The Simplex Method

Combinatorial Problem Solving (CPS)

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Global Idea

- The Fundamental Theorem of Linear Programming ensures it is sufficient to explore basic feasible solutions to find the optimum of a feasible and bounded LP.

- The simplex method moves from one basic feasible solution to another that does not worsen the objective function while optimality or unboundedness are not detected.
Bases and Tableaux

- Given a basis $B$, its tableau is the system of equations

$$x_B = B^{-1}b - B^{-1}Rx_R$$

which expresses values of basic variables in terms of non-basic variables

\[
\begin{align*}
\text{min} & \ -x - 2y \\
x + y + s_1 &= 3 \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

$\mathcal{B} = \{x, y, s_2\}$

\[
\begin{align*}
x &= 1 + s_3 - s_1 \\
y &= 2 - s_3 \\
s_2 &= 1 - s_3 + s_1
\end{align*}
\]
Basic Solution in a Tableau

- The basic solution can be easily obtained from the tableau by looking at independent terms

\[
\begin{align*}
    x &= 1 + s_3 - s_1 \\
    y &= 2 - s_3 \\
    s_2 &= 1 - s_3 + s_1
\end{align*}
\]

Note that by definition of basic solution, the values for non-basic variables are null.
Detecting Optimality (1)

- Tableaux can be extended with the expression of the cost function in terms of the non-basic variables

\[
\begin{align*}
\min -x - 2y &\implies \min -5 + s_1 + s_3 \\
x &= 1 + s_3 - s_1 \\
y &= 2 - s_3 \\
s_2 &= 1 - s_3 + s_1
\end{align*}
\]

- Value of objective function at basic solution can be easily found by looking at independent term
Detecting Optimality (1)

- Tableaux can be extended with the expression of the cost function in terms of the non-basic variables

\[
\begin{align*}
\min -x - 2y &\Rightarrow \min -5 + s_1 + s_3 \\
x &= 1 + s_3 - s_1 \\
y &= 2 - s_3 \\
s_2 &= 1 - s_3 + s_1
\end{align*}
\]

- Value of objective function at basic solution can be easily found by looking at independent term

- Coefficients of non-basic variables in objective function after substitution with tableau are called reduced costs

- By convention, reduced costs of basic variables are 0
Tableaux can be extended with the expression of the cost function in terms of the non-basic variables

\[
\begin{align*}
\min -x - 2y & \implies \min -5 + s_1 + s_3 \\
x & = 1 + s_3 - s_1 \\
y & = 2 - s_3 \\
s_2 & = 1 - s_3 + s_1
\end{align*}
\]

- Value of objective function at basic solution can be easily found by looking at independent term
- Coefficients of non-basic variables in objective function after substitution with tableau are called reduced costs
- By convention, reduced costs of basic variables are 0
- Sufficient condition for optimality: all reduced costs are \( \geq 0 \)

The cost of any other feasible solution can’t improve on the basic solution

So the basic solution is optimal!
Detecting Optimality (2)

- If reduced costs $\geq 0$: sufficient condition for optimality but not necessary

- In the example, both bases are optimal but in one we cannot detect optimality!

$$\begin{align*}
\min & x + 2y \\
x + y &= 0 \\
x, y &\geq 0
\end{align*}$$

$$B = \{x\} \quad B = \{y\}$$

$$\begin{align*}
\min & y \\
x &= -y
\end{align*} \quad \begin{align*}
\min & -x \\
y &= -x
\end{align*}$$
Improving the Basic Solution

■ What to do when the tableau does not satisfy the optimality condition?

\[
\begin{align*}
\min & -x - 2y \\
x + y + s_1 &= 3 \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0 \\
\end{align*}
\]

\[\mathcal{B} = (s_1, s_2, s_3)\]

\[
\begin{align*}
\min & -x - 2y \\
s_1 &= 3 - x - y \\
s_2 &= 2 - x \\
s_3 &= 2 - y \\
\end{align*}
\]

■ E.g. variable \( y \) has a negative reduced cost

■ If we can get a new solution where \( y > 0 \) and the rest of non-basic variables does not worsen the objective value, we will get a better solution.

