An Algebraic Perspective on Boolean Function Learning

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Introduction

We can learn boolean functions represented in many ways:

Conjunctions, k-CNF, k-DNF, monotone DNF, Deterministic Finite Automata, k-term DNF, k-decision lists, read-once formulas, bounded rank decision trees, constant-degree polynomials, sparse polynomials, threshold gates, decision trees, CDNF formulas, multisymmetric concepts, conjunctions of Horn clauses, $O(\log n)$-term DNF, nested subspaces, counter languages, OBDD, Multiplicity (Weighted) Automata, ...

Introduction

Programs over monoids

- …*yet another* representation of boolean functions!!

yes, but

- gives context: detailed, deep taxonomies of monoids
- highlights a few unnoticed learnable classes
- suggests limits of current techniques

Summary

- Membership queries: algorithm for $MOD_p \circ MOD_m$ circuits
- Equivalence queries: decision lists over constant-degree polynomials over $\mathbb{F}_p$
- Membership + Equivalence:
  Maximal class of functions learnable as Multiplicity Automata

- Unifies many known results
- Does not capture: monotonicity, threshold circuits, read-k conditions, sensitivity to variable ordering

Background: Algebra and circuits

Semigroups

- A *semigroup* is set with a binary, associative operation
- A *monoid* is a semigroup with an identity
- A *group* is a monoid where each element has an inverse

- A monoid *A* *divides* a monoid *B* if *A* is a homomorphic image of a subsemigroup of *B*
- An *aperiodic* (aka group-free) monoid is one that is divided by no nontrivial group

Monoid products

- The direct product of A and B, $A \times B$ is defined by

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot a_2, b_1 \cdot b_2)$$

- A semidirect product of A and B is defined by choosing a function $f : A \times B \to A$ and

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 \cdot f(a_2, b_1), b_1 \cdot b_2)$$

- The *wreath product* of A and B, denoted $A \star B$, generalizes semidirect product by accounting for all choices of f

Decomposition theorem

Theorem [Krohn-Rhodes 62]

1. Every finite semigroup M divides a wreath product of finite simple groups and copies of the flip-flop monoid*

2. Only finite simple groups are required if M is a group

3. Only flip-flop monoids are required if M is aperiodic

* A particular 3-element aperiodic monoid

Boolean functions

Functions $f : \{0,1\}^n \to \{0,1\}$

- AND, OR, NOT, threshold gates
- Generalized MOD_m gates

$$MOD_m^A(x_1, \ldots, x_n) = 1 \quad \text{iff} \quad (\sum_{i=1}^{n} x_i) \in A$$

- Decision lists, decision trees
- Deterministic Finite Automata
- Weighted Automata or Multiplicity Automata over rings

  $M(x_1, \ldots, x_n) =$ sum over all paths consistent with $x_1 \ldots x_n$
  of product of labels in path

Programs over monoids

- An instruction over a monoid M is a triple (i, u, v)

  Interpreted as "read x_i and emit u if $x_i = 0$, v if $x_i = 1$"

- A *program* over M is a sequence of instructions $L = (I_1, \ldots, I_s)$ plus an accepting set $A \subseteq M$

$$(L, A)(x) = \begin{cases} 1 & \text{if } \prod_{i=1}^{s} I_i(x) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Programs over monoids (2)

- Each program P over M computes a boolean function $B(P)$
- $B(M)$ is the set of boolean functions computed by programs over M
- For a class of monoids $\mathscr{M}$

$$B(\mathscr{M}) = \bigcup_{M \in \mathscr{M}} B(M)$$

From monoids to boolean functions

Division: If M_1 divides M_2 then $B(M_1) \subseteq B(M_2)$

Direct product:

$$\begin{aligned} B(M_1 \times M_2) &\equiv \text{boolean combinations of } B(M_1) \text{ and } B(M_2) \\ &\equiv NC^0 \circ (B(M_1) \cup B(M_2)) \end{aligned}$$

Wreath product: For G a group,

$$B(M \star G) = B(M) \circ B(G)$$

Examples

Classical examples [Barrington 87, Barrington-Thérien 89]:

Monoidland	**Circuitland**
all monoids	NC^1
any nonsolvable group	NC^1
Abelian groups	boolean combinations of MOD gates
solvable groups	poly-size, constant-depth circuits made of MOD gates
aperiodic monoids	poly-size constant-depth circuits made of AND, OR, NOT gates

For learning we should remain well below NC^1

Dramatis personae, groups

Group	Description
Abelian groups	direct products cyclic groups
G_p or p-groups	groups of cardinality p^k
Nilpotent groups	direct products of p-groups
Solvable groups	wreath product of cyclic groups

Dramatis personae, groups

Groupland	**Circuitland**
Abelian groups	MOD_m, degree 1 polynomials over Z_m
G_p or p-groups	$MOD_{p^k} \circ NC^0$, $MOD_p \circ NC^0$, constant degree polynomials over $\mathbb{F}_p$
Nilpotent groups	$MOD_m \circ NC^0$, constant degree polynomials over Z_m
Solvable groups	constant-depth, poly-size modular circuits

Dramatis personae, aperiodic

DA monoids: $(stu)^k\, t\, (stu)^k = (stu)^k$ for some k

In circuitland [GT03]:

$$
\begin{aligned}
B(\mathbf{DA}) &= \bigcup_k \text{rank-}k \text{ decision trees} \\
B(\mathbf{DA}) \circ NC^0 &= \bigcup_k k\text{-decision lists}
\end{aligned}
$$

Borderline of expressivity in several contexts (descriptive complexity, communication complexity)

(Almost) nothing between $B(\mathbf{DA})$ and DNF, in monoidland

Membership queries

Negative results

Fact [GT06]

Learning programs over M requires 2^n Membership queries if

- M is *not* a group
- or M is a *nonsolvable* group

Reason: Can compute singletons in polynomial size

What about solvable groups?

Two subclasses of solvable groups *provably* weaker than NC^1:

- Nilpotent groups
  - Equivalent to polynomials of constant degree over some Z_m
  - Includes Abelian groups and G_p
- $G_p \star Abelian$
  - Equivalent to depth-2, MOD_p-of-MOD_m circuits

Group lower bounds

If G nilpotent, any two programs of length s over G differ on some assignment of weight c_G [PT88]

If $G \in G_p \star Abelian$, any two programs of length s over G differ on some assignment of weight $c_G \log s$ [BST89]

Learning strategy:

1. Ask Membership queries with all assignments of weight c_G (or weight $c_G \log s$)
2. Build *unique* program consistent with the answers

Part 2 is a purely computational problem

Abelian groups

Theorem

If G is Abelian, then $B(G)$ is learnable from Membership queries in $n^{O(1)}$ time

Equivalent to MOD_m gates and degree-1 polynomials over Z_m

Open: extend to degree-$O(1)$ polynomials (= nilpotent groups)

$G_p \star Abelian$

Theorem

If $G \in G_p \star Abelian$, then $B(G)$ is learnable from Membership queries in $n^{O(\log s)}$ time

Equivalent to MOD_p-of-MOD_m circuits

Known to be learnable in time $(n+s)^{O(1)}$ from Membership *and Equivalence* queries [BBTV97]

Equivalence queries

$DL \circ MOD_p \circ NC^0$

Theorem [from known results]

Decision lists having constant-degree polynomials over $\mathbb{F}_p$ at the nodes are learnable from $n^{O(1)}$ Equivalence queries

Combine:

- Tricks to make MOD_p gates 0/1-valued [Fermat,BT94]
- Subspace learning algorithm [HSW87]
- Decision list / nested difference algorithm [R87,HSW87]
- Composition theorem

$DL \circ MOD_p \circ NC^0$

$DL \circ MOD_p \circ NC^0$ subsumes:

- $DL \circ MOD_p$: nested differences of linear subspaces of $\mathbb{F}_p$
- $DL \circ NC^0$: k-DL, so rank-k DT's, k-CNF and k-DNF
- $MOD_p \circ NC^0$: constant-degree polynomials over $\mathbb{F}_p$
- strict width-2 branching programs [BBTV97]

Note: All these classes are nonuniversal

Algebraic equivalent

Theorem

1. $DL \circ MOD_p \circ NC^0 = B(\mathbf{DA} \star G_p)$
2. $\bigcup_m DL \circ MOD_m \circ NC^0 = B(\mathbf{DA} \star \mathit{Nilpotent})$

Hence $B(\mathbf{DA} \star G_p)$ learnable from $n^{O(1)}$ Equivalence queries

With Equivalence queries, $B(\mathbf{DA} \star \mathit{Abelian})$ learnable iff $B(\mathbf{DA} \star \mathit{Nilpotent})$ learnable

What's the ceiling?

