \( M'(p) \), for \( p \in \text{Pos}(t) \), as an election of a set satisfying that \( c_{M'(p)} \) belongs to \( \Sigma \), and \( \mathcal{U}_{\text{Reach}}(M'(p)) = \mathcal{U}_{\text{Reach}}(M(p)) \). Moreover, in the case where \( M(p) \) is singleton, we force \( M'(p) \) to be equal to \( M(p) \).

To see that this marking \( M' \) is valid, first consider a leaf position \( p \) of \( t \). Since \( t \) is ground, \( t|_p \) is never a variable, and hence, condition (2) of the definition of valid
marking is satisfied. Then, $t|_p$ is a constant $c_S \in \Sigma$. Since $M$ is valid, $M(p) = S$. In the case where $S$ is singleton, we have also $M'(p) = S$, and hence, condition (1) of the definition of valid marking is satisfied. Otherwise, if $S$ is not singleton, then, by Lemma 4.5 there is no $S'$ different from $S$ such that $c_S'$ belongs to $\Sigma$ and $\text{Reach}_R(S') = \text{Reach}_R(S)$. Thus, the election for $M'(p)$ must be $S = M(p)$. Again, condition (1) of the definition of valid marking is satisfied.

Now, for non-leaf positions $p$ in $t$, since $M$ is valid, it holds that $\text{Reach}_R(M(p)) \supseteq f(\text{Reach}_R(M(p.1)), \ldots, \text{Reach}_R(M(p.m)))$. By the definition of $M'$, it holds $\text{Reach}_R(M(p)) = \text{Reach}_R(M'(p))$ and $\text{Reach}_R(M(p,i)) = \text{Reach}_R(M'(p,i))$ for all $i \in \{1, \ldots, m\}$. Thus, $\text{Reach}_R(M'(p)) \supseteq f(\text{Reach}_R(M'(p.1)), \ldots, \text{Reach}_R(M'(p.m)))$, and hence, condition (3) of the definition of valid marking is also satisfied for $M'$. 

We let $B$ be the subset of rules of $R$ that are either of the form $c_S \rightarrow c_S'$ or of the form $c_S \rightarrow f(c_{S_1}, \ldots, c_{S_m})$.

**Lemma 4.19.** Let $t$ be a term in $T(\Sigma)$. Let $M : \text{Pos}(t) \rightarrow 2^\Sigma$ be a valid marking satisfying that $c_{M(p)}$ belongs to $\Sigma$, for each $p \in \text{Pos}(t)$.

Then, $c_{M(\lambda)} \not\rightarrow_B^* t$.

**Proof.** We prove it by induction on the height of $t$. If $t$ has height 0, then, since $t \in T(\Sigma)$, $t$ has to be a constant $c_S$. Since $M$ is a valid marking, $M(\lambda) = S$. Therefore, $c_{M(\lambda)} \not\rightarrow_B^* c_S = t$ follows.

If $t$ has height more than 0, then it is of the form $f(t_1, \ldots, t_m)$. Since $M$ is a valid marking, it holds that $\text{Reach}_R(M(\lambda)) \supseteq f(\text{Reach}_R(M(1)), \ldots, \text{Reach}_R(M(m)))$. By the assumptions of the lemma, all $c_{M(1)}, \ldots, c_{M(m)}$ belong to $\Sigma$. By the definition of $\text{Reach}_R$, the rule $c_{M(\lambda)} \rightarrow f(c_{M(1)}, \ldots, c_{M(m)})$ belongs to $\text{Reach}_R$. For each $i \in \{1, \ldots, m\}$, we can consider the marking $M_i : \text{Pos}(t_i) \rightarrow 2^\Sigma$ defined by $M_i(p) = M(p,i)$ for each $p \in \text{Pos}(t_i)$. This is a valid marking satisfying that $c_{M_i(p)}$ belongs to $\Sigma$ for each $p \in \text{Pos}(t_i)$. By induction hypothesis, for each $i$ in $\{1, \ldots, m\}$ it holds that $c_{M_i(\lambda)} \not\rightarrow_B^* t_i$. We conclude:

$$
c_{M(\lambda)} \not\rightarrow_B^* f(c_{M(1)}, \ldots, c_{M(m)}) \\
= \not\rightarrow_B^* f(c_{M_1(\lambda)}, \ldots, c_{M_m(\lambda)}) \\
= \not\rightarrow_B^* f(t_1, \ldots, t_m) \\
= t
$$

\[\square\]

**Lemma 4.20.** Let $t$ be a term in $T(\Sigma)$. Let $M_1, \ldots, M_k : \text{Pos}(t) \rightarrow 2^\Sigma$ be valid markings. Let $M : \text{Pos}(t) \rightarrow 2^\Sigma$ be defined as $M(p) = \bigcup_{i \in \{1, \ldots, k\}} M_i(p)$ for each $p$ in $\text{Pos}(t)$.

Then, $M$ is a valid marking.

**Proof.** First, consider a variable position $p$ of $t$, i.e., $t|_p = x$ for some variable $x$. Since $M_1, \ldots, M_k$ are valid, it holds that $M_1(p) = \ldots = M_k(p) = \{x\}$. Thus, $M(p) = \{x\}$, and hence, $M$ satisfies condition (2) of the definition of valid marking.

Now, consider a constant position $p$ of $t$, i.e., $t|_p = c_S$ for some $c_S \in \Sigma$. Since $M_1, \ldots, M_k$ are valid, it holds that $M_1(p) = \ldots = M_k(p) = S$. Thus, $M(p) = S$, and hence, $M$ satisfies condition (1) of the definition of valid marking.

Finally, consider a non-leaf position \( p \) of \( t \), i.e. \( t_p = f(t_1, \ldots, t_m) \).  
Since \( M_1, \ldots, M_k \) are valid, it holds that \( \text{Reach}_R(M_i(p)) \supseteq f(\text{Reach}_R(M_i(p)), \ldots, \text{Reach}_R(M_i(p))) \) for all \( i = 0, \ldots, k \). To conclude:

\[
\begin{align*}
\text{Reach}_R(M(p)) &= \text{Reach}_R(\bigcup_{i \in \{0, \ldots, k\}} M_i(p)) \\
&= \bigcap_{i \in \{0, \ldots, k\}} \text{Reach}_R(M_i(p)) \\
&\supseteq \bigcap_{i \in \{0, \ldots, k\}} f(\text{Reach}_R(M_i(p)), \ldots, \text{Reach}_R(M_i(p))) \\
&= f(\bigcap_{i \in \{0, \ldots, k\}} \text{Reach}_R(M_i(p)), \ldots, \bigcap_{i \in \{0, \ldots, k\}} \text{Reach}_R(M_i(p))) \\
&= f(\text{Reach}_R(M(p)), \ldots, \text{Reach}_R(M(p))) \\
\end{align*}
\]

\[
\square
\]

We are now ready to state and prove the claim that \( \overline{R} \) only conservatively extends \( R \).

**Corollary 4.21.** For all terms \( s, t \in T(\Sigma^\ell, \mathcal{V}) \), \( s \rightarrow^*_\overline{R} t \) iff \( s \rightarrow^*_R t \).

**Proof.** The \( \supseteq \) inclusion is straightforward from the fact that \( \overline{R} \supseteq R^\ell \). For the \( \subseteq \) inclusion we consider two terms \( s, t \in T(\Sigma, \mathcal{V}) \) such that \( s \rightarrow^*_\overline{R} t \). Then, prove that \( s \rightarrow^*_R t \), or equivalently, that \( s \rightarrow^*_R t \).

Since \( s \rightarrow^*_\overline{R} t \), by Lemma 4.17 there exists a valid marking \( M : \text{Pos}(t) \rightarrow 2^{\Sigma_0} \) of \( t \) such that \( \text{Reach}_R(M(\lambda)) \subseteq \text{Reach}_R(s) \). Let \( P = \{p_1, \ldots, p_n\} \) be the leaf positions of \( t \). Note that \( P \) is a set of disjoint positions of \( t \) which is also complete. By Lemma 4.10, \( t[\text{Reach}_R(M(p_1))]|_{p_1} \ldots [\text{Reach}_R(M(p_n))]|_{p_n} \subseteq \text{Reach}_R(M(\lambda)) \). Since \( t \in t[\text{Reach}_R(M(p_1))]|_{p_1} \ldots [\text{Reach}_R(M(p_n))]|_{p_n} \) and \( \text{Reach}_R(M(\lambda)) \subseteq \text{Reach}_R(s) \), we conclude \( s \rightarrow^*_R t \), and we are done. \( \square \)

**Corollary 4.22.** Let \( c_S \) be a constant in \( \Sigma \), and let \( t \) be a term in \( \text{Reach}_R(S) \). Then, there exists a derivation \( c_S \rightarrow^*_\overline{R} t \).

Moreover, any term \( s \in T(\Sigma, \mathcal{V}) \) \( B \)-reaches some term in \( T(\Sigma^\ell, \mathcal{V}) \).

**Proof.** The second fact is a direct consequence of the first: any constant term \( c_S \in \Sigma \) occurring in \( s \) \( B \)-reaches some term of the form \( t^\ell \) for some \( t \in T(\Sigma) \), since \( \text{Reach}_R(S) \neq \emptyset \) by the definition of \( \Sigma \).

For proving the first fact, let \( S \) be \( \{d_1, \ldots, d_n\} \). For each \( d_i \), it holds that \( \text{Reach}_R(d_i) \supseteq \text{Reach}_R(S) \). Thus, there exists a derivation \( d_i \rightarrow^*_R t \). Moreover, since \( \overline{R} \) includes \( R^\ell \), there exists a derivation \( c_{\{d_1, \ldots, d_n\}} \rightarrow^*_R t \). Furthermore, by Lemma 4.17, there exists a valid marking \( M : \text{Pos}(t) \rightarrow 2^{\Sigma_0} \) of \( t \) such that \( \text{Reach}_R(d_i) \supseteq \text{Reach}_R(M(\lambda)) \). Since this is true for each \( i \) in \( \{1, \ldots, n\} \), by Lemma 4.20, the marking \( M : \text{Pos}(t) \rightarrow 2^{\Sigma_0} \) defined by \( M(p) = \bigcup_{i \in \{1, \ldots, n\}} M_i(p) \), for each \( p \) in \( \text{Pos}(t) \), is also a valid marking of \( t \). Moreover, it holds that \( \text{Reach}_R(M(\lambda)) = \text{Reach}_R(\bigcup_{i \in \{1, \ldots, n\}} M_i(\lambda)) = \bigcap_{i \in \{1, \ldots, n\}} \text{Reach}_R(M_i(\lambda)) \subseteq \bigcap_{i \in \{1, \ldots, n\}} \text{Reach}_R(d_i) = \text{Reach}_R(\{d_1, \ldots, d_n\}) = \text{Reach}_R(S) \). By Lemma 4.18, there exists a valid marking \( M' : \text{Pos}(t) \rightarrow 2^{\Sigma_0} \) such that, for each \( p \) in \( \text{Pos}(t) \), it holds that \( \text{Reach}_R(M'(p)) = \text{Reach}_R(M(p)) \) and \( c_{M'(p)} \in \Sigma \). Hence,
by Lemma 4.19, $c_{M'(p)} \rightarrow_{B}^{*} t(1)$. Now, note that, since $\text{Reach}_{R}(M'(\lambda)) = \text{Reach}_{R}(M(\lambda)) \subseteq \text{Reach}_{R}(S)$, by the definition of $\Sigma$ and $R$, either $S$ is $M'(\lambda)$, or there exists a rule $c_{S} \rightarrow c_{M'(\lambda)}$ in $R$. This suffices to conclude $c_{S} \rightarrow_{B}^{*} t(1)$. 

**Corollary 4.23.** Let $c_{S_{1}}, \ldots, c_{S_{k}}$ be constants in $\Sigma$ and let $t$ be a term in $T(\Sigma, V)$ $\text{Reach}_{R}$-reachable from all these constants. Then, there exists $c_{S}$ in $\Sigma$ such that $c_{S} \rightarrow_{B}^{*} t$, and for all $i$ in $\{1, \ldots, k\}$, it holds that $c_{S_{i}} \rightarrow_{B}^{*} c_{S}$.

**Proof.** Let $S'$ be $\bigcup_{i \in \{1, \ldots, k\}} S_{i}$. We write $S'$ more explicitly as $\{d_{1}, \ldots, d_{n}\}$. Every of such $d_{i}$ belongs to a certain $S_{j}$. Thus, by the definition of $\Sigma$ and $R$, either $\{d_{i}\} = S_{j}$ or a rule $c_{d_{i}} \rightarrow c_{S_{j}}$ belongs to $B$. Hence, there exists a derivation $c_{d_{i}} \rightarrow_{R}^{*} t$, and by Lemma 4.17 there exists a valid marking $M_{i} : \text{Pos}(t) \rightarrow 2^{Z_{0}}$ of $t$ such that $\text{Reach}_{R}(M_{i}(\lambda)) \subseteq \text{Reach}_{R}(d_{i})$. Since this is true for all $i$ in $\{1, \ldots, n\}$, by Lemma 4.20, the marking $M : \text{Pos}(t) \rightarrow 2^{Z_{0}}$ defined by $M(p) = \bigcup_{i \in \{1, \ldots, n\}} M_{i}(p)$, for each $p$ in $\text{Pos}(t)$, is also a valid marking of $t$. Moreover, it holds that $\text{Reach}_{R}(M(\lambda)) = \text{Reach}_{R}(\bigcup_{i \in \{1, \ldots, n\}} M_{i}(\lambda)) = \bigcap_{i \in \{1, \ldots, n\}} \text{Reach}_{R}(M_{i}(\lambda)) \subseteq \bigcap_{i \in \{1, \ldots, n\}} \text{Reach}_{R}(d_{i}) = \text{Reach}_{R}(\{d_{1}, \ldots, d_{n}\}) = \text{Reach}_{R}(S(\lambda))$. By Lemma 4.18, there exists a valid marking $M' : \text{Pos}(t) \rightarrow 2^{Z_{0}}$ such that, for each $p$ in $\text{Pos}(t)$, it holds that $\text{Reach}_{R}(M'(p)) = \text{Reach}_{R}(M(p))$ and $c_{M'(p)} \in \Sigma$. Hence, by Lemma 4.19, $c_{M'(p)} \rightarrow_{B}^{*} t$. Now, note that $\text{Reach}_{R}(M'(\lambda)) = \text{Reach}_{R}(M(\lambda)) \subseteq \text{Reach}_{R}(S')$. Moreover, for each $i$ in $\{1, \ldots, k\}$, it holds that $\text{Reach}_{R}(S') \subseteq \text{Reach}_{R}(S_{i})$. Thus, $\text{Reach}_{R}(M'(\lambda)) \subseteq \text{Reach}_{R}(S)$. Since both $c_{M'(\lambda)}$ and $c_{S_{i}}$ are in $\Sigma$, then, by the definition of $\Sigma$ and $R$, either $S_{i}$ is $M'(\lambda)$ or there exists a rule $c_{S_{i}} \rightarrow c_{M'(\lambda)}$ in $B$. Therefore, by calling $S$ to $M'(\lambda)$, we conclude that, for each $i$ in $\{1, \ldots, k\}$, $c_{S_{i}} \rightarrow_{B}^{*} c_{S} \rightarrow_{B}^{*} t$ holds. 