■ In general, to improve the objective value:
  increase the value of a non-basic variable with negative reduced cost while the rest of non-basic variables are frozen to 0

E.g. increase \( y \) while keeping \( x = 0 \)
Improving the Basic Solution

- Let us increase value of variable \( y \)
  while satisfying non-negativity constraints on basic variables

\[
\begin{align*}
    s_1 &= 3 - x - y \quad \text{Limits new value to } \leq 3 \\
    s_2 &= 2 - x \quad \text{Does not limit new value} \\
    s_3 &= 2 - y \quad \text{Limits new value to } \leq 2
\end{align*}
\]

- Best possible new value for \( y \) is \( \min(3, 2) = 2 \)

- The bound due to \( s_3 \) is tight, i.e.,
  the constraint \( s_3 \geq 0 \) limits the new value for \( y \)
Improving the Basic Solution

- The new solution does not seem to be basic... but in fact it is. We need to change the basis.

- When increasing the value of the improving non-basic variable, all basic variables for which the bound is tight become 0

\[ y = 2 \rightarrow s_3 = 0 \]

- Choose a tight basic variable, here \( s_3 \), to be exchanged with the improving non-basic variable, here \( y \)

- We can get the tableau of the new basis by solving the non-basic variable in terms of the basic one and substituting:

\[
\begin{align*}
\text{min} & \quad -x - 2y \\
\text{s}_1 &= 3 - x - y \\
\text{s}_2 &= 2 - x \\
\text{s}_3 &= 2 - y \\
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad 4 - x + 2s_3 \\
\text{s}_1 &= 1 + s_3 - x \\
\text{s}_2 &= 2 - x \\
\text{s}_3 &= 2 - y \\
\end{align*}
\]
Improving the Basic Solution

\[ \begin{align*}
    x & \geq 0 \\
    y & \geq 0 \\
    x + y & \leq 3 \\
    x & \leq 2 \\
    y & \leq 2 \\
    x & \geq 0 \\
    y & \geq 0
\end{align*} \]

\[ \min -x - 2y \]
Improving the Basic Solution

- Let us now increase value of variable $x$

\[
\begin{align*}
\text{min} & \quad -4 - x + 2s_3 \\
\text{s}_1 &= 1 + s_3 - x \quad \text{Limits new value to } \leq 1 \\
\text{s}_2 &= 2 - x \quad \text{Limits new value to } \leq 2 \\
y &= 2 - s_3 \quad \text{Does not limit new value}
\end{align*}
\]

- Best possible new value for $x$ is $\text{min}(2, 1) = 1$

- Variable $s_1$ leaves the basis and variable $x$ enters

\[
\begin{align*}
\text{min} & \quad -4 - x + 2s_3 \\
\text{s}_1 &= 1 + s_3 - x \\
\text{s}_2 &= 2 - x \\
y &= 2 - s_3
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad -5 + s_1 + s_3 \\
x &= 1 + s_3 - s_1 \\
\text{s}_2 &= 1 - s_3 + s_1 \\
y &= 2 - s_3
\end{align*}
\]
Improving the Basic Solution

\[ \begin{align*}
    x & \leq 2 \\
    x + y & \leq 3 \\
    y & \leq 2 \\
    y & \geq 0 \\
    x & \geq 0 \\
    (0, 0) & \geq 0 \\
    \min -x - 2y & \geq 0 \\
    (0, 2) & \leq 0 \\
    (1, 2) & \leq 0
\end{align*} \]
Unboundedness is detected when the new value for the chosen non-basic variable is not bounded.

\[
\begin{align*}
\max & \quad x + y \\
2x + y & \geq 3 \\
x, y & \geq 0
\end{align*}
\]

\[
\downarrow
\]

\[
\begin{align*}
\min & \quad -x - y \\
-2x - y + s & = -3
\end{align*}
\]

\[
\downarrow
\]

\[
\begin{align*}
\min & \quad -3 + x - s \\
y & = 3 - 2x + s
\end{align*}
\]
Outline of the Simplex Algorithm

1. **Initialization**: Pick a feasible basis.

