Mixed Integer Linear Programming

Combinatorial Problem Solving (CPS)

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Mixed Integer Linear Programs

- A mixed integer linear program (MILP, MIP) is of the form

$$\begin{aligned} &\min\ c^T x \\ &Ax = b \\ &x \geq 0 \\ &x_i \in \mathbb{Z} \ \ \forall i \in \mathcal{I} \end{aligned}$$

- If all variables need to be integer,
  it is called a (pure) integer linear program (ILP, IP)

- If all variables need to be 0 or 1 (binary, boolean),
  it is called a $0-1$ linear program

Complexity: LP vs. IP

- Including integer variables increases enourmously the modeling power, at the expense of more complexity

- LP's can be solved in polynomial time with interior-point methods (ellipsoid method, Karmarkar's algorithm)

- Integer Programming is an NP-hard problem. So:
    - There is no known polynomial-time algorithm
    - There are little chances that one will ever be found
    - Even small problems may be hard to solve

- What follows is one of the many approaches (and one of the most successful) for attacking IP's

LP Relaxation of a MIP

- Given a MIP

$$
(IP) \qquad \begin{aligned} &\min\ c^T x \\ &Ax = b \\ &x \geq 0 \\ &x_i \in \mathbb{Z} \ \ \forall i \in \mathcal{I} \end{aligned}
$$

its linear relaxation is the LP obtained by dropping integrality constraints:

$$
(LP) \qquad \begin{aligned} &\min\ c^T x \\ &Ax = b \\ &x \geq 0 \end{aligned}
$$

- Can we solve IP by solving LP? By rounding?

Branch & Bound

- The optimal solution of

$$
\begin{array}{rcl}
\max \ x + y & & \\
-2x + 2y & \geq & 1 \\
-8x + 10y & \leq & 13 \\
x, y \geq 0 & & \\
x, y \in \mathbb{Z} & &
\end{array}
$$

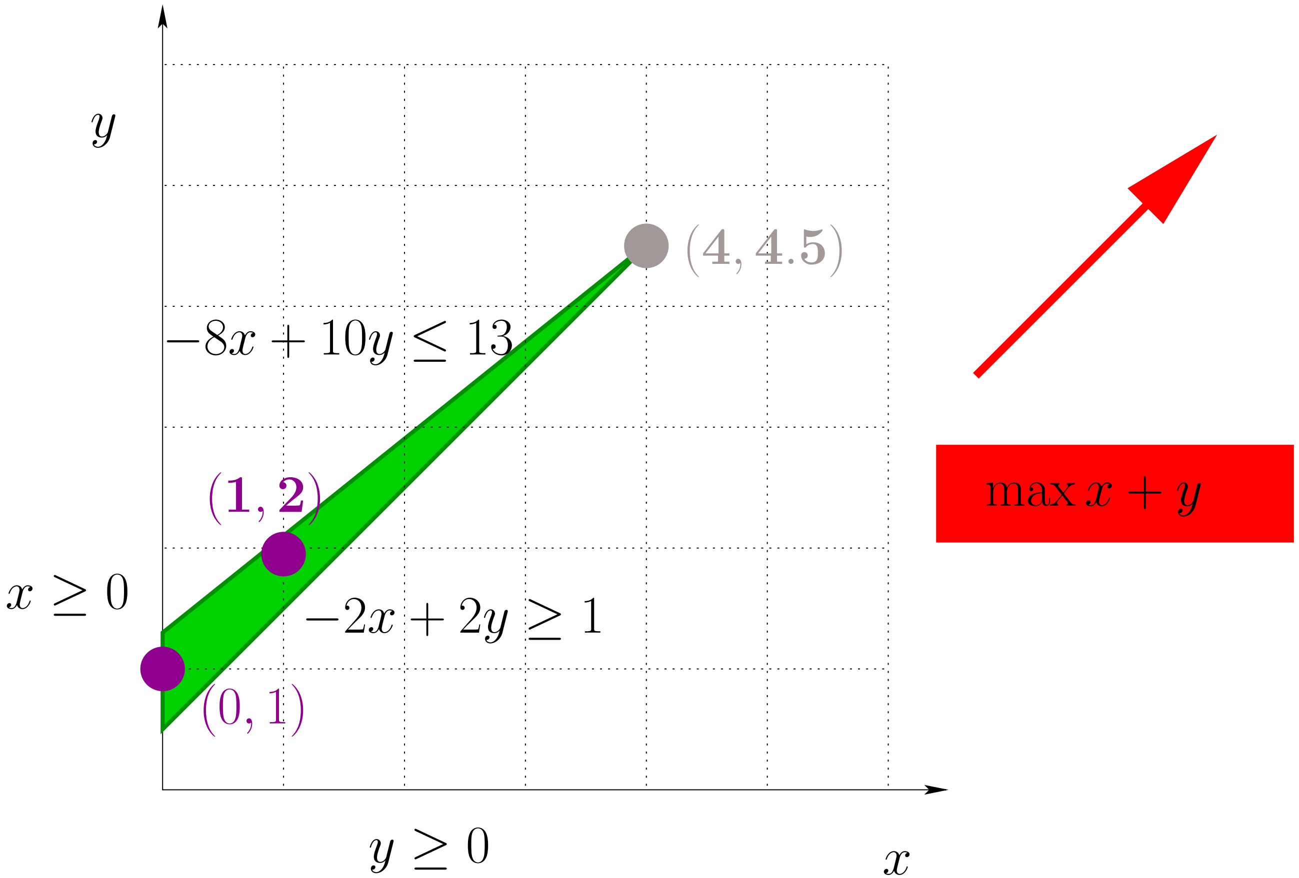
  is $(x, y) = (1, 2)$, with objective 3