As the particular case for $k = 1$ of the previous corollary we obtain the following result which establishes Property (A3).

**Corollary 4.24.** Let $c_{S}$ be a constant in $\Sigma$ and let $t$ be a term in $\text{Reach}_{R}(c_{S})$. Then, there exists a derivation $c_{S} \rightarrow_{B}^{*} t$.

We can also show that $R$ satisfies Property (A4).

**Corollary 4.25.** Let $t_{1}, \ldots, t_{n}$ be terms in $T(\Sigma, V)$ and let $s$ be a term $B$-reachable from $\{t_{1}, \ldots, t_{n}\}$. Then, there exists a term $s'$ in $T(\Sigma, V)$ $B$-reachable from $\{t_{1}, \ldots, t_{n}\}$ such that $s' \rightarrow_{B}^{*} s$ and $\text{Pos}(s') = \bigcup_{i \in \{1, \ldots, n\}} \text{Pos}(t_{i})$.

**Proof.** We prove it by induction on $|s|$. If all the $t_{i}$ are constants, the result follows from Corollary 4.23. If some $t_{i}$ is a variable then all $t_{i}$ must be the same variable and hence, the result follows again. Otherwise, no $t_{i}$ is a variable and some of the $t_{i}$ are not constants. This implies that $s$ has height greater than 0 and that $\bigcup_{i \in \{1, \ldots, n\}} \text{Pos}(t_{i})$ contains the positions $1, \ldots, m$.

Suppose that a concrete $t_{i}$ is still a constant. Since $t_{i} \rightarrow_{R}^{*} s$ holds, by Corollary 4.24 there is a derivation of the form $t_{i} \rightarrow_{B}^{*} s$. Hence, this derivation is of the form $t_{i} \rightarrow_{B} t_{i}' \rightarrow_{B}^{*} s$, where height$(t_{i}')$ is 1. Thus, we can replace all the $t_{i}$ which are constants by height 1 terms, and hence, we can assume that all the $t_{i}$ are not constants.

Now, for each position $j$ in $\{1, \ldots, m\}$, it holds that $s|_{j}$ is $B$-reachable from $\{t_{1}|_{j}, \ldots, t_{n}|_{j}\}$. By induction hypothesis there exists $s'_{j}$ in $T(\Sigma, V)$ $B$-reachable
from \( \{s_1, \ldots, s_m\} \) such that \( s'_j \rightarrow_R s_j \) and \( \text{Pos}(s'_j) = \bigcup_{i \in \{1, \ldots, m\}} \text{Pos}(t_{i,j}) \). Thus, the term \( s' \) defined as \( f(s'_1, \ldots, s'_m) \) satisfies the statement.

We now move on to proving that properties, such as confluence and weak normalization, are preserved.

**Lemma 4.26.** For any terms \( s, t \in T(\Sigma^{(1)}, \mathcal{V}) \), \( s \leftrightarrow_R^* t \) iff \( s \leftrightarrow_{R^{(1)}}^* t \).

**Proof.** The \( \supseteq \) inclusion is straightforward since \( \mathcal{R} \) includes \( R^{(1)} \). For the other direction, it suffices to observe that the new rules force every constant \( c_{d_1, \ldots, d_n} \in \Sigma \) to be in the same congruence class of every \( c_{d_i} \), and that the different \( R^{(1)} \)-classes are forced to be the same class by these new rules. But we can prove it more formally. Let \( T : \Sigma \rightarrow \Sigma \) be any function such that \( T(c_S) \) is some \( d \in S \), for all \( c_S \in \Sigma \). Now, let \( T \) be extended to any term in \( T(\Sigma, \mathcal{V}) \) as \( T(f(t_1, \ldots, t_m)) = f(T(t_1), \ldots, T(t_m)) \), and \( T(x) = x \) for any variable \( x \). Note that \( T \) is a transformation from \( T(\Sigma, \mathcal{V}) \) into \( T(\Sigma, \mathcal{V}) \), and that any \( s \in T(\Sigma, \mathcal{V}) \) satisfies \( T(s^{(1)}) = s \).

For rules of the form \( c_S \rightarrow c' \in \mathcal{R} \), the condition \( \theta \neq \text{Reach}_R(S') \) of \( \Sigma \) and the condition \( \text{Reach}_R(S') \subseteq \text{Reach}_R(S) \) of \( \mathcal{R} \) imply \( T(c_S) \rightarrow^* T(c') \). For rules of the form \( c_S \rightarrow f(c_{S_1}, \ldots, c_{S_m}) \in \mathcal{R} \) the condition \( c_{S_1}, \ldots, c_{S_m} \in \Sigma \) implies \( \theta \neq f(\text{Reach}_R(S_1), \ldots, \text{Reach}_R(S_m)) \), and by the definition of \( \mathcal{R} \) we also have \( f(\text{Reach}_R(S_1), \ldots, \text{Reach}_R(S_m)) \subseteq \text{Reach}_R(S) \). This implies \( T(c_S) \rightarrow^* T(f(c_{S_1}, \ldots, c_{S_m})) \).

Hence, the existence of a derivation \( s \leftrightarrow_R^* t \) implies the existence of a derivation \( T(s) \rightarrow^* R T(t) \). Since any term \( s \in T(\Sigma, \mathcal{V}) \) satisfies \( T(s^{(1)}) = s \), we conclude that \( \rightarrow^*_R \cap T(\Sigma^{(1)}, \mathcal{V}) \times T(\Sigma^{(1)}, \mathcal{V}) \subseteq \rightarrow^*_R \cap T(\Sigma^{(1)}, \mathcal{V}) \times T(\Sigma^{(1)}, \mathcal{V}) \). 

**Theorem 4.27.** \( R \) is confluent iff \( \mathcal{R} \) is confluent.

**Proof.**

\( \Rightarrow \) Assume \( \mathcal{R} \), and thus \( R^{(1)} \), is confluent and let \( s \) and \( t \) be two terms in \( T(\Sigma, \mathcal{V}) \) such that \( s \leftrightarrow^* R t \). By Corollary 4.22, there exist terms \( s_1 \) and \( t_1 \) in \( T(\Sigma^{(1)}, \mathcal{V}) \) that are \( \mathcal{R} \)-reachable from \( s \) and \( t \), respectively. Hence, we have \( s_1 \leftrightarrow^* R t_1 \), and by Lemma 4.26 \( s_1 \leftrightarrow^*_{R^{(1)}} t_1 \) also holds. By confluence of \( R \), there exists a term \( u \in T(\Sigma^{(1)}, \mathcal{V}) \) such that \( s_1 \rightarrow^* R t_1 \) and \( t_1 \rightarrow^* R t_1 \). Therefore, \( s \rightarrow^* R t_1 \) and \( t \rightarrow^* R t_1 \) concluding that \( \mathcal{R} \) is confluent.

\( \Leftarrow \) Now, assume that \( \mathcal{R} \) is confluent and let \( s \) and \( t \) be two terms in \( T(\Sigma, \mathcal{V}) \) such that \( s \leftrightarrow^* R t \). By Lemma 4.26, \( s^{(1)} \leftrightarrow^*_R t^{(1)} \), and by the confluence of \( \mathcal{R} \) there exists a term \( u \in T(\Sigma^{(1)}, \mathcal{V}) \) such that \( s^{(1)} \rightarrow^*_R u^{(1)} \) and \( t^{(1)} \rightarrow^*_R u^{(1)} \). By Corollary 4.22 there exists a term \( u_1 \in T(\Sigma, \mathcal{V}) \) such that \( u \rightarrow^* u_1^{(1)} \). Therefore, \( s^{(1)} \rightarrow^*_R u_1^{(1)} \) and \( t^{(1)} \rightarrow^*_R u_1^{(1)} \), and since all the three terms \( s^{(1)}, t^{(1)} \) and \( u_1^{(1)} \) are in \( T(\Sigma^{(1)}, \mathcal{V}) \), by Corollary 4.21, \( s^{(1)} \rightarrow^*_R u_1^{(1)} \) and \( t^{(1)} \rightarrow^*_R u_1^{(1)} \). Thus, \( s \rightarrow^*_R u_1 \) and \( t \rightarrow^*_R u_1 \), concluding that \( R \) is confluent.
Lemma 4.28. A term \( t \in T(\Sigma, \mathcal{V}) \) is a \( \overline{R} \)-normal form iff \( t \) is in \( T(\Sigma^1, \mathcal{V}) \) and it is a \( R^1 \)-normal form.

Proof.

\( \Rightarrow \) Let \( t \) be a \( \overline{R} \)-normal form. We proceed by contradiction by assuming that \( t \) is not in \( T(\Sigma^1, \mathcal{V}) \). Then, \( t \) has an occurrence of a constant \( c_S \) for a non-singleton set \( S \subseteq \Sigma \). Therefore, by Corollary 4.22, this constant \( \overline{R} \)-reaches a term in \( \Sigma^1 \). Thus, \( t \) is not \( \overline{R} \)-normal form, a contradiction. It follows that \( t \) is in \( T(\Sigma^1, \mathcal{V}) \), and since \( R^1 \subseteq \overline{R} \), we conclude that \( t \) is a \( R^1 \)-normal form.

\( \Leftarrow \) Let \( t \) be a term in \( T(\Sigma^1, \mathcal{V}) \) which is a \( R^1 \)-normal form. Since rules in \( R^1 \) cannot be applied on \( t \), and \( \overline{R} \) is the union of \( R^1 \), \( R^c \), and \( R' \), it suffices to see that no rule of \( R^c \) nor \( R' \) can be applied on \( t \). The constants of \( t \) are of the form \( c_\{d\} \), where \( d \) is a \( \overline{R} \)-normal form, and hence, it is enough to see that there is no rule with \( c_\{d\} \) as left-hand side in \( R^c \) nor \( R' \).

Suppose there exists a rule \( c_\{d\} \rightarrow c_S \) in \( R^c \). Then, \( R_{\overline{R}}(S) \subseteq R_{\overline{R}}(\{d\}) \) and \( S \neq \{d\} \). Since \( d \) is a \( \overline{R} \)-normal form, \( R_{\overline{R}}(\{d\}) = \{d\} \), and hence, \( R_{\overline{R}}(S) \) is either \( 0 \) or \( \{d\} \). By definition of \( \Sigma \), \( R_{\overline{R}}(S) \) cannot be \( 0 \), and hence, it is necessarily \( \{d\} \). The set \( S \) cannot be a singleton set \( \{e\} \), since \( S \neq \{d\} \) implies \( e \neq d \) and \( e \in R_{\overline{R}}(\{e\}) = R_{\overline{R}}(\{d\}) \), a contradiction with \( R_{\overline{R}}(S) \subseteq R_{\overline{R}}(\{d\}) \). Thus \( |S| \geq 2 \). But then, \( S \) is a set for which there exists a singleton set \( \{d\} \) satisfying \( R_{\overline{R}}(S) = R_{\overline{R}}(\{d\}) \), and hence, by Lemma 4.5 \( c_S \notin \Sigma \) thus \( c_\{d\} \rightarrow c_S \notin R^c \), a contradiction.

Suppose there exists a rule \( c_\{d\} \rightarrow f(c_{S_1}, \ldots, c_{S_m}) \) in \( R' \). Then, \( 0 \neq f(R_{\overline{R}}(S_1), \ldots, R_{\overline{R}}(S_m)) \subseteq R_{\overline{R}}(\{d\}) = \{d\} \), a contradiction.

\[ \square \]

Theorem 4.29. \( R \) is weakly normalizing iff \( \overline{R} \) is weakly normalizing.

Proof. This is equivalent to prove that \( R^1 \) is weakly normalizing iff \( \overline{R} \) is weakly normalizing.

\( \Rightarrow \) Assume that \( R^1 \) is weakly normalizing and let \( s \) be any term in \( T(\Sigma, \mathcal{V}) \). It suffices to prove that \( s \overline{R} \)-reaches a \( \overline{R} \)-normal form. By Corollary 4.22, \( s \overline{R} \)-reaches a term \( s' \) in \( \Sigma^1 \), and since \( \overline{R} \) includes \( R^1 \) and \( R^1 \) is weakly normalizing, it follows that \( s' \) reaches a \( \overline{R} \)-normal form \( t \). By Lemma 4.28, \( t \) is also a \( \overline{R} \)-normal form, and we are done.

\( \Leftarrow \) Assume that \( \overline{R} \) is weakly normalizing, and let \( s \) be any term in \( T(\Sigma^1, \mathcal{V}) \). It suffices to prove that \( s \overline{R} \)-reaches a \( \overline{R} \)-normal form. Since \( \overline{R} \) is weakly normalizing, there exists a \( \overline{R} \)-normal form \( s' \) \( \overline{R} \)-reachable from \( s \). By Lemma 4.28, \( s' \) is in \( T(\Sigma^1, \mathcal{V}) \) and is a \( R^1 \)-normal form. By Corollary 4.21, \( s \) also \( R^1 \)-reaches \( s' \), and we are done.

\[ \square \]

Theorem 4.30. \( R \) is uniquely normalizing iff \( \overline{R} \) is uniquely normalizing.