2. **Pricing**: If all reduced costs are $\geq 0$, then return **OPTIMAL**. Else pick a non-basic variable with reduced cost $< 0$.

3. **Ratio test**: Compute best value for improving non-basic variable respecting non-negativity constraints of basic variables. If best value is not bounded, then return **UNBOUNDED**. Else select basic variable for exchange with improving non-basic variable.

4. **Update**: Update the tableau and go to 2.
Finding an Initial Basis

- Note that to optimize

\[
\begin{align*}
\min c^T x \\
Ax &= b \\
x &\geq 0
\end{align*}
\]

initially we need a feasible basis at step 1.

Steps 2-4 of previous procedure are called phase II of simplex algorithm.

- Phase I looks for a feasible basis

- We can get a feasible basis with the same procedure by solving another LP for which phase I is trivial

- Let us assume wlog. that \( b \geq 0 \)

- Introduce new artificial variables \( z \) and solve

\[
\begin{align*}
\min 1^T z \\
Ax + z &= b \\
x, z &\geq 0
\end{align*}
\]
Finding an Initial Basis

\[
\min c^T x \quad \text{ min } 1^T z
\]

\[\begin{align*}
[LP] \quad & Ax = b \\
& x \geq 0
\end{align*}\]

\[\begin{align*}
[LP'] \quad & A x + z = b \\
& x, z \geq 0
\end{align*}\]

where \( b \geq 0 \)

- \( LP' \) is feasible, and a trivial feasible basis is \( B = (z) \)
- \( LP' \) cannot be unbounded: \( z \geq 0 \) implies \( 1^T z \geq 0 \)
  
  So \( LP' \) has optimal solution with objective value \( \geq 0 \)

- If \( x^* \) is feasible solution to \( LP \) then \((x, z) = (x^*, 0)\) is optimal solution to \( LP' \) with objective value \( 0 \)

- If \((x, z) = (x^*, z^*)\) is optimal solution to \( LP' \) with objective value \( 0 \) then \( z^* = 0 \) and so \( x^* \) is feasible solution to \( LP \)
Finding an Initial Basis

\[
\begin{align*}
\text{min } c^T x & \quad \text{min } 1^T z \\
[LP] & Ax = b \quad \implies [LP'] & Ax + z = b \quad \text{where } b \geq 0 \\
x \geq 0 & \quad x, z \geq 0
\end{align*}
\]

- \textit{LP} is feasible iff optimum of \textit{LP'} is 0
- Still: how can we get a feasible basis for \textit{LP}?
- Assume that optimum of \textit{LP'} is 0. Then:

1. If all artificial variables are non-basic, 
   then an optimal basis for \textit{LP'} is a feasible basis for \textit{LP}

2. Any basic artificial variable can be made non-basic 
   by Gaussian elimination (since \(A\) has full rank)
Finding an Initial Basis

- Improvement: use slack variables instead of artificial variables in the initial basis whenever possible.
- Alternative phase I approaches do not introduce new variables and work by minimizing the sum of infeasibilities:

\[
\min \left\{ \sum_{\beta_i < 0} \beta_i \mid \mathcal{B} \text{ basis with basic solution } \beta \right\}
\]
Finding an Initial Basis

\[
\begin{align*}
\text{min } & -x - 2y \\
1 & \leq x + y \leq 3 \\
0 & \leq x \leq 2 \\
0 & \leq y \leq 2
\end{align*}
\Rightarrow
\begin{align*}
\text{min } & -x - 2y \\
x + y + s_1 & = 3 \\
x + y - s_2 & = 1 \\
x + s_3 & = 2 \\
y + s_4 & = 2
\end{align*}
\Rightarrow
\begin{align*}
\text{min } & z_1 \\
x + y + s_1 & = 3 \\
x + y - s_2 + z_1 & = 1 \\
x + s_3 & = 2 \\
y + s_4 & = 2
\end{align*}
\]
Finding an Initial Basis

