# Encodings into SAT

## Combinatorial Problem Solving (CPS)

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# What is an encoding?

- Language of SAT solvers: CNF propositional formulas

- To solve combinatorial problems with SAT solvers, constraints have to be represented in this language

- An encoding of a constraint $C$ into SAT is a CNF $F$ that expresses $C$, so that there is a bijection

$$\text{solutions to } C \iff \text{models of } F$$

# Examples: AMO constraints

- An AMO constraint is of the form $x_0 + \ldots + x_{n-1} \leq 1$
  where each $x_i$ is 0-1

  (At Most One of the variables can be true)

- Quadratic encoding.

  - Variables: the same $x_0, \ldots, x_{n-1}$
  - Clauses: for $0 \leq i < j < n$, $\overline{x_i} \vee \overline{x_j}$
  - Requires $\binom{n}{2} = O(n^2)$ clauses

- Other encodings try to use fewer clauses,
  at the cost of introducing new variables

# Examples: AMO constraints

- Logarithmic encoding. Let $m = \lceil \log_2 n \rceil$. Then:
  - Variables: the $x_i$ and new variables $y_0, y_1, \ldots, y_{m-1}$
  - Clauses: for $0 \le i < n$, $0 \le j < m$
    - $\overline{x_i} \vee y_j$ if the $j$-th digit in binary of $i$ is 1
    - $\overline{x_i} \vee \overline{y_j}$ otherwise
  - Requires $O(\log n)$ new variables, $O(n \log n)$ clauses

# Examples: AMO constraints

- Heule encoding.
  - If $n \leq 3$, the encoding is the quadratic encoding.
  - If $n \geq 4$, introduce an auxiliary variable $y$ and encode $x_0 + x_1 + y \leq 1$ and $x_2 + \cdots + x_{n-1} + \overline{y} \leq 1$ (this one recursively).
  - Requires $O(n)$ new variables, $O(n)$ clauses
- Other encodings exist (see next)

# Consistency and Arc-Consistency

- Let us consider an encoding of a constraint $C$ such that there is a correspondence between assignments of the variables of $C$ to their domains, and partial assignments of the boolean variables of the encoding

- The encoding is consistent if whenever $M$ is not compatible with any solution to $C$, unit propagation on the boolean assignment of $M$ leads to a conflict

- The encoding is arc-consistent if

  - it is consistent, and
  - unit propagation discards arc-inconsistent values (i.e., values without a support)

- These are good properties for encodings: SAT solvers are very good at unit propagation!

# Consistency and Arc-Consistency

- In the case of an AMO constraint $x_0 + \ldots + x_{n-1} \leq 1$:
- Consistency $\equiv$ if there are two true vars $x_i$ in $M$ or more, then unit propagation should give a conflict
- Arc-consistency $\equiv$ Consistency $+$ if there is one true var $x_i$ in $M$, then unit propagation should set all others $x_j$ to false
- The quadratic, logarithmic and Heule encodings are all arc-consistent

# Cardinality Constraints

- A cardinality constraint is of the form $x_1 + \ldots + x_n \bowtie k$ where each $x_i$ is 0-1 and $\bowtie \in \{\leq, <, \geq, >, =\}$
- AMO are a particular case of card. constraints where $k = 1$ and $\bowtie$ is $\leq$
- Without loss of generality we may assume $\bowtie$ is $<$, i.e.,

$$x_1 + \ldots + x_n < k$$

- Naive encoding.
  - Variables: the same $x_1, \ldots, x_n$
  - Clauses: for all $1 \leq i_1 < i_2 < \ldots < i_k \leq n$,

$$\overline{x_{i_1}} \vee \overline{x_{i_2}} \vee \ldots \vee \overline{x_{i_k}}$$

  - This is $\binom{n}{k}$ clauses!

# Adders

- Again, other encodings try to use fewer clauses,
  at the cost of introducing new variables

- Adder encoding.
  Build an adder circuit by using bit-adders as building blocks:


$$
\begin{array}{lcl}
s & \leftrightarrow & \mathrm{XOR}(x, y, z) \\
c & \leftrightarrow & (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)
\end{array}
$$

where $\mathrm{XOR}(x, y, z)$ is short for

$$(x \wedge \overline{y} \wedge \overline{z}) \vee (\overline{x} \wedge y \wedge \overline{z}) \vee (\overline{x} \wedge \overline{y} \wedge z) \vee (x \wedge y \wedge z)$$