Proof. This is equivalent to prove that \( R^1 \) is uniquely normalizing iff \( \overline{R} \) is uniquely normalizing.
\(\Rightarrow\) Assume that \(R^1\) is uniquely normalizing and let \(s\) be any term in \(T(\Sigma, \nu)\). We proceed by contradiction by assuming that \(s\) can \(R\)-reach two different \(R\)-normal forms \(t_1\) and \(t_2\). By Lemma 4.28, \(t_1\) and \(t_2\) are in \(T(\Sigma^1, \nu)\) and are \(R^1\)-normal forms. Let \(s'\) be obtained from \(s\) by replacing every constant \(c_S\) by a corresponding constant \(c(d)\), where \(d\) is selected from the constants included in \(S\). By the definition of \(\Sigma\) and \(R\), \(s' \rightarrow_R^* s\). Thus, \(s' \rightarrow_R^* t_1\) and \(s' \rightarrow_R^* t_2\). Since the three terms \(s', t_1\) and \(t_2\) are in \(T(\Sigma^1, \nu)\), by Corollary 4.21, it holds \(s' \rightarrow_R^* t_1\) and \(s' \rightarrow_R^* t_2\), and this is in contradiction with the unique normalization of \(R^1\).

\(\Leftarrow\) Assume that \(R\) is uniquely normalizing and let \(s\) be any term in \(T(\Sigma^1, \nu)\). We proceed by contradiction by assuming that \(s\) can \(R^1\)-reach two different \(R^1\)-normal forms \(t_1\) and \(t_2\). By Lemma 4.28, \(t_1\) and \(t_2\) are also \(R\)-normal forms. Since \(\Sigma\) includes \(\Sigma^1\) and \(R\) includes \(R\), the three terms \(s, t_1\) and \(t_2\) are in \(T(\Sigma, \nu)\), and \(s\) \(R\)-reaches \(t_1\) and \(t_2\), a contradiction with the unique normalization of \(R\).

For proving preservation of the termination property, we need in particular to show that termination of \(R\) implies termination of \(R^1\). We argue this direction progressively by the following lemmas.

**Lemma 4.31.** *If \(R\) is terminating, then \(R^c\) is terminating.*

**Proof.** Assume \(R\) is terminating but \(R^c\) is not. Then, there is an infinite derivation \(c_S, \rightarrow_R c_S, \rightarrow_R \ldots\). By the definition of \(R^c\), for every \(i \geq 1\) it holds that \(\text{Reach}_R(S_{i+1}) \subseteq \text{Reach}_R(S_i)\). Moreover, by Lemma 4.5 and the definition of \(R\), if one of \(S_{i+1}\) or \(S_i\) is not a singleton set, then this relation is strict, i.e., \(\text{Reach}_R(S_{i+1}) \subset \text{Reach}_R(S_i)\).

Suppose that for infinite \(i\)'s it holds that \(S_i\) is not singleton. It follows the existence of an infinite subsequence \(\text{Reach}_R(S_{i_1}) \supset \text{Reach}_R(S_{i_2}) \supset \ldots\). In particular, all these sets \(S_{i_1}, S_{i_2}, \ldots\) are different. But this is impossible, since all of them are subsets of \(\Sigma\). Hence, we conclude that only a finite number of such \(S_i\)'s are not singleton sets. Thus, there exists an \(n\) satisfying that all \(S_i\) for \(i \geq n\) are singleton sets. Thus, for all \(i \geq 0\), \(S_{n+i}\) and \(S_{n+i+1}\) are singleton sets \(\{d_{n+i}\}\) and \(\{d_{n+i+1}\}\), and hence, there is a rule \(c_{\{d_{n+i+1}\}} \rightarrow c_{\{d_{n+i}\}}\) in \(R^c\). Since \(d_{n+i+1} \in \text{Reach}_R(\{d_{n+i+1}\}) \subseteq \text{Reach}_R(\{d_{n+i}\})\), and, by the definition of \(R^c\), \(d_{n+i+1} \neq d_{n+i}\), it follows that \(d_{n+i+1}\) is \(R\)-reachable from \(d_{n+i}\) in one or more steps. Therefore, there exists an infinite derivation \(\longrightarrow^+_R d_{n+i} \rightarrow^+_R \ldots\), contradicting the termination of \(R\). \(\square\)

**Lemma 4.32.** *If \(R\) is terminating, then \(R^c \cup R^f\) is terminating.*

**Proof.** Assume that \(R\) is terminating but \(R^c \cup R^f\) is not. Then, any derivation with \(R\) from a constant \(d \in \Sigma\) terminates. Let \(N\) be a bound for the size of any term \(R\)-reachable from any constant in \(\Sigma\). Since \(R^c \cup R^f\) does not terminate, there exists an infinite derivation with \(R^c \cup R^f\) starting from some term in \(\Sigma\). By the form of the rules of \(R^c \cup R^f\), we can assume this derivation to start from a constant \(c_S\). Thus, there exists an infinite derivation of the form \(c_S \rightarrow_R t_1 \rightarrow_R t_2 \rightarrow_R \ldots\). By Lemma 4.31, \(R^c\) terminates, and hence, there exist infinite rewrite steps with
$R^l$ in this derivation: otherwise, a queue of this derivation would be an infinite derivation with $R^l$, a contradiction. Since application of a rule in $R^c$ preserves the size of a term, and application of a rule in $R^l$ increases the size of a term, it follows that, in the previous derivation, terms with arbitrarily large size occur. Therefore, $c_S \rightarrow_R^* t$ for some term $t$ with size greater than $N$. By Corollary 4.22, $t$ $B$-reaches a term $s^{(1)} \in T(\Sigma^{(1)})$ for some $s \in T(\Sigma)$. Since $B$-rules either preserve or increase the size, it holds that the size of $s^{(1)}$ is greater than $N$. Now, let $d$ be any constant in $S$. Either $\{d\} = S$ or a rule $c_{\{d\}} \rightarrow c_S$ belongs to $R^c$. In any case, it follows that $c_{\{d\}} \rightarrow_R^* s^{(1)}$. By Corollary 4.21, we have $d \rightarrow^{*}_R s$, and since $|s| > N$, this contradicts the election of $N$.\end{proof} 

**Definition 4.33.** Let $s$ be a term in $T(\Sigma, \mathcal{V})$. We say that a term $t \in T(\Sigma, \mathcal{V})$ is an election of $s$ if $s \rightarrow^{*}_R t^{(1)}$. 

**Lemma 4.34.** Let $u \rightarrow^{*}_R v$ be a derivation with $k$ steps using rules from $R^{(1)}$. Then, there exist elections $u'$ and $v'$ obtained from $u$ and $v$, respectively, and a derivation $u' \rightarrow^{*}_R v'$ with at least $k$ steps. 

**Proof.** We prove it by induction on the length of $u \rightarrow^{*}_R v$. If this derivation has length 0 then $u = v$ and the result follows trivially by choosing the same election $u' = v'$ (note that, by Corollary 4.22, an election always exists for any term). Otherwise, assume that $u \rightarrow^{*}_R v$ is of the form $u_1 \rightarrow^{*}_R u_2 \rightarrow^{*}_R u_3 \rightarrow^{*}_R \ldots \rightarrow^{*}_R u_n = v$ for $n \geq 2$. Let $k'$ be the number of steps using $R^{(1)}$ in $u_2 \rightarrow^{*}_R u_n$. By induction hypothesis, there exist elections $u_2'$ and $v_2'$ obtained from $u_2$ and $u_n$, respectively, such that $u_2' R$-reaches $u_n'$ by a derivation with at least $k'$ steps. 

If $u_1 \rightarrow^{*}_R u_2$ is done with a rule in $R - R^{(1)}$, then it is a $B$-step and hence, $k = k'$, $u_2'$ is also an election for $u_1$, and the result follows. Otherwise, assume that $u_1 \rightarrow^{*}_R u_2$ uses a rule $r^{(1)} \rightarrow r^{(1)}$ from $R^{(1)}$ at a position $p$. Let $u_2''$ be the term obtained by replacing every position $p.p'$ in $u_2'$, satisfying that $r_{|p'}$ is a constant, by this constant $r_{|p'}$. Note that each of such positions $p.p'$ satisfies $u_2|_{p.p'} = c_{\{r_{|p'}\}}$. By Corollary 4.21, since $c_{\{r_{|p'}\}} B$-reaches $u_2'|_{p.p'}$, it follows that $u_2''|_{p.p'} = r_{|p'} R$-reaches $u_2'|_{p.p'}$. Therefore, $u_2'' R$-reaches $u_2'$. To conclude, it suffices to define $u_1'$ as any election of $u_1$ such that $u_1'$ rewrites to $u_2''$ using $l \rightarrow r$ at position $p$. To this end, $u_1'$ must satisfy the following conditions. It has to be equal to $u_2'$ at all positions $p'$ that are either disjoint with $p$, or a proper prefix of $p$. Moreover, $u_1'|_{p}$ has to be $\sigma(l)$, for any substitution $\sigma$ satisfying that $\sigma(r)$ is $u_2''|_{p}$, and such that, for any position $p''$ where $l_{|p''}$ is a variable not occurring in $r$, it holds that $u_1'|_{p,p''}$ is an election of $u_1|_{p,p''}$. It is straightforward to check that all such conditions can be satisfied. \end{proof} 

**Theorem 4.35.** $R$ is terminating iff $\overline{R} = R^c \cup R^l \cup R^{(1)}$ is terminating. 

**Proof.** The right-to-left direction is trivial, since $\overline{R}$ contains $R^{(1)}$, and $R^{(1)}$ is homomorphic to $R$. For the left-to-right direction, assume that $R$ is terminating but $\overline{R}$ is not. Then, any derivation with $R$ from a constant $d \in \Sigma$ terminates. Let $N$ be a bound for the size of any term $R$-reachable from any constant in $\Sigma$. Let $t = t_1 \rightarrow^{*}_R t_2 \rightarrow^{*}_R \ldots$ be an infinite derivation. Similarly to before, since $R$ terminates, let $M$ be a bound for the length of a derivation (i.e., number of rewrite steps) with

starting from any term in $T(\Sigma, \mathcal{V})$ of size smaller than or equal to $|t|(N + 1)$. By Lemma 4.32, $R^c \cup R^l$ terminates, and hence, the derivation $t = t_1 \rightarrow_\mathcal{R} t_2 \rightarrow_\mathcal{R} \ldots$ has infinite steps using $R^l$: otherwise, a queue of this derivation would be an infinite derivation with $R^c \cup R^l$, a contradiction. Let $t_1 \rightarrow_\mathcal{R}^* t_n$ be a prefix of this infinite derivation with more than $M$ steps using rules of $R^l$. By Lemma 4.34, there exist elections $t'_1$ and $t'_n$ of $t_1$ and $t_n$, respectively, and a derivation $t'_1 \rightarrow_\mathcal{R}^* t'_n$ with more than $M$ rewrite steps. By the definition of election, $t'_1$ is obtained from $t_1$ by replacing constants $c_S$ by corresponding terms $s$ in $T(\Sigma)$ such that $s^l$ is $B$-reachable from $c_S$. For every constant $d$ in each of such $S$, either a rule $c_{d} \rightarrow c_S$ belongs to $R^c$ or $\{d\} = S$. In any case it also holds that $c_{d} B$-reaches $s^l$. By Corollary 4.21, $d$ $R$-reaches $s$. Thus, every of such terms $s$ satisfies $|s| < N$. We conclude that $|t'_1| \leq |t| + |t|N = |t|(N + 1)$, which is the bound on terms used in the election of $M$. In summary, $t'_1$ $R$-reaches $t'_n$ via a derivation with more than $M$ steps, and $t'_1$ has a size bounded by $|t|(N + 1)$. This is in contradiction with the election of $M$. □

**Theorem 4.36.** $\mathcal{R}$ satisfies (A3) and (A4), and is terminating (confluent, weakly normalizing, uniquely normalizing) if and only if $R$ is.

5. REWRITE CLOSURE

In this section, we assume that $R$ has Properties (A0), (A1), (A2), (A3) and (A4), $B$ is the subset of $R$ containing all rules with constants on the left-hand side, and show how to construct a rewrite closure presentation for $R$ using constrained rules.

**Definition 5.1.** A constrained rule is an expression of the form $l \rightarrow r|C$ satisfying the following conditions:

—$l$ and $r$ are linear terms which do not share variables.

—$\text{height}(l) \geq 1 \geq \text{height}(r)$.

—$C$ is a set of elements of the form $\{\{\alpha_1, \ldots, \alpha_n\}, x\}$ or $\{\{\alpha_1, \ldots, \alpha_n\}\}$ where each $\alpha_i$ is either a constant or a variable occurring in $l$, and $x$ is a variable occurring in $r$.

—Every variable of $\text{Vars}(l) \cup \text{Vars}(r)$ occurs once in $C$.

A solution of $C$ is a substitution $\sigma$ such that, for every $\{\{\alpha_1, \ldots, \alpha_n\}, x\}$ or $\{\{\alpha_1, \ldots, \alpha_n\}\}$ in $C$, it holds that $\{\sigma(\alpha_1), \ldots, \sigma(\alpha_n)\}$ is $B$-joinable, and for the case of $\{\{\alpha_1, \ldots, \alpha_n\}, x\}$, $\sigma(x)$ is $B$-reachable from $\{\sigma(\alpha_1), \ldots, \sigma(\alpha_n)\}$ and $\text{Pos}(\sigma(x)) = \bigcup_{i \in \{1, \ldots, n\}} \text{Pos}(\sigma(\alpha_i))$.

The term $s$ rewrites in one step to $t$ with a rule $l \rightarrow r|C$ at position $p$ if there exists a solution $\sigma$ of $C$ such that $s|_p$ is $\sigma(l)$, and $t$ is $s|\sigma(r)|_p$.

For the rewrite closure presentation, we keep the $B$ rules and transform the rest of rules into constrained rules.

**Definition 5.2.** Let $l \rightarrow r$ be a right-flat right-linear rule with $\text{height}(l) > 0$. The constrained version of $l \rightarrow r$ is a constrained rule $l' \rightarrow r'|C$ such that there is a substitution $\sigma$ mapping variables to variables and satisfying $\sigma(l') = l$, $\sigma(r') = r$ and $\sigma(C)$ is a set of elements of the form $\{\{x, \ldots, x\}, x\}$ or $\{\{x, \ldots, x\}\}$. 