\[
\begin{align*}
\begin{cases}
\min & 1 - x - y + s_2 \\
& s_1 = 3 - x - y \\
& z_1 = 1 - x - y + s_2 \\
& s_3 = 2 - x \\
& s_4 = 2 - y
\end{cases}
\Rightarrow
\begin{cases}
\min & z_1 \\
& s_1 = 2 + z_1 - s_2 \\
& x = 1 - z_1 - y + s_2 \\
& s_3 = 1 + z_1 + y - s_2 \\
& s_4 = 2 - y
\end{cases}
\end{align*}
\]

Feasible tableau

\[
\begin{align*}
\begin{cases}
& s_1 = 2 - s_2 \\
& x = 1 - y + s_2 \\
& s_3 = 1 + y - s_2 \\
& s_4 = 2 - y
\end{cases}
\end{align*}
\]
Finding an Initial Basis

\[
\begin{align*}
    x & \leq 2 \\
    y & \leq 2 \\
    x + y & \geq 1 \\
    x + y & \leq 3 \\
    x & \leq 2 \\
    y & \geq 0 \\
    x & \geq 0
\end{align*}
\]

(0, 0) \quad (1, 0)
Big $M$ Method

- Alternative to phase I + phase II approach
- LP is changed as follows, where $M$ is a “big number”

$$
\begin{align*}
\min c^T x & \quad \min c^T x + M \cdot 1^T z \\
Ax = b & \quad \Rightarrow Ax + z = b \quad \text{where } b \geq 0 \\
x \geq 0 & \quad x, z \geq 0
\end{align*}
$$

- Again by taking the artificial variables we get an initial feasible basis
- The search of a feasible basis for the original problem is not blind wrt. cost
- Problems:
  - If $M$ is a fixed big number, then the algorithm becomes numerically unstable
  - If $M$ is kept symbolically, then handling costs becomes more expensive
Big \( M \) Method

\[
\begin{align*}
\begin{aligned}
\text{min } & -x - 2y \\
1 \leq & x + y \leq 3 \\
0 \leq & x \leq 2 \\
0 \leq & y \leq 2 \\
\end{aligned}
\Rightarrow
\begin{aligned}
\text{min } & -x - 2y \\
x + y + s_1 = & 3 \\
x + y - s_2 = & 1 \\
x + s_3 = & 2 \\
y + s_4 = & 2 \\
\end{aligned}
\Rightarrow
\begin{aligned}
\text{min } & -x - 2y + Mz \\
x + y + s_1 = & 3 \\
x + y - s_2 + z = & 1 \\
x + s_3 = & 2 \\
y + s_4 = & 2 \\
\end{aligned}
\end{align*}
\]
Big $M$ Method

\[
\begin{align*}
\min M + (-1 - M)x + (-2 - M)y + Ms_2 \\
 s_1 &= 3 - x - y \\
 s_2 &= z = 1 - x - y + s_2 \\
 s_3 &= 2 - x \\
 s_4 &= 2 - y \\
\end{align*}
\]

$\implies$

\[
\begin{align*}
\min x - 2 - 2s_2 + (M + 2)z \\
 s_1 &= 2 + z - s_2 \\
 s_2 &= y = 1 - x - z + s_2 \\
 s_3 &= 2 - x \\
 s_4 &= 1 + z + x - s_2 \\
\end{align*}
\]

Then we could drop the artificial variable $z$ and continue the optimization.
Termination and Complexity

- At each step of the simplex algorithm:
  \[ \text{cost improvement} = \text{reduced cost} \cdot \text{increment} \] (of chosen non-basic var)

- A step of the simplex algorithm is **degenerate** if the increment of the chosen non-basic variable is 0.

