Satisfiability Modulo Difference Logic

Combinatorial Problem Solving (CPS)

Albert Oliveras   Enric Rodríguez-Carbonell

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Basic Definitions

- Given a directed graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$, the weight $w(p)$ of a path $p = (v_0, \ldots, v_k)$ is

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

- The distance $\delta(u, v)$ from u to v is

$$\delta(u, v) = \begin{cases} \min\{w(p) : u \overset{p}{\leadsto} v\} & \text{if there is a path from } u \text{ to } v \\ \infty & \text{otherwise} \end{cases}$$

- A shortest path from u to v is
any path p such that $u \overset{p}{\leadsto} v$ and $w(p) = \delta(u, v)$

Difference Constraints

- A difference constraint
  is a linear constraint of the form $x - y \bowtie k$, where:
  - $\bowtie \in \{\leq, \geq, <, >, =, \neq\}$
  - x and y are integer/real variables
  - $k \in \mathbb{Z}$ or $\mathbb{R}$
- Equivalent forms: $x \bowtie y + k$, $x + k \bowtie y$
- Ex. Let s_a, s_b be the starting times of two tasks a and b.
  - $s_a + T \leq s_b$:
    task b cannot start earlier than T minutes after task a
    (they use same resources, etc.)
  - $s_a \leq s_b + T$:
    task a cannot start later than T minutes after task b
    (the product of b to be used by a expires, etc.)

Difference Logic

- Lits in Difference Logic (DL) are difference constraints
- Some transformations are performed at parsing time
- If domain is $\mathbb{Z}$ replace $x - y < k$ by $x - y \leq k - 1$
- If domain is $\mathbb{R}$ replace $x - y < k$ by $x - y \leq k - \delta$
  - δ is a sufficiently small real
  - δ is not computed but used symbolically (like in De Moura's & Dutertre's approach for LRA)

Difference Logic

- Note any solution to a set of DL literals can be shifted (i.e. if σ is a solution then so is $\sigma'(x) = \sigma(x) + k$)
- This allows one to handle bounds $x \leq k$
  - Introduce fresh variable $zero$
  - Convert all bounds $x \leq k$ into $x - zero \leq k$
  - Given a solution σ, shift it so that $\sigma(zero) = 0$

Difference Logic

- $x - y = k$ is replaced by $x - y \leq k \ \wedge \ x - y \geq k$
- $x - y \neq k$ is replaced by $x - y < k \ \vee \ x - y > k$
- If we allowed (dis)equalities as literals, then:
  - If domain is $\mathbb{R}$, then consistency check is polynomial
  - If domain is $\mathbb{Z}$, then consistency check is NP-hard (k-colorability)
    - $1 \leq c_i \leq k$ with $i = 1 \ldots |V|$ encodes colors for vertexs
    - $c_i \neq c_j$ if $(i, j) \in E$ encodes colorability constraint
- Hence we can assume all literals are $x - y \leq k$

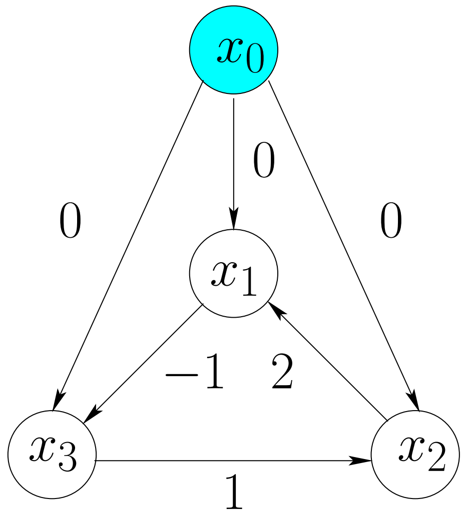
Constraint Graph

- Given n variables x_1, x_2, ..., x_n and
  a system S of m difference constraints $x_i - x_j \leq k_{ij}$
  we can construct the constraint graph $G = (V, E)$ where:

  - $V = \{x_0, x_1, ..., x_n\}$
    (each vertex corresponds to a var plus extra vertex x_0)
  - $E = \{(x_j, x_i) \mid x_i - x_j \leq k_{ij} \in S\} \cup \{(x_0, x_i) \mid 1 \leq i \leq n\}$
    (each edge corresponds to a constraint,
    plus extra edges from x_0 to variables)
    Moreover, $w(x_j, x_i) = k_{ij}$ and $w(x_0, x_i) = 0$
  - G has $n + 1$ vertices and $n + m$ edges
  - Note that $\delta(x_0, x_i) < \infty$ for any x_i
  - But $\delta(x_0, x_i)$ may not be well-defined if x_i belongs to a negative cycle

Constraint Graph

$$x_1 - x_2 \leq 2$$
$$x_2 - x_3 \leq 1$$
$$x_3 - x_1 \leq -1$$



Systems of Difference Constraints

Theorem. Given S a system of difference constraints,
let $G = (V, E)$ be the corresponding constraint graph.