Example 5.3. The constrained version of the rule \( f(x, f(g(x), y)) \rightarrow y \) is the rule \( f(x_1, f(g(x_2), y_1)) \rightarrow y_2 \{ \{ y_1 \}, y_2 \}, \{ x_1, x_2 \} \}.

The rewrite closure of \( R \) is obtained by transforming the rules in \( R - B \) to their constrained versions and then saturating this new set of rules under the following inference steps:

\[
\begin{align*}
(R1) & \quad \frac{c \rightarrow d}{l[c]_p \rightarrow r \mid C} \\
(R2) & \quad \frac{c \rightarrow f(c_1, \ldots, c_m)}{l[f(c_1, \ldots, c_m)]_p \rightarrow r \mid \sigma(C)}
\end{align*}
\]

where \( |p| \geq 1 \) and, in Rule (R2), \( \sigma \) is the most general unifier of \( f(c_1, \ldots, c_m) \) and \( f(\alpha_1, \ldots, \alpha_m) \).

Note that we only add new constrained rules, and the set of \( B \) rules is unchanged.

Example 5.4. (Extends Example 5.3) By saturating the set of rules \( \{ f(x_1, f(g(x_2), y_1)) \rightarrow y_2 \{ \{ y_1 \}, y_2 \}, \{ x_1, x_2 \} \} \), \( d \rightarrow e \), \( c \rightarrow h(c') \), \( a \rightarrow f(b, c) \), \( b \rightarrow g(d) \) we add the new rules \( f(x_1, f(b, y_1)) \rightarrow y_2 \{ \{ y_1 \}, y_2 \}, \{ x_1, d \} \} \) and \( f(x_1, a) \rightarrow y_2 \{ \{ c \}, y_2 \}, \{ x_1, d \} \} \).

Let \( F \) denote the saturated set of constrained rules by rewrite closure. Since there are only a bounded number of new \( F \) rules that can be possibly added, the saturation process finitely terminates.

**Lemma 5.5.** The rewrite relations \( \rightarrow^*_R \) and \( \rightarrow^*_{F \cup B} \) are identical.

**Proof.** The constrained version \( l' \rightarrow r' \mid C \) of a rule \( l \rightarrow r \) is more expressive than the rule \( l \rightarrow r \) itself, but can be simulated by one or more steps by \( R \). Thus, when we replace each rule in \( R - B \) by its constrained version the relation \( \rightarrow^*_R \) is preserved. Each inference step adds a new rule which can be simulated by two previous ones. Thus, the relation \( \rightarrow^*_R \) is preserved again.

**Corollary 5.6.** The rewrite relations \( \rightarrow^*_R \) and \( \rightarrow^*_{F \cup B} \) are identical.

The benefit of rewrite closure is that it allows us to transform any rewrite derivation \( s \rightarrow^*_R t \) into a “valley derivation” of the form \( s \rightarrow^*_F u \rightarrow^*_B t \). In order to prove this property, we first argue that “local peaks” \( s \rightarrow^*_{B - F} t \) can be committed to either \( s \rightarrow^*_B t \) or \( s \rightarrow^*_{F - B} t \).

**Lemma 5.7.** Let \( s \) and \( t \) be terms satisfying \( s \rightarrow^*_B t \). Then, either \( s \rightarrow^*_B t \) or \( s \rightarrow^*_{F - B} t \) holds.

**Proof.** Consider the rewrite derivation \( s \rightarrow^*_{B - F} u \rightarrow^*_{C \cap \sigma_2} t \) where we have made explicit the intermediate term, the rule, the substitution and the involved position. We have to consider different cases depending on the positions \( p_1, p_2 \).

**Case 1: Nonoverlap** Suppose \( p_1 \) and \( p_2 \) are disjoint positions. In this case, the original rewrite derivation is of the form

\[
\begin{align*}
s & := s[c]_{p_1}[\sigma(l)]_{p_2} \rightarrow^*_{v, p_1} u := s[v]_{p_1}[\sigma(l)]_{p_2} \\
& \rightarrow^*_{v, p_1, C, p_2} t := s[v]_{p_1}[\sigma(r)]_{p_2}.
\end{align*}
\]

By commuting the two steps, we get a new rewrite derivation

\[ s := s[c_{p_1}[\sigma(l)]_{p_2} \rightarrow_{t \rightarrow r|C_{p_2}} u' := s[c_{p_1}[\sigma(r)]_{p_2} \rightarrow_{c \rightarrow v, p_1} t := s[v]_{p_1}[\sigma(r)]_{p_2}. \]

(Case 2: Variable Overlap) Suppose \( p_1 = p_2, p'_2, p_3 \) and \( \ell|_{p'_2} = x \). Let \( \{\{x = \alpha_1, \alpha_2, \ldots, \alpha_n\}, y\} \) be the constraint in \( C \) for \( x \). (The case where there is no \( y \) is simpler and follows using the same argument as given below.) In this case, the old rewrite derivation is of the form

\[ s := s[\sigma(l)[c]_{p'_2, p_3|p_2} \rightarrow_{c \rightarrow v, p_1} u := s[\sigma(l)[v]_{p'_2, p_3|p_2} = s[\sigma(l)]_{p_2} \rightarrow_{t \rightarrow r|C_{p_2}} t := s[\sigma(r)]_{p_2}. \]

We will again commute the two steps. We first need a new substitution \( \sigma' \) such that \( \sigma'(l) = s[p_2] \), and \( \sigma' \) is a solution of \( C \). We let \( \sigma' \) be the same as \( \sigma \), except for the variables \( x \) and \( y \). We let \( \sigma'(x) := \sigma(x)[c]_{p_3} \) and we will define \( \sigma'(y) \) later. Using the fact that \( c \rightarrow v \in B \) and that \( \sigma \) is a solution of \( C \), we immediately conclude that \( \{\sigma'(x), \sigma'(\alpha_2), \ldots, \sigma'(\alpha_n)\} \) is \( B \)-joinable. In fact, \( \sigma(y) \) is \( B \)-reachable from \( \{\sigma'(x), \sigma'(\alpha_2), \ldots, \sigma'(\alpha_n)\} \). However, \( \text{Pos}(\sigma(y)) \) may be larger than \( \cup_{i \in \{1, \ldots, n\}} \text{Pos}(\sigma'(\alpha_i)) \) because \( v \) may have more positions than \( c \). We can now use Assumption (A4) to infer the existence of a term \( w \) that is \( B \)-reachable from \( \{\sigma'(x), \sigma'(\alpha_2), \ldots, \sigma'(\alpha_n)\} \), \( w \) \( B \)-reaches \( \sigma(x) \) and \( \text{Pos}(w) = \cup_{i \in \{1, \ldots, n\}} \text{Pos}(\sigma'(\alpha_i)) \).

We set \( \sigma'(y) \) to \( w \) and now \( \sigma' \) is a solution of \( C \). Using \( \sigma' \), we can get the following new rewrite derivation:

\[ s := s[\sigma(l)[c]_{p'_2, p_3|p_2} \rightarrow_{t \rightarrow r|C_{p_2}} u' := s[\sigma'(r)]_{p_2} \rightarrow_{c \rightarrow v, p_1} t := s[\sigma(r)]_{p_2}. \]

Recall that the existence of the required \( B \)-rules is guaranteed by the construction of \( \overline{T} \) (specifically, the rules \( R' \)).

(Case 3. Critical Overlap) In this case, \( p_1 = p_2, p'_2 \) and \( \ell|_{p'_2} \) is not a variable. The original rewrite derivation is of the form

\[ s = s[\sigma(l)[c]_{p'_2, p_3|p_2} \rightarrow_{c \rightarrow v, p_1} u := s[\sigma(l)[v]_{p'_2, p_3|p_2} \rightarrow_{t \rightarrow r|C_{p_2}} t := s[\sigma(r)]_{p_2}. \]

Note that \( s[\sigma(l)[v]_{p'_2, p_3|p_2} \) is, in fact, \( s[\sigma(l)]_{p_2} \).

(a) If \( |p'_2| \geq 1 \), then we get a new single step rewrite derivation using the rule

\[ l[c]_{p'_2} \rightarrow r|C' \]

deduced using either Rule (R1) or Rule (R2) applied to the rules \( c \rightarrow v \) and \( l \rightarrow r|C \): Rule (R1) if \( v \) is a constant and Rule (R2) if \( v \) is of the form \( f(c_1, \ldots, c_m) \). The new rewrite derivation is

\[ s = s[\sigma(l)[c]_{p'_2, p_3|p_2} \rightarrow_{l[c]_{p'_2} \rightarrow r|C'_{p_2}} t = s[\sigma(r)]_{p_2}. \]

(b) If \( |p'_2| = 0 \), then \( s[p_2] \) is \( c \), and \( \ell|_{p'_2} \) is \( (F \cup B) \)-reachable from \( c \). Thus, since \( F \)-rules can be simulated by \( R \)-rules, \( \ell|_{p_2} \) is \( R \)-reachable from \( c \), and by Assumption (A3), \( c \rightarrow_B \ell|_{p_2} \) holds, and hence, \( s \rightarrow_B t \) follows. □

Example 5.8. (Continues Example 5.4) The derivation \( f(e, a) \rightarrow_B f(e, f(b, c)) \rightarrow_B f(e, f(g(d), c)) \rightarrow_B f(e, f(g(d), h(e')))) \rightarrow_F h(e') \) can be
commuted to $f(e, a) \rightarrow_B f(e, f(b, c)) \rightarrow_B f(e, f(g(d), c)) \rightarrow_F c \rightarrow_B h(e')$ by using the same $F$-rule $f(x_1, f(g(x_2), y_1)) \rightarrow_B [\{y_1\}, y_2, \{x_1, x_2\}]$ one step before.

The previous derivation can be commuted to $f(e, a) \rightarrow_B f(e, f(b, c)) \rightarrow_F c \rightarrow_B h(e')$ by applying $f(x_1, f(b, y_1)) \rightarrow_B [\{y_1\}, y_2, \{x_1, y_1\}]$ instead of $f(x_1, f(g(x_2), y_1)) \rightarrow_B [\{y_1\}, y_2, \{x_1, x_2\}]$.

The latter derivation can be commuted to $f(e, a) \rightarrow_F c \rightarrow_B h(e')$ by applying $f(x_1, a) \rightarrow_B [\{c\}, y_2, \{x_1, d\}]$ instead of $f(x_1, f(b, y_1)) \rightarrow_B [\{y_1\}, y_2, \{x_1, d\}]$.

Note that we have removed all “local peaks”, obtaining a derivation of the form $s \rightarrow_F^* v \rightarrow_B^* t$.

**Lemma 5.9.** Let $s$ and $t$ be terms. Then, $s \rightarrow_F^* t$ if and only if there exists $v$ such that $s \rightarrow_F^* v \rightarrow_B^* t$.

**Proof.** The right-to-left implication follows from the fact that $F$-rules can be simulated by $R$-rules, and $B$ is included in $R$.

For proving the left-to-right implication we first note that any derivation $s \rightarrow_F^* t$ can be transformed into a derivation $s \rightarrow_{F \cup B}^* t$ by replacing every rewrite step $u \rightarrow_{l \rightarrow} v$ with $\text{height}(l) \geq 1$ by $u \rightarrow_{l' \rightarrow C} v$, where $l' \rightarrow r' \rightarrow C$ is the constrained version of $l \rightarrow r$. Now, it rests to prove that $s \rightarrow_{F \cup B}^* t$ implies $s \rightarrow_{F}^* \rightarrow_B^* t$. We do it by induction on (i) the number of $F$-steps in $s \rightarrow_{F \cup B}^* t$, and (ii) in case of identical $F$-steps, the number of consecutive $B$-steps at the beginning of the derivation $s \rightarrow_{F \cup B}^* t$. We distinguish the following cases:

1. If $s \rightarrow_{F \cup B}^* t$ is of the form $s \rightarrow_{F} s' \rightarrow_{F \cup B}^* t$, then, $s' \rightarrow_{F \cup B}^* t$ has less $F$-steps, and by induction hypothesis, $s' \rightarrow_{F}^* \rightarrow_{B}^* t$ follows, from which $s \rightarrow_{F}^* \rightarrow_{B}^* t$ follows.
2. If $s \rightarrow_{F \cup B}^* t$ is of the form $s \rightarrow_{B} t$, then it is already of the form $s \rightarrow_{F}^* \rightarrow_{B}^* t$.
3. If $s \rightarrow_{F \cup B}^* t$ is of the form $s \rightarrow_{B} s' \rightarrow_{F} t' \rightarrow_{F \cup B}^* t$, then, by Lemma 5.7, there is another derivation of the form either $s \rightarrow_{B}^* s' \rightarrow_{B} t' \rightarrow_{F \cup B}^* t$ or $s \rightarrow_{B}^* s' \rightarrow_{F}^* t' \rightarrow_{F \cup B}^* t$. This new derivation has either less $F$-steps, or the same number of $F$-steps but less number of consecutive $B$-steps at the beginning.

Thus, by induction hypothesis, $s \rightarrow_{F}^* \rightarrow_{B}^* t$ follows.

Note how the constrained rules helped us in handling variable overlaps in the presence of non-linear left-hand side terms. It is well-known that non-linearity causes problems during asymmetric completion.

The following technical definitions and lemmas related to the rewrite closure will help when proving decidability of the weak normalization problem. In particular, the following lemma ensures that a term $s$ can reach a term with smaller size than $s$, whenever a certain derivation of the form $s \rightarrow_{B}^* t$ exists.

**Lemma 5.10.** Let $s, t$ and $u$ be terms such that $s \rightarrow_{B}^* t \rightarrow_{l \rightarrow r | C, \lambda} u$, where $l \rightarrow r | C \in F$. Let $i$ be a position in $\{1, \ldots, m\}$ satisfying $\text{height}(s|_i) > 0$ and $\text{height}(l|_i) > 0$.

Then, there exists a rule $l' \rightarrow r' | C' \in F$ that rewrites $s$ at position $\lambda$ and holding $\text{height}(l|_i) > 0$.