- If the step is degenerate then there is no cost improvement.

- Degenerate steps can only happen with degenerate bases.

- Assume **no degenerate bases** occur.

  Then there is a **strict improvement** from a base to the next one.

  So **simplex terminates**, as bases cannot be repeated.

  No. steps is at most **exponential**: there are \( \leq \binom{n}{m} \) bases.

  Tight bound for pathological cases (Klee-Minty cube).

  In practice the cost is polynomial.
Termination and Complexity

- When there is degeneracy, simplex may loop forever.
- Termination guaranteed with anticycling rules, e.g., Bland’s rule:

  Assume there is a fixed ordering of variables.

  **Pricing:** among non-basic vars with reduced cost $< 0$, take the least one.
  **Ratio test:** among tight basic vars, take the least one.
Termination with Bland’s Rule

PROOF:
States of simplex algorithm determined by bases.
To prove termination, enough to prove we can’t repeat bases
Let us prove termination by contradiction.
Assume there is a cycle: $B_k, ..., B_t, B_{t+1}$ such that $B_k = B_{t+1}$
Var $x_j$ is fickle if it is in some, but not all, bases of the cycle
For all ratio tests in cycle, entering variable takes value 0
Hence pivoting steps do not change basic solution:
basic solution is the same for all bases of the cycle
So fickle variables have value 0 in basic solution
Termination with Bland’s Rule

Let $x_r$ be the largest fickle variable

Let $l \in \{k, ..., t\}$ be such that $x_r \in B_l$ and $x_r \in R_{l+1}$

Let $x_r = \sum_{x_j \in R_l} \lambda_j x_j$ be the respective row in $B_l$'s tableau

Let $x_s \in R_l$ be the non-basic variable that is swapped with $x_r$ in $B_l$

Let $d_l(x_j)$ be the reduced cost of a variable $x_j$ in $B_l$

Since $x_s$ is entering the basis, $d_l(x_s) < 0$ and $\lambda_s < 0$

Moreover, $x_s$ is fickle too, and hence $x_s \prec x_r$
Termination with Bland’s Rule

Let $\mathcal{B}_p$ be the first basis after $\mathcal{B}_{l+1}$ where $x_r$ gets basic again:
$x_r \in \mathcal{R}_p$ and $x_r \in \mathcal{B}_{p+1}$

Let $d_p(x_j)$ be the reduced cost of a variable $x_j$ in $\mathcal{B}_p$

Since $x_r$ is entering the basis, $d_p(x_r) < 0$

Moreover $d_p(x_s) \geq 0$:

- If $x_s \in \mathcal{R}_p$: by Bland’s rule and $x_s \prec x_r$
- If $x_s \in \mathcal{B}_p$: reduced costs of basic vars are null
Termination with Bland’s Rule

Let \( \gamma_l \) be the value of the objective function at the basic solution of \( B_l \)
Then for any \( x \) such that \( Ax = b \):  \( c^T x = \gamma_l + \sum d_l(x_j)x_j \)

Let \( \gamma_p \) be the value of the objective function at the basic solution of \( B_p \)
Then for any \( x \) such that \( Ax = b \):  \( c^T x = \gamma_p + \sum d_p(x_j)x_j \)

As basic solution is the same all the time:  \( \gamma_l = \gamma_p \)
Hence for any \( x \) such that \( Ax = b \):  \( \sum d_l(x_j)x_j = \sum d_p(x_j)x_j \)

If \( x_s = t \) and \( x_j = 0 \) for all \( x_j \in R_l, j \neq s \) then \( x_{B_l} = B_l^{-1}b - B_l^{-1}a_st. \) So:

\[
\sum_{x_j \in B_l} d_l(x_j)x_j + \sum_{x_j \in R_l} d_l(x_j)x_j = \sum_{x_j \in B_l} d_p(x_j)x_j + \sum_{x_j \in R_l} d_p(x_j)x_j
\]

\[0 + d_l(x_s)t = \sum_{x_{k_i} \in B_l} d_p(x_{k_i})(\beta_i - \alpha_i^t) + d_p(x_s)t\]

where \( \beta = B_l^{-1}b \) and \( \alpha_s = B_l^{-1}a_s \)
Termination with Bland’s Rule