# Adders

- Encodings of this kind are not arc-consistent.
- Consider $x + y + z \leq 0$.
  Then unit propagation should propagate $\overline{x}, \overline{y}, \overline{z}$.
- Let us encode the constraint with a full adder
- The encoding is the Tseitin transformation of $\overline{s}, \overline{c}$ and

$$
\begin{array}{rcl}
s & \leftrightarrow & \mathrm{XOR}(x, y, z) \\
c & \leftrightarrow & (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)
\end{array}
$$

- Note that

$$
\begin{array}{rcl}
\overline{s} & \rightarrow & (\overline{x} \vee y \vee z) \wedge (x \vee \overline{y} \vee z) \wedge (x \vee y \vee \overline{z}) \wedge (\overline{x} \vee \overline{y} \vee \overline{z}) \\
\overline{c} & \rightarrow & (\overline{x} \vee \overline{y}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{y} \vee \overline{z})
\end{array}
$$

- Unit propagation cannot propagate anything!

# Sorting Networks

- Sorting Network encoding.

  Pass $x_1, \ldots, x_n$ as inputs to
  a circuit that sorts (say, decreasingly) $n$ bits.

  Let $y_1, \ldots, y_n$ be the outputs of this circuit.

  Then if the constraint to be encoded is

  - $\sum_{i=1}^{n} x_i \geq k$, then add clause $y_k$
  - $\sum_{i=1}^{n} x_i \leq k$, then add clause $\overline{y_{k+1}}$
  - $\sum_{i=1}^{n} x_i = k$, then add clauses $y_k$, $\overline{y_{k+1}}$

# Sorting Networks

- How to build such a sorting circuit?
- A possibility is to implement mergesort
- In what follows: so-called odd-even sorting networks
- The basic block of odd-even sorting networks are 2-comparators

# 2-comparators

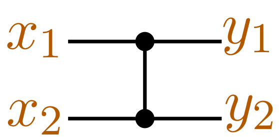
- A 2-comparator is a sorting network of size 2:
  - it has 2 input variables ($x_1$ and $x_2$)
  - it has 2 output variables ($y_1$ and $y_2$)
  - $y_1$ is true if and only if at least one of the input variables is true (i.e., it is the maximum or disjunction)
  - $y_2$ is true if and only if both two input variables are true (i.e., it is the minimum or conjunction)

# 2-comparators

- Clauses:

$$\begin{array}{ccccccccc} x_1 & \leftarrow & y_2, & x_2 & \leftarrow & y_2, & x_1 \vee x_2 & \leftarrow & y_1, \\ x_1 & \rightarrow & y_1, & x_2 & \rightarrow & y_1, & x_1 \wedge x_2 & \rightarrow & y_2 \end{array}$$

- Graphical representation:



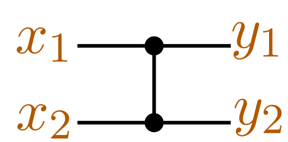

- Some simplifications are possible:
  - For $\geq$ constraints: top three clauses suffice
  - For $\leq$ constraints: bottom three clauses suffice
  - For $=$ constraints: all clauses needed

# 2-comparators

- Clauses:

$$
\begin{aligned}
x_1 &\leftarrow y_2, & x_2 &\leftarrow y_2, & x_1 \vee x_2 &\leftarrow y_1, \\
\overline{x}_1 &\leftarrow \overline{y}_1, & \overline{x}_2 &\leftarrow \overline{y}_1, & \overline{x}_1 \vee \overline{x}_2 &\leftarrow \overline{y}_2
\end{aligned}
$$

- Graphical representation:

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# Merge Networks

- From now on we assume that $n$ is a power of two
(if not, pad with variables set to false)

- A merge network takes as input two ordered sets of variables of size $n$ and produces an ordered output of size $2n$.

- Let $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ be the inputs.
We recursively define a merge network as follows:

- If $n = 1$, a merge network is a 2-comparator:

$$\mathrm{Merge}(x_1; x'_1) := \text{2-Comp}(x_1, x'_1).$$

# Merge Networks

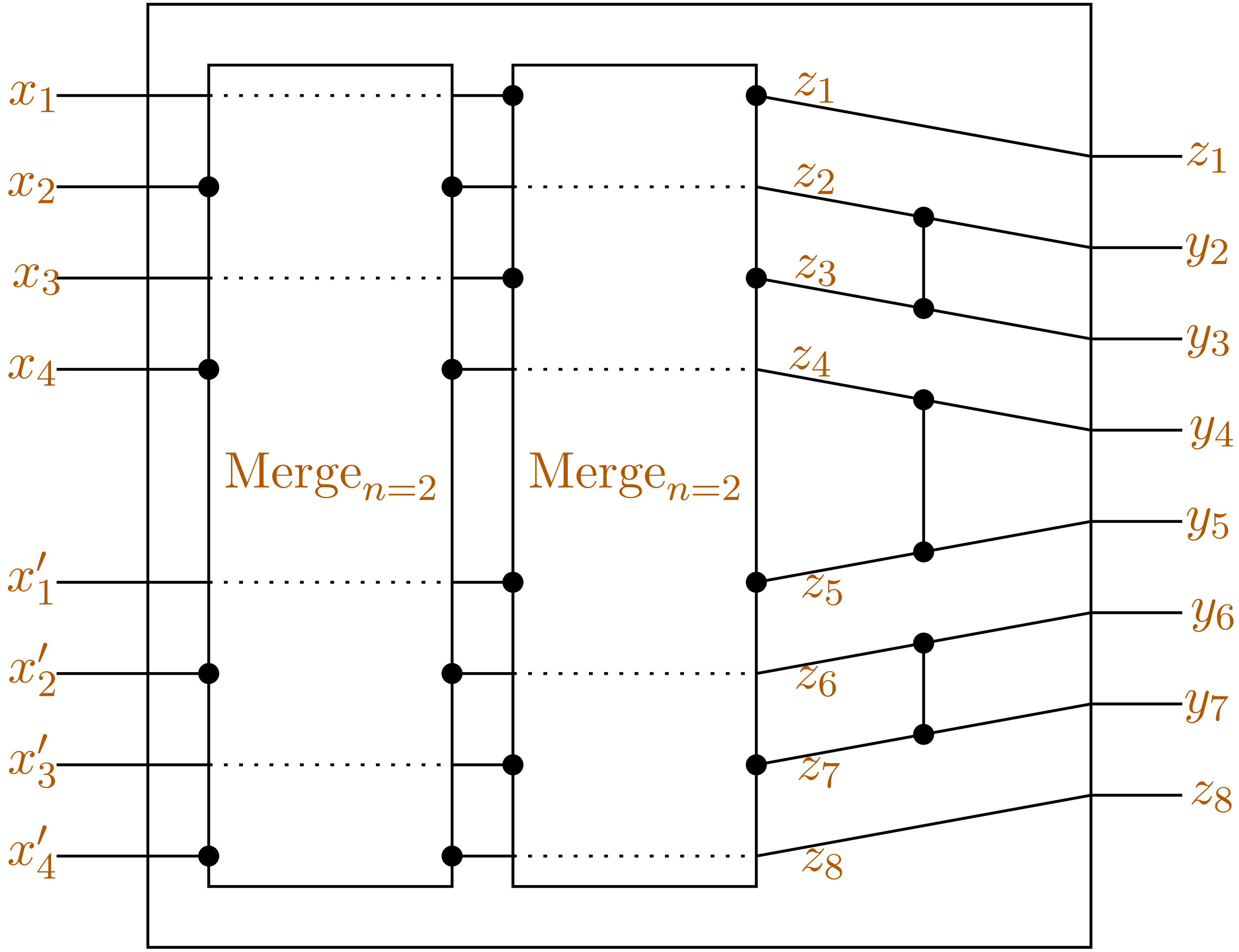
- For $n > 1$: Let us define

$$\begin{aligned}
(z_1, z_3, \ldots, z_{2n-1}) &= \text{Merge}(x_1, x_3, \ldots, x_{n-1}; x'_1, x'_3, \ldots x'_{n-1}), \\
(z_2, z_4, \ldots, z_{2n}) &= \text{Merge}(x_2, x_4, \ldots, x_n; x'_2, x'_4, \ldots, x'_n), \\
(y_2, y_3) &= \text{2-Comp}(z_2, z_3), \\
(y_4, y_5) &= \text{2-Comp}(z_4, z_5), \\
&\ldots \\
(y_{2n-2}, y_{2n-1}) &= \text{2-Comp}(z_{2n-2}, z_{2n-1})
\end{aligned}$$

Then,

$$\text{Merge}(x_1, x_2, \ldots, x_n; x'_1, x'_2, \ldots, x'_n) := (z_1, y_2, y_3, \ldots, y_{2n-1}, z_{2n})$$

# Merge Networks



# Merge Networks

Sketch of the proof of correctness of $\mathrm{Merge}$:

By IH: $\{x_1, x_3, \ldots, x_{n-1},\ x'_1, x'_3, \ldots, x'_{n-1}\} = \{z_1, z_3, \ldots, z_{2n-1}\}$