Proof. We prove it by induction on the length \( n \) of the derivation \( s \rightarrow^*_F t \). When \( n = 0 \), the result is easily verified. If \( n > 0 \), then we can write the rewrite derivation \( s \rightarrow_B t \rightarrow^*_B t' \rightarrow_{l \rightarrow r} C, \lambda u \) as:

\[
s \rightarrow_B t' \rightarrow^*_B t' \rightarrow_{l \rightarrow r} C, \lambda u.
\]

Since \( \text{height}(s|_i) > 0 \), it follows that \( \text{height}(t'|_i) > 0 \). Therefore, we can use the induction hypothesis to conclude that \( t' \rightarrow_{l \rightarrow r} C', \lambda u' \) and \( \text{height}(l'|_i) > 0 \). The new rewrite derivation,

\[
s \rightarrow_B t' \rightarrow_{l \rightarrow r} C', \lambda u',
\]

is a critical overlap, as defined in the proof of Lemma 0???. Moreover, since \( \text{height}(l'|_i) > 0 \), we actually are in a special case of Case (a) of critical overlap in the proof of Lemma 5.9. In other words, we can conclude that there is an \( F \)-rule \( t'' \rightarrow r''|C'' \) – deduced using Rule (R1) or Rule (R2) – that can be used to reduce \( t' \) at position \( \lambda \).

We finally argue that \( \text{height}(l''|_i) > 0 \) by contradiction. Suppose \( \text{height}(l''|_i) = 0 \). Since \( \text{height}(t'|_i) > 0 \), \( l''|_i \) necessarily has to be a variable. However, since \( t'' \rightarrow r''|C'' \) is deduced from \( t' \rightarrow r'|C' \) using Rule (R1) or Rule (R2), and since \( \text{height}(l'|_i) > 0 \), we conclude that either \( \text{height}(l'|_i) \) is a constant or \( \text{height}(l''|_i) > 0 \). This contradicts the claim that \( l''|_i \) is a variable.

Definition 5.11. Let \( s \) be a term. We define the magnitude of \( s \), denoted \( \|s\| \), as the pair \( (\|s\|, \text{VarPos}(s)) \). We compare pairs by the lexicographic extension of the usual ordering on natural numbers. Thus, \( \|s\| < \|t\| \) means that either \( |s| \) is smaller than \( |t| \), or \( |s| \) equals \( |t| \) and \( s \) has less occurrences of variables than \( t \).

Definition 5.12. Let \( s \) and \( t \) be terms satisfying \( s \rightarrow_R t \). This rewrite step is called a \( P \)-step (preserving step), denoted \( s \rightarrow_P t \), if it is done using a rule in \( F \) and \( \|s\| = \|t\| \). It is called a \( D \)-step (decreasing step), denoted \( s \rightarrow_D t \), if it is done using a rule in \( F \) and \( \|s\| > \|t\| \).

Example 5.13. The \( F \)-rule \( f(x_1, y_1, y_2) \rightarrow g(a, b, y_3) \) can be used in the \( P \)-step \( f(a, a, a) \rightarrow_F g(a, b, a) \), but also in the \( D \)-step \( f(h(a), h(a), h(a)) \rightarrow_F g(a, b, h(a)) \).

Lemma 5.14. Let \( s \) and \( t \) be terms, let \( l \rightarrow r|C \) be a rule in \( F \), let \( \sigma \) be a substitution and let \( p \) be a position in \( \text{Pos}(s) \) such that \( s \rightarrow_{l \rightarrow r|C, \sigma, p} t \). Then, this \( F \)-step is a \( P \)-step if and only if \( l \) and \( r \) are flat, and for each \( i \) in \( \{1, \ldots, m\} \), if \( s|_{p, i} \) is not a constant, then \( l|_i \) is a variable occurring in \( C \) in a tuple of the form \( \langle \{l|_1, \alpha_2, \ldots, \alpha_n\}, x \rangle \) for \( n \geq 1 \), where all \( \alpha(\alpha_2), \ldots, \alpha(\alpha_n) \) are constants.

Proof. It suffices to observe that both conditions are equivalent to \( \|s\| = \|t\| \).

Lemma 5.15. Let \( s \) and \( t \) be terms, and let \( p \) be a position of \( s \) satisfying \( s \rightarrow_{F, p} t \). If \( s \rightarrow_{P, p} t \), then, for each variable \( x \in \mathcal{V} \), \( \|s\|_x = \|t\|_x \). Otherwise, if \( s \rightarrow_{D, p} t \), then, for each variable \( x \in \mathcal{V} \), \( \|s\|_x \geq \|t\|_x \).

Proof. It suffices to define an exhaustive partial mapping \( \tau : \text{VarPos}(s) \rightarrow \text{VarPos}(t) \) satisfying, for each \( q \) in \( \text{Dom}(\tau) \), \( s|_q = t|_{\tau(q)} \) (note that, in the case of a \( P \)-step, since \( \|s\| = \|t\| \), this mapping will be total and injective).

We define \( \tau \) as follows, for each \( q \) in \( \text{VarPos}(s) \). If \( q \) is disjoint with \( p \), then \( \tau(q) = q \). Otherwise, the only possible case for a variable position \( q \) in \( s \) is when it is of the form \( p_1.p_2 \) for some positions \( p_1 \) and \( p_2 \) such that \( \|l_{p_1}\| \) is a variable, where \( l \to r \) \( C \) is the used rule in the above rewrite step. If \( \|l_{p_1}\| \) occurs in a tuple of \( C \) of the form \( \{ \alpha_1, \ldots, \alpha_n \} \), then we leave \( \tau(q) \) undefined. Otherwise, if it occurs in a tuple of \( C \) of the form \( \{ \alpha_1, \ldots, \alpha_n, x \} \), then, being \( q' \) the position where \( x \) occurs in \( r \), we define \( \tau(q) = p_1.q'.p_2 \).

By construction, \( \tau \) satisfies the required conditions (note that applying \( B \)-rules in a term leaves the variables unchanged). \( \Box \)

**Lemma 5.16.** Let \( s \) and \( t \) be terms such that \( s \to_F t \), and let \( \sigma \) be a substitution. Then, \( \sigma(s) \to_F \sigma(t) \).

**Proof.** Let \( \varphi \) be the substitution used in \( s \to_F t \). Let \( l \to f(\alpha_1, \ldots, \alpha_m) \) \( C \) be the rule used in \( s \to_F t \). The constraint \( C \) has tuples of the form \( \{ \{ \beta_1, \ldots, \beta_n \}, x \} \) or \( \{ \{ \beta_1, \ldots, \beta_n \} \} \). By using that \( \varphi \) is a solution of \( C \), it suffices to see that \( \sigma(\varphi) \) is also a solution of \( C \).

For a tuple \( \{ \{ \beta_1, \ldots, \beta_n \} \} \), there exists a term \( u \) satisfying \( \varphi(\beta_1) \to_B^* u, \ldots, \varphi(\beta_n) \to_B^* u \). By instantiating these derivations with \( \sigma \) we obtain \( \sigma(\varphi(\beta_1)) \to_B^* \sigma(u), \ldots, \sigma(\varphi(\beta_n)) \to_B^* \sigma(u) \), and hence, \( \sigma \) is a solution for each of these tuples.

For a tuple \( \{ \{ \beta_1, \ldots, \beta_n \}, x \} \), it holds \( \varphi(\beta_1) \to_B^* \varphi(x), \ldots, \varphi(\beta_n) \to_B^* \varphi(x) \). By instantiating these derivations with \( \sigma \) we obtain \( \sigma(\varphi(\beta_1)) \to_B^* \sigma(\varphi(x)), \ldots, \sigma(\varphi(\beta_n)) \to_B^* \sigma(\varphi(x)) \). It rests to show that any position \( p \) in \( \text{Pos}(\sigma(\varphi(x))) \) is also a position in \( \bigcup_{i \in \{1, \ldots, n\}} \text{Pos}(\sigma(\varphi(\beta_i))) \).

If \( p \) is also in \( \text{Pos}(\varphi(x)) \), then, since \( \varphi \) is a solution of \( C \), \( p \) is in some \( \text{Pos}(\varphi(\beta_i)) \). Thus \( p \) is in \( \text{Pos}(\sigma(\varphi(\beta_i))) \).

Otherwise, if \( p \) is not in \( \text{Pos}(\varphi(x)) \), then, \( p \) is of the form \( p_1.p_2 \), where \( \varphi(x)|_{p_1} \) is a variable \( y \) and \( p_2 \) is a position in \( \sigma(y) \). Since \( \varphi \) is a solution of \( C \), \( p_1 \) is in some \( \text{Pos}(\varphi(\beta_i)) \) for some \( i \) in \( \{1, \ldots, n\} \) (in fact, this is true for all \( i \) because \( B \) is ground and \( y \) is a variable, and thus \( y \) is only \( B \)-reachable from itself, but we do not need it). Since \( \varphi(\beta_i) \to_B^* \varphi(x) \), and \( B \)-rules are ground, necessarily \( \varphi(\beta_i)|_{p_1} \) is \( y \). Thus \( p_1.p_2 \) is a position of \( \sigma(\varphi(\beta_i)) \), and we are done. \( \Box \)

**Lemma 5.17.** Let \( s \) and \( t \) be terms, and let \( \sigma \) be a substitution.

(a) If \( s \to_B t \), then \( \sigma(s) \to_B \sigma(t) \).
(b) If \( s \to_F t \), then \( \sigma(s) \to_F \sigma(t) \).
(c) If \( s \to_D t \) and for each \( x \) in \( V \) it holds that \( \sigma(x) \) is either \( x \) or satisfies \( |\sigma(x)| > 1 \), then \( \sigma(s) \to_D \sigma(t) \).

**Proof.** Case (a) is trivial, since \( B \)-rules are plain (non-constrained) rules.

For case (b), assume \( s \to_F t \) holds. In particular, we have \( s \to_F t \), and hence, by Lemma 5.16, it holds \( \sigma(s) \to_F \sigma(t) \). It rests to see \( |\sigma(s)| = |\sigma(t)| \), which follows from the following facts. Since \( s \to_F t \), by the definition of \( P \)-step \( |s| = |t| \) and both \( s \) and \( t \) have the same number of occurrences of variables. Moreover, by Lemma 5.15, for each variable \( x \in V \), \( |s|_x = |t|_x \), from which \( |\sigma(s)| = |\sigma(t)| \) follows.
For case (c), assume \( s \rightarrow_D t \) and for each \( x \) in \( \mathcal{V} \) it holds that \( \sigma(x) \) is either \( x \) or satisfies \( |\sigma(x)| > 1 \). Hence, \( s \rightarrow_F t \), and by Lemma 5.16, \( \sigma(s) \rightarrow_F \sigma(t) \). It rests to see that \( \|\sigma(s)\| > \|\sigma(t)\| \). Since \( s \rightarrow_D t \), by Lemma 5.15, for each variable \( x \) in \( \mathcal{V} \) it holds \( |s_x| \geq |t_x| \). Combining this fact with the definition of \( D \)-step, it follows that either (i) \( |s| > |t| \), or (ii) \( |s| = |t| \) and there is a concrete variable \( y \) in \( \mathcal{V} \) for which \( |s_y| > |t_y| \). In any case, we have that, for each variable \( x \) in \( \mathcal{V} \), it holds \( |\sigma(s)|_x \geq |\sigma(t)|_x \). In case (i), \( |\sigma(s)| > |\sigma(t)| \) holds. In case (ii), we just know \( |\sigma(s)| \geq |\sigma(t)| \). If \( |\sigma(s)| > |\sigma(t)| \) also holds, then \( \|\sigma(s)\| > \|\sigma(t)\| \) and we are done. Otherwise, if \( |\sigma(s)| = |\sigma(t)| \) holds, then, since \( |s_y| > |t_y| \), it follows \( |\sigma(y)| = 1 \). Thus, by the assumptions for case (c), the variable \( y \) satisfies \( \sigma(y) = y \). Since for any variable \( x \) we have \( |s_x| \geq |t_x| \), it follows \( |\sigma(s)|_y > |\sigma(t)|_y \). Thus, \( \|\sigma(s)\| > \|\sigma(t)\| \) and we are done. \( \Box \)

6. DECIDING WEAK NORMALIZATION

In this section, \( R \) is assumed to be the constrained TRS \( F \cup B \) obtained by previous sections. In Subsection 6.1 we prove that, when \( R \) is not weakly normalizing, there exists a particular non-normalizing term satisfying some special properties. We will call a witness to this term. The existence of a computationally bounded witness is given in Subsection 6.3, thus proving decidability of weak normalization. Just before that, in Subsection 6.2 we give intuition about how witnesses are managed in order to prove that computational bound.

6.1 Witnesses to weak normalization

In order to prove decidability of weak normalization, we will show that, when \( R \) is not weakly normalizing, there exists a special kind of terms which are not normalizing. The following definition describes this kind of terms.

Definition 6.1. A term \( t = f(t_1, \ldots, t_m) \) is a witness if it satisfies the following conditions:

(W1) \( t \) is not \( R \)-normalizing.
(W2) Each \( t_i \) is either a variable or \( R \)-reachable from a constant, for \( i \) in \( \{1, \ldots, m\} \).
(W3) Each \( t_i \) satisfying \( \text{height}(t_i) > 0 \) is a \( R \)-normal form, for \( i \) in \( \{1, \ldots, m\} \).
(W4) \( t \) cannot \( R \)-reach a term \( s \) satisfying \( \|t\| > \|s\| \).

Under the assumption that \( R \) is not weakly normalizing, we want to prove the existence of a witness. Our intention is to transform an arbitrary non-\( R \)-normalizing term into a witness by applying \( F \)-rules. The following lemma will be useful to this end since it guarantees the preservation of condition \( (W_3) \) by application of these rules.

Lemma 6.2. Let \( s \) be a term satisfying condition \( (W_3) \). Let \( t \) be a term with \( \text{height} \) greater than \( 0 \) and satisfying \( s \rightarrow_F t \). Then \( t \) also satisfies condition \( (W_3) \).