- If $M \in \mathbf{DA} \star G_p$ then M is *not* universal
- If $M \notin \mathbf{DA} \star \mathit{Nilpotent}$ then M is universal [1]
- For M in between, we don't know; basic first question

[1] subtle lie here; see proceedings

Membership and Equivalence queries

Multiplicity Automata

Theorem [BBBKV00]

Functions $\Sigma^* \to \mathbb{F}_p$ computed by Multiplicity Automata over $\mathbb{F}_p$ are polynomial-time learnable from Membership and Equivalence queries.

Subsumes, besides DFA:

- polynomials over $\mathbb{F}_p$
- unambiguous DNF (hence decision trees and k-term DNF)
- MOD_p-of-MOD_m circuits

Algebraic characterization

$L\mathbf{G_p} \circledcirc \mathbf{Com}$ [Weil 87]

The value of $m_1 \ldots m_s$ can be determined by counting mod p the number of factorizations of the form $a_0 L a_1 L a_2 \ldots a_{k-1} L a_k$, for L a commutative language (bool comb of)

Theorem

1. $B(L\mathbf{G_p} \circledcirc \mathbf{Com})$ is polynomially simulated by MA over $\mathbb{F}_p$
2. unambiguous DNF, polynomials, and MOD_p-of-MOD_q circuits are polynomially simulated in $B(L\mathbf{G_p} \circledcirc \mathbf{Com})$

Order sensitivity

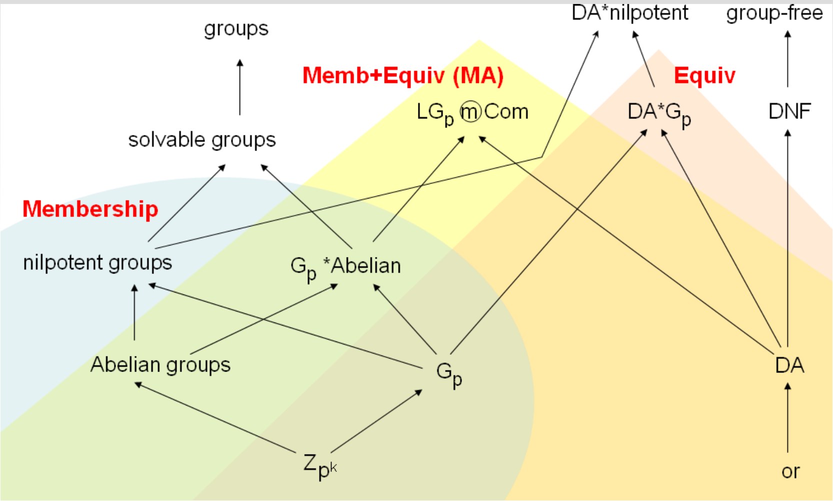
Conjecture

$L\mathbf{G_p} \textcircled{m} \mathbf{Com}$ is the largest class of monoids that is polynomially simulated by MA

Intuition: If $M \notin L\mathbf{G_p} \textcircled{m} \mathbf{Com}$ there is a function $f \in B(M)$ such that f has MA of size $poly(n)$ but the smallest MA for some $f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ has size $2^{\Omega(n)}$

There is an explicit characterization [TT07] of monoids *not* in $L\mathbf{G_p} \textcircled{m} \mathbf{Com}$

In summary



groups
DA*nilpotent
group-free
Memb+Equiv (MA)
Equiv
LGp ⓜ Com
DA*Gp
DNF
solvable groups
Membership
nilpotent groups
Gp *Abelian
Abelian groups
Gp
DA
Zpk
or

Conclusions

- Many learning results can be unified into 3 algorithms for learning large classes of monoids
- Extending to larger classes seems to require either proving new lower bounds or learning DNF
- Open problem: Efficiently learn one $MOD_m \circ NC^0$ gate