By IH: $\{x_2, x_4, \ldots, x_n,\quad x'_2, x'_4, \ldots, x'_n\}\quad = \{z_2, z_4, \ldots, z_{2n}\}$

Hence $\{x_1, x_2, \ldots, x_n,\quad x'_1, x'_2, \ldots, x'_n\}\quad = \{z_1, z_2, \ldots, z_{2n}\}$

And

$(y_2, y_3) = \text{2-Comp}(z_2, z_3)$ implies $\{y_2, y_3\} = \{z_2, z_3\}$

$(y_4, y_5) = \text{2-Comp}(z_4, z_5)$ implies $\{y_4, y_5\} = \{z_4, z_5\}$

$\ldots$

$(y_{2n-2}, y_{2n-1}) = \text{2-Comp}(z_{2n-2}, z_{2n-1})$ implies $\{y_{2n-2}, y_{2n-1}\} = \{z_{2n-2}, z_{2n-1}\}$

So $\{x_1, x_2, \ldots, x_n,\ x'_1, x'_2, \ldots, x'_n\} = \{z_1, y_2, y_3, \ldots, y_{2n-2}, y_{2n-1}, z_{2n}\}$

# Merge Networks

Let us prove outputs are sorted decreasingly. For $1 \leq i < n-1$ let us see:

- $z_{2i} \geq z_{2(i+1)+1}$:

  Let us see $z_{2(i+1)+1} = 1$ implies $z_{2i} = 1$

  If $z_{2(i+1)+1} = z_{2i+3} = z_{2(i+2)-1} = 1$ there are $\geq i+2$ 1's in odd $x, x'$

  Let $p$ be the number of 1's in odd $x$

  Let $q$ the number of 1's in odd $x'$

  Then $p + q \geq i + 2$

  As $x, x'$ is ordered decreasingly,

  there are $\geq p-1$ 1's in even $x$, $\geq q-1$ 1's in even $x'$

  So there are $\geq (p-1) + (q-1) = p + q - 2 \geq i$ 1's in even $x, x'$

  Hence $z_{2i} = 1$

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# Merge Networks

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- $z_{2i} \geq z_{2(i+1)+1}$: proved
- $z_{2i} \geq z_{2(i+1)}$: by IH
- $z_{2i+1} \geq z_{2(i+1)+1}$: by IH
- $z_{2i+1} \geq z_{2(i+1)}$: similar to above

So $\min(z_{2i}, z_{2i+1}) \geq \max(z_{2(i+1)}, z_{2(i+1)+1})$

But $y_{2i+1} = \min(z_{2i}, z_{2i+1})$ and $y_{2(i+1)} = \max(z_{2(i+1)}, z_{2(i+1)+1})$

So $y_{2i+1} \geq y_{2(i+1)}$

And $y_{2i} \geq y_{2i+1}$ for being outputs of $2\text{-Comp}$

Altogether $z_1, y_2, y_3, \ldots, y_{2n-2}, y_{2n-1}, z_{2n}$ is sorted decreasingly

# Sorting Networks

- A sorting network of size $n$ takes an input of size $n$ and sorts it (decreasingly).
- We can build a sorting network by successively applying merge networks (as in mergesort).
- Let $x_1, \ldots, x_n$ be the inputs.
  We recursively define a sorting network as follows:
- If $n = 2$, a sorting network is a 2-comparator:

$$\mathrm{Sorting}(x_1, x_2) := \text{2-Comp}(x_1, x_2)$$

# Sorting Networks

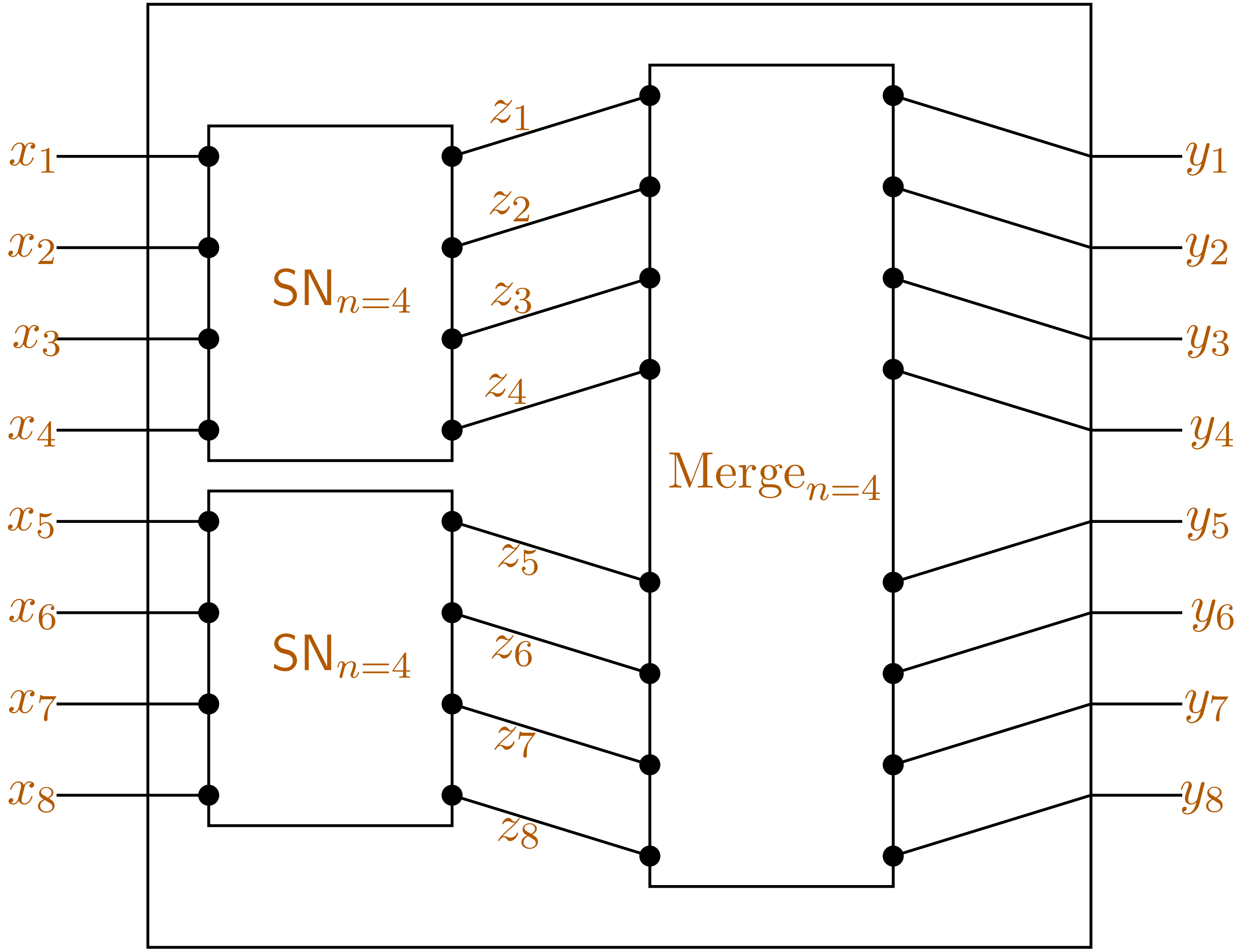
- For $n > 2$: Let us define

$$\begin{aligned} (z_1, z_2, \dots, z_{n/2}) &= \mathrm{Sorting}(x_1, x_2, \dots, x_{n/2}), \\ (z_{n/2+1}, z_{n/2+2}, \dots, z_n) &= \mathrm{Sorting}(x_{n/2+1}, x_{n/2+2}, \dots, x_n), \\ (y_1, y_2, \dots, y_n) &= \mathrm{Merge}(z_1, z_2, \dots, z_{n/2}; z_{n/2+1} \dots, z_n) \end{aligned}$$

Then,

$$\mathrm{Sorting}(x_1, x_2, \dots, x_n) := (y_1, y_2, \dots, y_n)$$

# Sorting Networks



# Sorting Networks

- This encoding of cardinality constraints is arc-consistent
- It uses $O(n \log^2 n)$ new variables and $O(n \log^2 n)$ clauses
- Several improvements are possible:
  - Only the first $k$ outputs suffice:
    cardinality networks use $O(n \log^2 k)$ vars and clauses
  - No need to assume that $n$ is a power of two:
    merges can be defined for inputs of different sizes

# Bibliography


- N. Eén, N. Sörensson: Translating Pseudo-Boolean Constraints into SAT. JSAT 2(1-4): 1-26 (2006)
- R. Asín, R. Nieuwenhuis, A. Oliveras, E. Rodríguez-Carbonell: Cardinality Networks: a theoretical and empirical study. Constraints 16(2): 195-221 (2011)
- I. Abío, R. Nieuwenhuis, A. Oliveras, E. Rodríguez-Carbonell: A Parametric Approach for Smaller and Better Encodings of Cardinality Constraints. Principles and Practice of Constraint Programming, 2013
- I. Abío: Solving hard industrial combinatorial problems with SAT. PhD Thesis (2013)