Proof. First, note that such a step can only be applied at the root of \( s \), since all proper subterms of \( s \) are either height 0 terms or \( R \)-normal forms. Let \( l \rightarrow r \) \( C \) and \( \sigma \) be the rule and substitution used in this rewrite step. Note that \( r \) is flat, and for each variable \( x \) in \( \text{Vars}(r) \), there is a tuple in \( C \) of the form \( \{a_1, \ldots, a_n, x\} \). To conclude, it suffices to prove, for each of such \( x \)'s, that \( \sigma(x) \) is either a constant or a
non-constant $R$-normal form. If all $\sigma(\alpha_i)$ are constants, by definition of constrained rewriting, $x$ is also a constant and we are done. Thus, assume that some $\sigma(\alpha_i)$ is not a constant. Then, $\alpha_i$ is a variable occurring in $l$ at some position $p$ such that $\vert p \vert \geq 1$ and $s^*_p = \sigma(\alpha_i)$ is not a constant. Since $s$ satisfies condition (W$_3$), $s^*_p$ is a non-constant $R$-normal form. By definition of constrained rewriting, $\sigma(x)$ is $B$-reachable from $\sigma(\alpha_i)$. Therefore, $\sigma(x)$ equals $\sigma(\alpha_i)$. Thus, $\sigma(x)$ is a non-constant normal form, and we are done. □

Under the assumption of the existence of a witness, we also want to prove the existence of a specific one whose size is bounded by a computable function. To this end we will replace some subterms of a witness by other smaller subterms. These new subterms must satisfy the properties described by the following two definitions in order to preserve the conditions of the witness definition.

**Definition 6.3.** We call $H$ to the maximum height of a term in the rules of $R$. Let $u$ be a term. We define $P(u)$ as $\{ p \mid p \in \text{Pos}(u) \land \vert p \vert \leq H \}$.

**Definition 6.4.** Let $s$ and $t$ be terms. We say that $s$ and $t$ are $H$-equivalent, denoted $s \sim_H t$, if the following conditions hold:

1. $P(s) = P(t)$.
2. For each $p$ in $P(s)$ it holds $\text{root}(s^*_p) = \text{root}(t^*_p)$.
3. For each $p$ in $P(s)$ and each constant $c$, it holds $c \rightarrow_R^* s^*_p$ if and only if $c \rightarrow_R^* t^*_p$ (i.e., $s^*_p$ and $t^*_p$ are $R$-reachable from the same constants).
4. For each $p$ in $P(s)$, it holds that $s^*_p$ is a $R$-normal form if and only if $t^*_p$ is a $R$-normal form.
5. For each $p$ and $q$ in $P(s)$, it holds $s^*_p = s^*_q$ if and only if $t^*_p = t^*_q$.

The following technical lemma will be used to ensure that the transformation of a witness into a smaller term preserves the witness conditions.

**Lemma 6.5.** Let $p_1, \ldots, p_n, q_1, \ldots, q_k$ be a permutation of $1, \ldots, m$. Let $s$ be a term of the form $f[s_1[p_1], \ldots, s_n[p_n], c_1[1], \ldots, c_k[m]]$ for some non-constant $R$-normal forms $s_1, \ldots, s_n$ and constants $c_1, \ldots, c_k$. Let $t$ be a term such that $s \sim_R^* t$. Let $u$ be a term $H$-equivalent to $s$.

Then, $t$ is of the form $f[u_1[p_1], \ldots, u_n[p_n], c_1', \ldots, c_k']$, for some permutation $p_1', \ldots, p_n', q_1', \ldots, q_k'$ of $1, \ldots, m$, and some constants $c_1', \ldots, c_k'$. Moreover, $u$ is of the form $f[u_1[p_1], \ldots, u_n[p_n][c_1[1], \ldots, c_k[m]]$ for some non-constant $R$-normal forms $u_1, \ldots, u_n$. Furthermore, the term $u = f[u_1[p_1''], \ldots, u_n[p_n''], c_1'', \ldots, c_k'']$ is $P$-reachable (in one or more steps in case $s \sim_R^* t$) from $u$ and $H$-equivalent to $t$.

**Proof.** If suffices to prove this fact for one step derivations, since then it inductively extends to any length. Thus, assume $s \sim_R^* t$. This $P$-step is necessarily done at position $\lambda$, since $s$ has only constants and normal forms at its depth 1 positions. Let $l \rightarrow r$ $C$ and $\sigma$ be the used rule and substitution in the above rewrite step. By Lemma 5.14, $l$ and $r$ are flat, and for each $p_i$ in $\{ p_1, \ldots, p_n \}$, $l[p_i]$ is a variable occurring in $C$ in a tuple of the form $(\{l[p_i], \alpha_2, \alpha_3, \ldots \}, x)$, where all $\sigma(\alpha_2), \sigma(\alpha_3), \ldots$ are constants. For each of such $p_i$, the corresponding $x$ occurs once in $r$ at a certain position $p_i'$. Moreover, since $\sigma(l[p_i]) = s_i$ is a $R$-normal form, by definition of constrained rewriting $\sigma(x)$ is the same $R$-normal form. Thus, $l[p_i']$ equals $s_i$. It follows
that, for a certain permutation \( \{ q_1', \ldots, q_k' \} \) of \( \{ 1, \ldots, m \} \) -- \( \{ p_1', \ldots, p_n' \} \), \( t \) is of the form \( f[s_1][p_1] \cdots [s_n][p_n][c_1][q_1] \cdots [c_k][q_k] \) for some constants \( c_1', \ldots, c_k' \).

Since \( u \) is \( H \)-equivalent to \( s \), it is necessarily of the form \( f[u_1][p_1] \cdots [u_n][p_n][c_1][q_1] \cdots [c_k][q_k] \), for some non-constant \( R \)-normal forms \( u_1, \ldots, u_n \) satisfying the following conditions for each \( i \) in \( \{ 1, \ldots, n \} \), where we call \( P'(s_i) \) to \( \{ p \mid p \in \text{Pos}(s_i) \land |p| \leq H - 1 \} \) and \( P'(u_i) \) to \( \{ p \mid p \in \text{Pos}(u_i) \land |p| \leq H - 1 \} \):

\[
- P'(s_i) = P'(u_i).
- For each \( p \in P'(s_i) \) it holds \( \text{root}(s_i|p) = \text{root}(u_i|p) \).
- For each \( p \in P'(s_i) \) and each constant \( c \), it holds \( c \rightarrow_R^* s_i|p \) if and only if \( c \rightarrow_R^* u_i|p \).
- For each \( p \in P'(s_i) \), it holds that \( s_i|p \) is a \( R \)-normal form if and only if \( u_i|p \) is a \( R \)-normal form.
- For each \( p \) and \( q \) in \( P'(s_i) \), it holds \( s_i|p = s_i|q \) if and only if \( u_i|p = u_i|q \).

By applying the same rule \( l \rightarrow r|c \) to \( u \) we obtain \( v = f[u_1][p_1] \cdots [u_n'][p_n'][c_1][q_1] \cdots [c_k'][q_k] \). By the above conditions, \( v \) is \( H \)-equivalent to \( t \). □

As a particular case of previous lemma we have the following.

**Corollary 6.6.** Let \( p_1, \ldots, p_n, q_1, \ldots, q_k \) be a permutation of \( 1, \ldots, m \). Let \( s \) be a term of the form \( f[t_1][p_1] \cdots [t_n][p_n][c_1][q_1] \cdots [c_k][q_k] \) for some non-constant \( R \)-normal forms \( t_1, \ldots, t_n \) and constants \( c_1, \ldots, c_k \). Let \( t \) be a term such that \( s \rightarrow_R^* t \).

Then, \( t \) is of the form \( f[t_1][p_1] \cdots [t_n][p_n'][c_1'][q_1] \cdots [c_k'][q_k] \), for some permutation \( p_1', \ldots, p_n', q_1', \ldots, q_k' \) of \( 1, \ldots, m \), and some constants \( c_1', \ldots, c_k' \).

**Proof.** It follows from Lemma 6.5 by defining \( u \) as \( s \). □

**Lemma 6.7.** Assume that \( R \) is not weakly normalizing but each constant can \( R \)-reach some \( R \)-normal form. Then, there exists a witness \( t \).

**Proof.** Let \( s \) be a minimal term in magnitude among all the non-\( R \)-normalizing terms. Since \( R \)-constants are \( R \)-normalizing, \( s \) is of the form \( f(s_1, \ldots, s_m) \). By the minimality of \( s \), each \( s_i \) can \( R \)-reach a \( R \)-normal form \( s_i' \). Note that \( s' = f(s_1', \ldots, s_m') \) satisfies conditions \((W_1)\) and \((W_3)\).

Let \( u' \) be a minimal term in magnitude among all the terms \( R \)-reachable from \( s' \). By Lemma 5.9, \( s' \rightarrow_R^* u \rightarrow_R^* u' \) for some term \( u \). Since \( R \)-steps always preserve or increase the magnitude, \( \|u\| \) is smaller than or equal to \( \|u'\| \). But in fact, by the minimality of \( u' \), it holds that \( \|u\| \) is also minimal in magnitude among the terms \( R \)-reachable from \( s' \).

Any term \( R \)-reachable from a non-\( R \)-normalizing term is also non-\( R \)-normalizing. Thus, \( u \) also satisfies condition \((W_1)\). Hence, since constants are \( R \)-normalizing and variables are always \( R \)-normal forms, the root of \( u \) is necessarily \( f \). Moreover, since \( s' \rightarrow_R^* u \) and \( s' \) satisfies condition \((W_3)\), by several applications of Lemma 6.2, \( u \) also satisfies this property. Finally, by its minimality, \( u \) also satisfies property \((W_4)\). If \( u \) satisfies \((W_2)\) then it is a witness and we are done by calling \( t \) to \( u \).

Thus, from now on assume that some subterm at depth \( 1 \) in \( u \) has height greater than \( 0 \) and is not \( R \)-reachable from a constant.

In order to finish the proof, we define a new term \( t \) using \( u \) and show that it is a witness. To this end, we write \( u \) more explicitly as 
\[
f[u_1|p_1, \ldots, [u_n|p_n|v_1|\delta_1, \ldots, [v_e|\delta_e|c_1|q_1, \ldots, [c_k|q_k],
\]
where \( p_1, \ldots, p_n, \delta_1, \ldots, \delta_e, q_1, \ldots, q_k \) is a permutation of \( 1, \ldots, m, u_1, \ldots, u_n \) are \( R \)-normal forms with height greater than 0 and non-\( R \)-reachable from a constant (i.e. they are the ones making \( u \) to satisfy \((W_2)\)). \( v_1, \ldots, v_e \) are variables or non-constant \( R \)-normal forms \( R \)-reachable from a constant, and \( c_1, \ldots, c_k \) are constants. Now, \( t \) is defined as 
\[
f[x_1|p_1, \ldots, [x_n|p_n|x_1|\delta_1, \ldots, [x_e|\delta_e|c_1|q_1, \ldots, [c_k|q_k],
\]
where \( x_1, \ldots, x_n \) are (new) variables which do not occur in \( u \), and such that \( x_i = x_j \) if \( u_i = u_j \) for each \( 1 \leq i < j \leq n \). Note that, by defining a substitution \( \sigma \) as \( \sigma(x_i) = u_i \) for each \( i \) in \( \{1, \ldots, n\} \), and \( \sigma(y) = y \) for any other variable \( y \), it holds \( \sigma(t) = u \).

By construction \( t \) satisfies \((W_2)\) and \((W_3)\). It rests to see that \( t \) satisfies \((W_1)\) and \((W_4)\). We start proving \((W_4)\) by contradiction. Thus, assume that there exists a term \( t' \) \( R \)-reachable from \( t \) and satisfying \( \|t\| > \|t'\| \). By Lemma 5.9, there exists a derivation of the form \( t \rightarrow^*_p w \rightarrow^*_B t' \) for some term \( w \). Since \( B \)-steps always preserve or increase the size, it follows \( \|t\| > \|w\| \). Moreover, the derivation \( t \rightarrow^*_p w \) contains at least one \( D \)-step. By Lemmas 5.16 and 5.17, there is a derivation \( u = \sigma(t) \rightarrow^*_p \sigma(w) \). By the same lemma and the definition of \( \sigma \), there is at least one \( D \)-step in this derivation. Thus, \( \|u\| = \|\sigma(t)\| > \|\sigma(w)\| \), contradicting the fact that \( u \) satisfies \((W_4)\).

Finally, we prove that \( t \) satisfies \((W_1)\) by contradiction. Thus, assume there exists a \( R \)-normal form \( w \) \( R \)-reachable from \( t \). By Lemma 5.9, \( t \rightarrow^*_p t' \rightarrow^*_B w \) for some term \( t' \). But since \( t \) satisfies \((W_4)\), it holds \( t \rightarrow^*_p t' \rightarrow^*_B w \). By Corollary 6.6, \( t' \) is of the form \( f[x_1|p'_1, \ldots, [x_n|p'_n|x_1|\delta'_1, \ldots, [v_e|\delta'_e|c'_1|q'_1, \ldots, [c'_k|q'_k] \) for some permutation \( p'_1, \ldots, p'_n, \delta'_1, \ldots, \delta'_e, q'_1, \ldots, q'_k \) of \( 1, \ldots, m, \) and some constants \( c'_1, \ldots, c'_k \). Since \( t' \rightarrow^*_B w \), \( w \) is of the form \( f[x_1|x'_1, \ldots, [x_n|x'_n|x_1|\delta|x'_1, \ldots, [v_e|\delta|c|w_1|q_1, \ldots, [w_k|q_k] \) for some \( R \)-normal forms \( w_1, \ldots, w_k \), \( R \)-reachable from \( c'_1, \ldots, c'_k \), respectively. Since \( t \rightarrow^*_p t' \rightarrow^*_B w \), by Lemma 5.17, \( u = \sigma(t) \rightarrow^*_p \sigma(t') \rightarrow^*_B \sigma(w) \). Note that \( \sigma(w) \) is \( f[u_1|p'_1, \ldots, [u_n|p'_n|v_1|\delta|c|w_1|q_1, \ldots, [w_k|q_k] \). This term cannot be a \( R \)-normal form because \( u \) satisfies \((W_1)\). Thus, there exists a rule \( l \rightarrow r(C) \) applicable on \( \sigma(w) \). This rule can only be applied at \( \lambda \), since all \( u_1, \ldots, u_n, v_1, \ldots, v_e, w_1, \ldots, w_k \) are \( R \)-normal forms.