Hence \( d_l(x_s) = - \sum_{x_{ki} \in B_l} d_p(x_{ki}) \alpha_s^i + d_p(x_s) \)

As \( d_l(x_s) < 0 \) and \( d_p(x_s) \geq 0 \), it must be \( \sum_{x_{ki} \in B_l} d_p(x_{ki}) \alpha_s^i > 0 \)

There must exist \( x_{ki} \in B_l \) such that \( d_p(x_{ki}) \alpha_s^i > 0 \)

So \( d_p(x_{ki}) \neq 0 \) and \( x_{ki} \not\in B_p \). As \( x_{ki} \in B_l \), \( x_{ki} \) is fickle. Now:

- \( x_{ki} = x_r \): \( d_p(x_r) < 0 \) and \( \alpha_s^i > 0 \) implies \( d_p(x_{ki}) \alpha_s^i < 0 \) !!!
- \( x_{ki} < x_r \): as we didn’t chose \( x_{ki} \) to enter \( B_p \), \( d_p(x_{ki}) \geq 0 \)

Since \( d_p(x_{ki}) \alpha_s^i > 0 \), we have \( d_p(x_{ki}) > 0 \) and \( \alpha_s^i > 0 \)

But \( x_{ki} \) is fickle, so its basic value at \( B_l \) is 0

By the ratio rule, \( x_{ki} \) has ratio 0, so it could leave \( B_l \)

Contradiction! \( x_{ki} < x_r \) and \( x_r \) was chosen to leave \( B_l \)
Pricing Strategies

1. **Full pricing**
   Choose the variable with the most negative reduced cost

2. **Partial pricing**
   Make a list with the indices of the $P$ variables with the most negative reduced costs.
   In following iterations choose variables from the list until reduced costs are all $\geq 0$
3. **Best-improvement pricing**

Let $\theta_k$ be the increment for a non-basic variable $x_k$ with reduced cost $d_k < 0$. Choose the variable $j$ such that

$$|d_j| \cdot \theta_j = \max \{|d_k| \cdot \theta_k \text{ such that } d_k < 0, k \in \mathcal{R}|$$

4. **Normalized pricing.**

Let $n_k = ||\alpha_k||$ (in practice $n_k = \sqrt{1 + ||\alpha_k||^2}$) where $\alpha_k$ is the column in the tableau of variable $x_k$.

Take criteria 1. or 2. but using $\frac{d_k}{n_k}$ instead of $d_k$

5. **Other more sophisticate normalized pricing strategies:**

steepest edge, devex
Bounded Variables

LP solvers implement a variant of the simplex algorithm that handles bounds more efficiently for LP’s of the form

$$\min c^T x$$
$$Ax = b$$
$$\ell \leq x \leq u$$

- $\ell_i$ may be $-\infty$ and/or $u_i$ may be $+\infty$
- Bounds are incorporated into pricing and ratio test
- Now non-basic variables will take values at the lower or the upper bound
Bounded Variables

\[
\begin{align*}
\text{min } & -x - 2y \\
& x + y \leq 3 \\
& 0 \leq x \leq 2 \\
& 0 \leq y \leq 2
\end{align*}
\]

\[
\begin{align*}
\text{min } & -x - 2y \\
& x + y + s = 3 \\
& 0 \leq x \leq 2 \\
& 0 \leq y \leq 2 \\
& s \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{min } & -x - 2y \\
& s = 3 - x - y \\
& 0 \leq x \leq 2 \\
& 0 \leq y \leq 2 \\
& s \geq 0
\end{align*}
\]