1. If G contains a negative cycle, then S is infeasible.
2. Otherwise $x_i \to \delta(x_0, x_i)$ is a solution to S.

Proof.

Let us prove 1.

Let $c = (v_0, ..., v_k)$ be a negative cycle, which corresponds to constraints
$x(v_i) - x(v_{i-1}) \leq w(v_{i-1}, v_i), \ (1 \leq i \leq k)$ in S.
(note x_0 cannot be in the cycle, as it has no entering edges)
By adding all these constraints we get the constraint $0 \leq \sum_{i=1}^{k} w(v_{i-1}, v_i)$,
which is trivially false as RHS is < 0.

Systems of Difference Constraints

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let $G = (V, E)$ be the corresponding constraint graph.

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Proof.

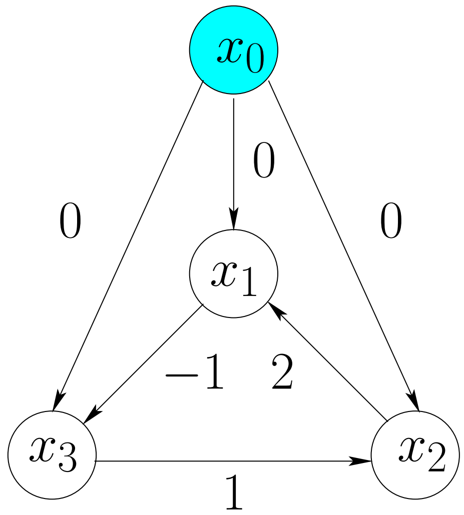
Let us prove 2.

If G does not contain any negative cycle,
then for all $1 \leq i \leq n$, we have $-\infty < \delta(x_0, x_i) < \infty$.

By the triangle inequality, $x_i \to \delta(x_0, x_i)$ is a solution.

Example (I)

$$x_1 - x_2 \leq 2$$
$$x_2 - x_3 \leq 1$$
$$x_3 - x_1 \leq -1$$

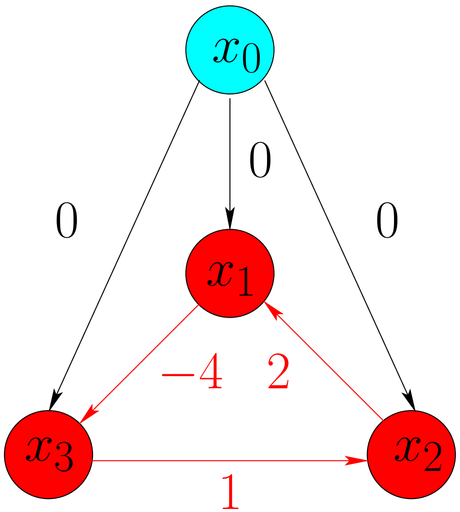


$$(x_1, x_2, x_3) = (\delta(x_0, x_1), \delta(x_0, x_2), \delta(x_0, x_2)) = (0, 0, -1)$$

is a solution!

Example (II)

$$x_1 - x_2 \leq 2$$
$$x_2 - x_3 \leq 1$$
$$x_3 - x_1 \leq -4$$



Infeasible!

Consistency Checks

- Consistency checks can be performed using Bellman-Ford in time $O(|V| \cdot |E|)$
- Other more efficient variants exist
- Inconsistency explanations are negative cycles (minimal wrt. set inclusion)

Bibliography - Further reading

- Chao Wang, Franjo Ivancic, Malay K. Ganai, Aarti Gupta. *Deciding Separation Logic Formulae by SAT and Incremental Negative Cycle Elimination.* LPAR 2005: 322-336

- Scott Cotton, Oded Maler. *Fast and Flexible Difference Constraint Propagation for DPLL(T)*. SAT 2006: 170-183