Assume first that all \( l|p'_i \) are variables, for \( i \) in \( \{1, \ldots, n\} \). Consider one of these \( l|p'_i \), and let \((l|p'_i, a_2, \ldots, x)\) or \((l|p'_i, a_2, \ldots, x)\) be the tuple where it occurs. Since \( \sigma(w)|p'_i = u_i \) is not \( R \)-reachable from a constant, the elements in \( \{a_2, \ldots, x\} \) cannot be constants nor variables at positions different from \( p'_1, \ldots, p'_n \) in \( l \). Moreover, if some \( l|p'_j \) occurs in this tuple, being \( j \) in \( \{1, \ldots, n\} \) and different from \( i \), then, since \( u_i \) and \( u_j \) are \( R \)-normal forms and must be \( B \)-joinable, it follows \( u_i = u_j \). But also, by the deletion of the variables \( x_1, \ldots, x_n \), it follows \( x_i = x_j \). Altogether implies that \( l \rightarrow r(C) \) can also be applied on \( w \), contradicting the fact that it is a \( R \)-normal form, and we are done.

Now, assume that some \( l|p'_i \) is not a variable. Since \( u_i \) is not \( R \)-reachable from a constant, it follows that \( \text{height}(l|p'_i) > 0 \). By Lemma 5.10, there exists a rule \( t' \rightarrow r'(C) \) in \( F \) which rewrites \( \sigma(t') = f[u_1|p'_1, \ldots, [u_n|p'_n|x_1|\delta|x'_1, \ldots, [c'_k|q'_k] \) at position \( \lambda \) and \( \text{height}(l|p'_i) > 0 \). Thus, \( u \) \( R \)-reaches a term with smaller size than \( u \), contradicting the fact that \( u \) satisfies \((W_4)\), and we are done. \( \square \)
6.2 Intuition for the computational approach

Once we know the existence of a witness, it rests to see the existence of one with a computationally bounded size. This is done in the following subsection by using a sequence of computational bounds \( \mathcal{C}_0 \leq \mathcal{C}_1 \leq \mathcal{C}_2 \leq \ldots \) with respect to \( R \). We briefly summarize here how this is done for a given witness \( f(s_1, s_2, s_3) \).

If all \( |s_1|, |s_2|, |s_3| \) are greater than \( \mathcal{C}_0 \) then it can be proved the existence of terms \( s'_1, s'_2, s'_3 \) satisfying that \( f(s'_1, s'_2, s'_3) \) is a witness and all \( |s'_i| \) are bounded by \( \mathcal{C}_0 \). Thus, in this case, we are done.

Suppose that some \( |s_i| \), say \( |s_3| \), is bounded by \( \mathcal{C}_0 \), and that \( |s_1|, |s_2| \) are greater than \( \mathcal{C}_1 \). Then it can be proved the existence of terms \( s'_1, s'_2 \) satisfying that \( f(s'_1, s'_2, s_3) \) is a witness and all \( |s'_1|, |s'_2| \) are bounded by \( \mathcal{C}_1 \), and we are done.

Suppose again that \( |s_3| \) is bounded by \( \mathcal{C}_0 \), but some of \( |s_1|, |s_2| \), say \( |s_2| \), is bounded by \( \mathcal{C}_1 \), and \( |s_1| \) is greater than \( \mathcal{C}_2 \). Then it can be proved the existence of a term \( s'_1 \) satisfying that \( f(s'_1, s_2, s_3) \) is a witness and \( |s'_1| \) is bounded by \( \mathcal{C}_2 \), and we are done.

Finally, in the case where \( |s_3| \) is bounded by \( \mathcal{C}_1 \), \( |s_2| \) is bounded by \( \mathcal{C}_1 \), and \( |s_1| \) bounded by \( \mathcal{C}_2 \), we are done again.

In any case, we have proved the existence of a term \( f(s'_1, s'_2, s'_3) \), where all \( |s'_i| \) are bounded by \( \mathcal{C}_2 \). Thus, \( 3 \cdot \mathcal{C}_2 + 1 \) is a computational bound for a witness.

6.3 Bounding witnesses

The following lemma is the base for achieving the goal of obtaining a witness with a computationally bounded size. It uses results on reduction automata [Dauchet et al. 1995].

**Lemma 6.8.** There exists an increasing and computable function \( C : \mathbb{N} \to \mathbb{N} - \{0, 1\} \) satisfying the following condition.

For any permutation \( p_1, \ldots, p_n, q_1, \ldots, q_k \) of \( 1, \ldots, m \), if \( s = f(s_{p_1}, \ldots, s_{p_n}, w_{i_1}, \ldots, w_{i_k}) \) is a term satisfying conditions (W2) and (W3), then there exist terms \( s'_1, \ldots, s'_n \) such that:

- \( s' = f(s'_{p_1}, \ldots, s'_{p_n}, w_{i_1}, \ldots, w_{i_k}) \) is \( H \)-equivalent to \( s \).
- \( s' \) satisfies conditions (W2) and (W3).
- For each \( i \) in \( \{1, \ldots, k\} \), \( |w_i| \leq C(|w_1| + \ldots + |w_k|) \).
- For each \( i \) in \( \{1, \ldots, n\} \), \( |s'_{p_i}| \leq C(|w_1| + \ldots + |w_k|) \).

**Proof.** In [Dauchet et al. 1995], emptiness is shown to be decidable for languages recognized by deterministic reduction automata. In particular, it is proved that there exists a computational bound \( B : \mathbb{N} \to \mathbb{N} \) such that, given a reduction automaton \( A \), \( \mathcal{L}(A) \) is not empty if and only if there exists a term \( t \) recognized by \( A \) and whose size is smaller than or equal to \( B(|A|) \). Hence, the idea is to represent the set

\[ S = \{ s' = f(s'_{p_1}, \ldots, s'_{p_n}, w_{i_1}, \ldots, w_{i_k}) \mid s' = s \wedge s' \text{ satisfies (W2) and (W3)} \} \]

using a reduction automaton. Note that, in this set definition, \( s'_{p_1}, \ldots, s'_{p_n} \) are the variables. Nevertheless, due to some technical reasons we will not be able to represent \( S \) directly, but an alternative set similar to \( S \), and for which the same bounds apply.
One of the difficulties to represent $S$ is that reduction automata recognize ground terms, while the above set may have variables. This problem can be avoided by assuming an extension of our signature $\Sigma$ to $\Sigma' = \Sigma \cup \{ h, d \}$, where $h$ is a new unary function symbol, and $d$ is a new constant. Note that both $h$ and $d$ do not occur in $R$, and hence, terms of the form $h(h(\ldots (h(d)) \ldots ))$ cannot be rewritten with $R$. With these new symbols, we can codify arbitrary terms of $T(\Sigma, V)$ using ground terms of $T(\Sigma')$ as follows. If a term $t$ contains $l$ different variables $x_1, \ldots, x_l$, we can apply the substitution $\sigma \{ x_1 \mapsto h(d), x_2 \mapsto h(h(d)), \ldots, x_l \mapsto h^l(d) \}$ to $t$ (terms of the form $h^l(d)$ codify variables) obtaining a ground term. Thus, assume that $s$ is, in fact, a term of $T(\Sigma')$ satisfying the following: for each position $p$ in $\mathsf{Pos}(s)$ such that $\mathsf{root}(s_{|p}) = h$, it holds that $\mathsf{root}(s_{|p,1})$ is either $h$ or $d$. We will callVis codified to any term satisfying this condition.

Now we discuss another difficulty for the representation of $S$. In order to check $H$-equivalence with a deterministic reduction automaton, in particular we need to check that $s$ and $s'$ satisfy $(s_{|p_1} = s_{|p_2}) \iff (s'_{|p_1} = s'_{|p_2})$ for each $p_1, p_2$ holding $|p_1|, |p_2| \leq H$. A first approach for representing this statement could be the following. We define the reduction automaton $\langle Q, \Sigma', F, \Delta, > \rangle$ where $Q = \{ q, q_{\text{accept}} \}$, $F$ is $\{ q_{\text{accept}} \}$, $q > q_{\text{accept}}$ and $\Delta$ is the set of rules $\{ a \rightarrow q \mid a \in \Sigma'_0 \cup \{ h(q) \rightarrow q \} \cup \{ f(q, \ldots, q) \rightarrow q, f(q, \ldots, q) \rightarrow q_{\text{accept}} \}$, where $c$ is the constraint:

$$\bigwedge_{|p_1|, |p_2| \leq H} (p_1 = p_2) \land \bigwedge_{|p_1|, |p_2| \leq H} (p_1 \neq p_2) \quad \text{with} \quad s_{|p_1} = s'_{|p_2}$$

Unfortunately this reduction automaton is non-deterministic, and in fact, it can be proved that there is no way of determinizing it. The problem is that the reduction automaton does not know when it reaches the root position, that is when it has to check satisfaction of equality constraints. For this reason, we will consider a variation of our problem by adding a new symbol $f$ of arity $m$ which can appear only at the root. By replacing the rule $f(q, \ldots, q) \rightarrow q_{\text{accept}}$ by $f(q, \ldots, q) \rightarrow q_{\text{accept}}$, where $c$ is the same as above, the obtained reduction automaton is now deterministic.

Hence, returning to our general problem, we will prove the following variation, which implies the statement of the lemma.

**Claim 6.9.** There exists an increasing and computable function $C : \mathbb{N} \rightarrow \mathbb{N} - \{0, 1\}$ satisfying the following condition.

For any permutation $p_1, \ldots, p_n, q_1, \ldots, q_k$ of $1, \ldots, m$, if $s = f[s_1]_{p_1} \ldots [s_n]_{p_n} [w_1]_{q_1} \ldots [w_k]_{q_k}$ is a term satisfying conditions $(W_2)$ and $(W_3)$, where all $s_i, w_i$ are codified (in particular, they are in $T(\Sigma')$, i.e. no $f$ occurs in them), then there exist well-codified terms $s'_1, \ldots, s'_n$ such that:

- $s' = f[s'_1]_{p_1} \ldots [s'_n]_{p_n} [w_1]_{q_1} \ldots [w_k]_{q_k}$ is $H$-equivalent to $s$.
- $s'$ satisfies conditions $(W_2)$ and $(W_3)$.
- For each $i$ in $\{1, \ldots, k\}$, $|w_i| \leq C(|w_1| + \ldots + |w_k|)$.
- For each $i$ in $\{1, \ldots, n\}$, $|s'_i| \leq C(|w_1| + \ldots + |w_k|)$.
Since deterministic reduction automata are closed by intersection, it suffices to provide a deterministic reduction automaton for each one of the conditions for $s'$. The last condition of $H$-equivalence has been expressed above with deterministic reduction automata. Thus, it rests to see how to express the rest of conditions.

In [Comon and Jacquemard 2003], it is shown that deterministic tree automata with disequality constraints, which are a particular case of deterministic reduction automata, can recognize the normal forms of a given TRS. Hence, the condition that the terms $s_1', \ldots, s_n'$ are $R$-normal forms can be expressed with deterministic reduction automata. Moreover, in [Nagaya and Toyama 2002], it is shown that right-shallow and right-linear TRSs are regularity preserving. In particular, we can construct a tree automaton recognizing the set of reachable terms from a concrete constant $a$. Since regular languages are closed by boolean operations, we can construct a tree automaton recognizing the set of terms which are reachable from a set of constants $C$, and which are not reachable from constants in $\Sigma_0 \setminus C$. Hence, the requirement that subterms at positions smaller than or equal to $H$ are $R$-reachable from certain constants and are not $R$-reachable from other constants, can also be described using deterministic reduction automata.

Summarizing, the set $S$ is accepted by a deterministic reduction automaton obtained by intersecting all the deterministic reduction automata mentioned above. The size $K$ of this automaton can be bounded by a function of $|w_1| + \ldots + |w_k|$ and $H$. Therefore, each $|s'|$ is bounded by $\mathcal{B}(K(H, |w_1| + \ldots + |w_k|))$. □

As a technical fact, we also need that, when a term can be reduced in size by rewriting, any other term $H$-equivalent to it also can.

**Lemma 6.10.** Let $s$ and $s'$ be $H$-equivalent terms satisfying condition $(W_3)$. Let $t$ be a term satisfying $s \rightarrow_{D, \lambda} t$. Then, there exists a term $t'$ satisfying $s' \rightarrow_{D, \lambda} t'$.

**Proof.** Let $l \rightarrow r|C$ and $\sigma$ be the involved rule and substitution in $s \rightarrow_{D, \lambda} t$. We define a substitution $\sigma'$ satisfying $s' \rightarrow_{D, \sigma', \lambda} t'$ as follows. For each variable occurring at some position $p$ in $l$, $\sigma'$ is defined by $\sigma'(l|p) = s'|p$. For each variable occurring at some position $p$ in $r$, let $\{\alpha_1, \ldots, \alpha_n\}, \{r|p\}$ be the tuple of $C$ where $r|p$ occurs. If all $\sigma(\alpha_i)$ are constants, then we define $\sigma'(r|p)$ as $\sigma(r|p)$. Otherwise, if a certain $\sigma(\alpha_i)$ is not a constant, then $\alpha_i$ is a variable. Let $q$ be the position at $l$ where $\alpha_i$ occurs. Note that $\sigma(\alpha_i) = s|q$ and $\sigma'(\alpha_i) = s'|q$. Note also that, since $s$ and $s'$ are $H$-equivalent and satisfy Condition $(W_3)$, both $s|q$ and $s'|q$ are $R$-normal forms. In this case we define $\sigma'(r|p)$ as $s'|q$.

The fact that $\sigma'$ is well defined and satisfies $s' \rightarrow_{D, \sigma', \lambda} t'$ for some term $t'$ easily follows from the fact that $s$ and $s'$ are $H$-equivalent and $s \rightarrow_{D, \sigma, \lambda} t$. □

We want to reduce the size of a witness by replacing some of its subterms by new ones. The following definition and lemma show that the size of the new subterms has a bound depending only on the size of the non-replaced subterms.

**Definition 6.11.** Let $C$ be the computable function of Lemma 6.8. We define the computable function $\overline{C} : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ as $\overline{C}(n, M) = C(n \cdot C(M) + M)$, for $n, M \in \mathbb{N}$.