- Initially non-basic variables \(x, y\) are at lower bound
- We choose variable \(x\) in pricing
Bounded Variables

\[
\begin{aligned}
\min \ -x - 2y \\
\text{s.t. } \ s = 3 - x - y \\
0 \leq x \leq 2 \\
0 \leq y \leq 2 \\
s \geq 0
\end{aligned}
\]

- Limits new value to \(\leq 3\) as \(s \geq 0\)
- Limits new value to \(\leq 2\) as \(x \leq 2\)

Best possible new value for \(x\) is \(\min(3, 2) = 2\)

Bound flip: \(x\) is still non-basic, but is now at upper bound

\[
\begin{aligned}
\min \ -x - 2y \\
\text{s.t. } \ s = 3 - x - y \\
0 \leq x \leq 2 \\
0 \leq y \leq 2 \\
s \geq 0
\end{aligned}
\]
Bounded Variables

- Pricing considers the bound status of non-basic variables
- A non-basic variable \( x_j \) with reduced cost \( d_j \) can improve the cost function
  - if \( x_j \) is at lower bound and \( d_j < 0 \); or
  - if \( x_j \) is at upper bound and \( d_j > 0 \)
- Choose \( y \) in pricing:

\[
\begin{align*}
\min -x - 2y \\
s = 3 - x - y \\
0 \leq x \leq 2 \\
0 \leq y \leq 2 \\
s \geq 0
\end{align*}
\]

- Limits new value to \( \leq 1 \) as \( s \geq 0 \)
- Limits new value to \( \leq 2 \) as \( y \leq 2 \)

- Best possible new value for \( y \) is \( \min(1, 2) = 1 \)
Bounded Variables

Usual pivoting step now:

\[ s = 3 - x - y \implies y = 3 - x - s \]

\[
\begin{align*}
\min & -x - 2y \\
\text{s.t.} & s = 3 - x - y \\
& 0 \leq x \leq 2 \\
& 0 \leq y \leq 2 \\
& s \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\min & -6 + x + 2s \\
\text{s.t.} & y = 3 - x - s \\
& 0 \leq x \leq 2 \\
& 0 \leq y \leq 2 \\
& s \geq 0 \\
\end{align*}
\]
Choose $x$ in pricing. To respect bounds for $y$:

\[
0 \leq y(x) \leq 2 \\
0 \leq 3 - x \leq 2
\]

(since $x$ decreases its value, $0 \leq y(x)$ is OK)

\[
3 - x \leq 2 \\
1 \leq x
\]

\[
\begin{align*}
\min -6 + x + 2s \\
y &= 3 - x - s & \text{Limits new value to } \geq 1 \\
0 \leq x \leq 2 & \text{Limits new value to } \geq 0 \\
0 \leq y \leq 2 \\
s \geq 0
\end{align*}
\]

Best possible new value for $x$ is $\max(1, 0) = 1$
Bounded Variables

Usual pivoting step now:

\[ y = 3 - x - s \quad \Rightarrow \quad x = 3 - y - s \]
Bounded Variables

- Usual pivoting step now:

\[ y = 3 - x - s \quad \Rightarrow \quad x = 3 - y - s \]

\[
\begin{aligned}
\min & -6 + x + 2s \\
y &= 3 - x - s \\
0 &\leq x \leq 2 \\
0 &\leq y \leq 2 \\
s &\geq 0 \\
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\min & -3 + s - y \\
x &= 3 - y - s \\
0 &\leq x \leq 2 \\
0 &\leq y \leq 2 \\
s &\geq 0 \\
\end{aligned}
\]

- Since upper bound of \( y \) was tight, now \( y \) is set to its upper bound

- Optimal solution: \((x, y, s) = (1, 2, 0)\) with cost \(-5\)

- Now reading the basic solution and its cost is more involved!
Bounded Variables

\[ \min -x - 2y \]

\[ x + y \leq 3 \]

\[ x \leq 2 \]

\[ y \geq 0 \]

\[ x \geq 0 \]

\[ y \leq 2 \]