**Lemma 6.12.** Let $p_1, \ldots, p_n, \delta_1, \ldots, \delta_e, q_1, \ldots, q_k$ be a permutation of $1, \ldots, m$. Let $s = f[s_1]_{p_1} \ldots [s_n]_{p_n}[w_1]_{s_n} \ldots [w_e]_{c_1}[c_1]_{q_1} \ldots [c_k]_{q_k}$ be a witness, where all $s_1, \ldots, s_n, w_1, \ldots, w_e$ are non-constant normal forms, and $c_1, \ldots, c_k$ are constants.
Let $M$ be $|w_1| + \ldots + |w_e| + k$. Suppose that, for each $i$ in $\{1, \ldots, n\}$, it holds $|s_i| > C(n, M)$.

Then, there exist terms $s'_1, \ldots, s'_n$ satisfying:

$- s' = f[s'_1|p_1| \ldots |s'_n|p_n|w_1|s_1|w_1|s_1|c_1|q_1| \ldots |c_k|q_k]_q$ is a witness.

For each $i$ in $\{1, \ldots, n\}$, $|s'_i| \leq C(M)$.

**Proof.** Since $s$ is a witness, it satisfies the conditions of Lemma 6.8. Thus, we can choose $s'_1, \ldots, s'_n$ to be the ones of Lemma 6.8. They satisfy the following conditions:

- $s'$ satisfies conditions (W₂) and (W₃).
- For each $i$ in $\{1, \ldots, k\}$, $|w_i| \leq C(M)$.
- For each $i$ in $\{1, \ldots, n\}$, $|s'_i| \leq C(M)$.

Hence, in order to conclude, it suffices to see that $s'$ satisfies conditions (W₁) and (W₄). We start proving condition (W₄) by contradiction. Thus, assume there exists a derivation of the form $s' \rightarrow^p_D u' v'$. By Lemma 6.5, there exists a term $u$ $H$-equivalent to $u'$ and satisfying $s \rightarrow^p_D u$. By Lemma 6.10, there exists a term $v$ satisfying $u \rightarrow_D v$. This is in contradiction with the fact that $s$ satisfies condition (W₁).

Now, we prove that $s'$ satisfies condition (W₁) by contradiction. Thus, assume there exists a derivation from $s'$ into a $R$-normal form $v'$. By Lemma 5.9 and since $s'$ satisfies condition (W₁), this derivation is necessarily of the form $s' \rightarrow^p_D u' v'$. By Lemma 6.5, $u'$ is of the form $f[s'_1|p'_1| \ldots |s'_n|p'_n|w_1|s_1|w_1|s_1|c'_1|q'_1| \ldots |c'_k|q'_k]$ for some $p'_1, \ldots, p'_n, s'_1, \ldots, s'_n, c'_1, \ldots, c'_k$ which is a permutation of $1, \ldots, m$, and some constants $c'_1, \ldots, c'_k$. Furthermore, by the same lemma, there exists a term $u$ $P$-reachable from $s$, $H$-equivalent to $u'$, and of the form $f[s'_1|p'_1| \ldots |s'_n|p'_n|w_1|s_1|w_1|s_1|c'_1|q'_1| \ldots |c'_k|q'_k]_q$.

Since $u' \rightarrow^p_D v'$, then $v'$ is of the form $f[s'_1|p'_1| \ldots |s'_n|p'_n|w_1|s_1|w_1|s_1|c'_1|q'_1| \ldots |c'_k|q'_k]$ for some terms $u'_1, \ldots, u'_{k}$.

Recall that $v'$ is a $R$-normal form. Moreover, by construction, $v'$ satisfies conditions (W₂) and (W₃). Furthermore, $|s'_1| + \ldots + |s'_n| \leq n \cdot C(M)$, and $|w_1| + \ldots + |w_e| \leq |w_1| + \ldots + |w_e| + k = M$. Therefore, by applying Lemma 6.8, we deduce the existence of terms $u_1, \ldots, u_k$ satisfying the following conditions:

- $v'' = f[s'_1|p'_1| \ldots |s'_n|p'_n|w_1|s_1|w_1|s_1|c'_1|q'_1| \ldots |c'_k|q'_k]$ is $H$-equivalent to $v'$.
- $v''$ satisfies conditions (W₂) and (W₃).
- For each $i$ in $\{1, \ldots, n\}$ and each $j$ in $\{1, \ldots, e\}$, $|s'_i|, |w_j| \leq C(n \cdot C(M) + M) = C(n, M)$.
- For each $i$ in $\{1, \ldots, k\}$, $|w_i| \leq C(n, M)$.

Now, we argue that $v''$ is a $R$-normal form proceeding by contradiction.

- Suppose that a $B$-rule can be applied on $v''$. Since $v''$ satisfies (W₂), this $B$-step must be done at a depth $1$ position. But since $v''$ is $H$-equivalent to $v'$, the same $B$-step can be applied on $v'$, contradicting the fact that $v'$ is a $R$-normal form.
— Suppose that a $P$-step or a $D$-step can be applied on $v''$. Since $v''$ and $v'$ are $H$-equivalent and satisfy (W3), by Lemma 6.5 in the case of a $P$-step, and by Lemma 6.10 in the case of a $D$-step, another rewrite step is applicable on $v'$, thus contradicting the fact that it is a $R$-normal form.

Again since $v''$ is $H$-equivalent to $v'$, each $v_i$ is $R$-reachable from the same constants as $v'_i$, for $i$ in $\{1, \ldots, k\}$. Hence, there exists a derivation $u' \rightarrow^*_B v''$. We define a new term $v$ as $f[s_1|p'_1| \ldots |s_n|p'_n][w_1|s'_1| \ldots |w_e|s'_e][v_1|q'_1| \ldots |v_k|q'_k]$. Clearly, $v$ is $B$-reachable from $u$ (recall that $u$ is $f[s_1|p'_1| \ldots |s_n|p'_n][w_1|s'_1| \ldots |w_e|s'_e][v_1|q'_1| \ldots |v_k|q'_k]$ and $P$-reachable from $s$).

Recall that $s$ is a witness, and hence, $v$ is not a $R$-normal form. But it is a $B$-normal form because $v$ only differs from $v''$ in the terms $s_i$, which are $R$-normal forms, and $v''$ is a $B$-normal form (in fact, a $R$-normal form). Since $v$ is a $B$-normal form but not a $R$-normal form, there exists a $F$-rule $l \rightarrow r|C$ applicable on $v$ with a substitution $\sigma$. This rule must be applied at position $\lambda$ since all subterms of $v$ at depth 1 are $R$-normal forms. For the same reason, when two variables $x$ and $y$ occurring in $l$ appear in the same constraint of $C$, $\sigma(x)$ and $\sigma(y)$ must be identical. We distinguish the following cases depending on the form of this rule.

— Suppose the existence of two variables $x, y$ satisfying the following conditions: $x$ occurs in $l$ at a position in $\{p'_1, \ldots, p'_n\}$, $y$ occurs in $l$ at a position with a prefix in $\{s'_1, \ldots, s'_e, q'_1, \ldots, q'_k\}$, and both $x$ and $y$ occur in the same constraint of $C$. Note that the sizes of the terms $s'_1, \ldots, s'_e$ are bigger than $C(n, M)$, and that the sizes of the terms $v_1, \ldots, v_k$ are smaller than or equal to $C(n, M)$. Thus, $\sigma(x) = \sigma(y)$ is not possible, a contradiction.

— Suppose that previous case does not hold and all $l|p'_i$ for $i$ in $\{1, \ldots, n\}$ are variables. Then, the same rule can be applied on $v''$, contradicting the fact that it is a $R$-normal form.

— Suppose that for some $p'_i$ in $\{p'_1, \ldots, p'_n\}$, height($l|p'_i$) > 0 holds. By Lemma 5.10, there exists a rule $l' \rightarrow r'|C'$ in $F$ such that it rewrites $u$ at position $\lambda$ and height($l'|p'_i$) > 0. But this is a $D$-step on $u$, contradicting the fact that $s$ cannot reach a smaller term.

\[\square\]

We finally prove the existence of a computational bound for a witness with the following definition and lemma.

**Definition** 6.13. For each $i$ in $\{0, \ldots, m\}$, we define $\overline{C}_i$ inductively as follows:

$-\ \overline{C}_0 = C(m, 0)$

$-\ \overline{C}_i = \overline{C}(m - i, \overline{C}_{i-1})$, for $i$ in $\{1, \ldots, m\}$.

**Lemma 6.14.** If there exists a witness, then there exists a witness with size bounded by $m \cdot \overline{C}_m + 1$.

**Proof.** By the assumptions, there exists a witness $s$. We write $s$ more explicitly as $f(s_1, \ldots, s_n)$.

Assume that each $|s_i|$, for $i$ in $\{1, \ldots, m\}$, satisfies $|s_i| > \overline{C}(m, 0) = \overline{C}_0$. Then, by the definition of $C$ and $\overline{C}$, all $s_i$ are not constants. Thus, by Lemma 6.12, there
exist terms $s'_1, \ldots, s'_m$ satisfying that $f(s'_1, \ldots, s'_m)$ is a witness and $s'_i \leq \mathcal{C}(0) \leq \overline{\mathcal{C}}(m, 0) = \overline{\mathcal{C}}_0$, for each $i$ in $\{1, \ldots, m\}$. Thus, $|f(s'_1, \ldots, s'_m)| \leq m \cdot \overline{\mathcal{C}}_0 + 1 \leq m \cdot \overline{\mathcal{C}}_m + 1$, and the statement follows.

Otherwise, assume that for a certain $i$ in $\{1, \ldots, m\}$, say $i = m$ without loss of generality, $|s_m| \leq \overline{\mathcal{C}}_0$. At this point, assume also that each $|s_i|$, for $i$ in $\{1, \ldots, m - 1\}$, satisfies $|s_i| \leq \mathcal{C}(m - 1, \overline{\mathcal{C}}_0) = \overline{\mathcal{C}}_1$. Then, by Lemma 6.12, there exist terms $s'_1, \ldots, s'_{m-1}$ satisfying that $f(s'_1, \ldots, s'_{m-1}, s_m)$ is a witness and $s'_i \leq \mathcal{C}(\overline{\mathcal{C}}_0) \leq \overline{\mathcal{C}}(m - 1, \overline{\mathcal{C}}_0) = \overline{\mathcal{C}}_1$, for each $i$ in $\{1, \ldots, m - 1\}$. Thus, $|f(s'_1, \ldots, s'_{m-1}, s_m)| \leq m \cdot \overline{\mathcal{C}}_1 + 1 \leq m \cdot \overline{\mathcal{C}}_m + 1$, and the statement follows.

By iterating the previous argument, we conclude that either there exists an $i$ in $\{1, \ldots, m\}$ and terms $s'_1, \ldots, s'_i$ satisfying that $f(s'_1, \ldots, s'_i, s_{i+1}, \ldots, s_m)$ is a witness and $|f(s'_1, \ldots, s'_i, s_{i+1}, \ldots, s_m)| \leq m \cdot \overline{\mathcal{C}}_{m-i} + 1 \leq m \cdot \overline{\mathcal{C}}_m + 1$, or finally that all $s_i$ are bounded by $\overline{\mathcal{C}}_m$. In the first case, the statement follows. In the second case, it results that $f(s_1, \ldots, s_m)$ is a witness satisfying $|f(s_1, \ldots, s_m)| \leq \overline{\mathcal{C}}_m + 1$, and the statement follows again. □

**Theorem 6.15.** The weak normalization property is decidable for right-shallow right-linear TRS.

**Proof.** Since right-shallow right-linear TRS are effectively regularity preserving, for any term $s$ we can compute a tree automaton $A_s$ recognizing the $R$-reachable terms from $s$. Moreover, since the set of $R$-normal forms can be recognized by a deterministic tree automaton with disequality constraints $A_R$, and this class is closed by intersection with regular sets (like $\mathcal{L}(A_s)$), there exists a deterministic tree automaton with disequality constraints recognizing the set of normal forms $R$-reachable from $s$. Finally, since emptiness for this class is decidable [Comon and Jacquemard 2003], it is also decidable whether a given term $s$ is $R$-normalizing.

Thus, in order to conclude, it suffices to prove a computable bound for the size of an existing non-$R$-normalizing term whenever it exists. By Lemmas 6.7 and 6.14, either some constant is not $R$-normalizing, or there exists a witness $t$ with size bounded by $m \cdot \overline{\mathcal{C}}_m + 1$, and we are done. □

7. CONCLUSIONS AND FURTHER WORK

We have obtained a rewrite closure procedure for right-shallow right-linear TRS. The computed rewrite closure preserves nice properties like termination, confluence, weak normalization and unique normalization. Using the constructed rewrite closure, we have proved decidability of weak normalization for right-shallow right-linear TRS. This result required the use of some decidability results on tree automata with constraints.

Although we have used reduction automata, these kind of automata are perhaps too powerful for our purpose. These automata allow checking for a finite number of equalities at each branch in a run on a given term. We only need to check a finite number of equalities locally to the root. Thus, a less expressive class of automata is enough for proving our decidability result on weak normalization. It would be interesting to study whether decidability of the emptiness problem for this more restricted class is better than the one given in [Dauchet et al. 1995] for reduction automata. This could be possible, since the emptiness problem is
EXPTIME-complete for the even more restricted class of tree automata with disequality constraints [Comon and Jacquemard 2003].

Since our rewrite closure preserves confluence, it opens a door for proving decidability of confluence for right-shallow right-linear TRS. In the RTA-2004 conference in Aachen, Michio Oyamaguchi conjectured that right-shallow right-linear TRS should be the larger class, in this setting, for which confluence is decidable.

Our rewrite closure also preserves unique normalization. This property is undecidable for right-shallow right-linear TRS [Godoy and Tison 2007] (and even for more restricted TRS). Since weak normalization is decidable, it could be interesting to study whether unique normalization of a given TRS R is decidable under the assumption that R is weakly normalizing. A positive answer to this question implies decidability of whether each term reaches one and only one normal form. This problem is related to the confluence problem: when a TRS is weakly normalizing, it is confluent if and only if it is uniquely normalizing.

Our rewrite closure also preserves termination. This property is already known decidable for right-shallow right-linear TRS [Godoy et al. 2007]. It should be studied whether our rewrite closure transformation allows to obtain a better cost for deciding this problem.

REFERENCES