- The optimal solution of its LP relaxation is $(x, y) = (4, 4.5)$, with objective 9.5

- No direct way of getting from $(4, 4.5)$ to $(1, 2)$ by rounding!

- Something more elaborate is needed: branch & bound

Branch & Bound



Branch & Bound

- Assume variables are bounded, i.e., have lower and upper bounds
- Let P_0 be the initial problem, $\mathrm{LP}(P_0)$ be the LP relaxation of P_0
- If in optimal solution of $\mathrm{LP}(P_0)$ all integer variables take integer values then it is also an optimal solution to P_0
- Else
  - Let x_j be integer variable
    whose value β_j at optimal solution of $\mathrm{LP}(P_0)$ is such that $\beta_j \notin \mathbb{Z}$.

    Define

    $$\begin{array}{rcl} P_1 & := & P_0 \wedge x_j \leq \lfloor \beta_j \rfloor \\ P_2 & := & P_0 \wedge x_j \geq \lceil \beta_j \rceil \end{array}$$

  - $\mathrm{feasibleSols}(P_0) \;=\; \mathrm{feasibleSols}(P_1) \cup \mathrm{feasibleSols}(P_2)$
  - Idea: solve P_1, solve P_2 and then take the best

Branch & Bound

- Let x_j be integer variable
  whose value β_j at optimal solution of $\mathrm{LP}(P_0)$ is such that $\beta_j \notin \mathbb{Z}$.

  Each of the problems

  $$P_1 := P_0 \wedge x_j \leq \lfloor \beta_j \rfloor \qquad P_2 := P_0 \wedge x_j \geq \lceil \beta_j \rceil$$

  can be solved recursively

- We can build a binary tree of subproblems
  whose leaves correspond to pending problems still to be solved

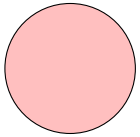
- This procedure terminates as integer vars have finite bounds and,
  at each split, the domain of x_j becomes strictly smaller

- If $\mathrm{LP}(P_i)$ has optimal solution where integer variables take integer values
  then solution is stored

- If $\mathrm{LP}(P_i)$ is infeasible then P_i can be discarded (pruned, fathomed)

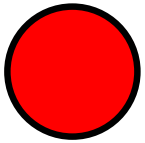
Example

$$\begin{aligned}
&\min\ -x - y\\
&-2x + 2y \geq 1\\
&-8x + 10y \leq 13\\
&x, y \geq 0\\
&x, y \in \mathbb{Z}
\end{aligned}$$



Example

$$\min \ -x - y$$
$$-2x + 2y \geq 1$$
$$-8x + 10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$

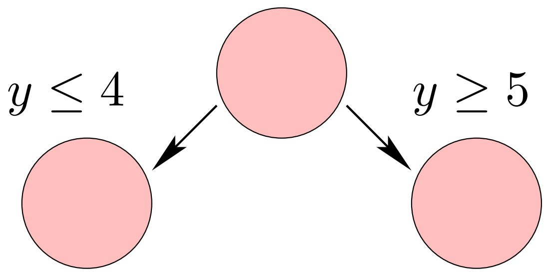


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
End
====================================================================

CPLEX> optimize
Primal simplex - Optimal:  Objective = - 8.5000000000e+00
Solution time =    0.00 sec.  Iterations = 0 (0)
Deterministic time = 0.00 ticks  (0.37 ticks/sec)

CPLEX> display solution variables x
Variable Name           Solution Value
x                             4.000000

CPLEX> display solution variables y
Variable Name           Solution Value
y                             4.500000
```

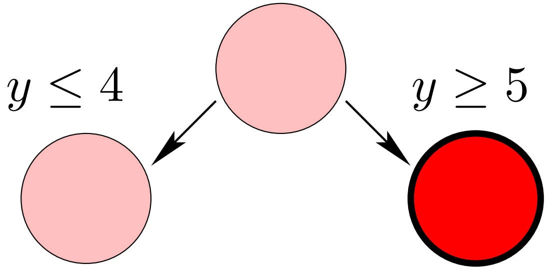
Example

$$\min \ -x - y$$
$$-2x + 2y \geq 1$$
$$-8x + 10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$



Example

$$\min\ -x - y$$
$$-2x + 2y \geq 1$$
$$-8x + 10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$

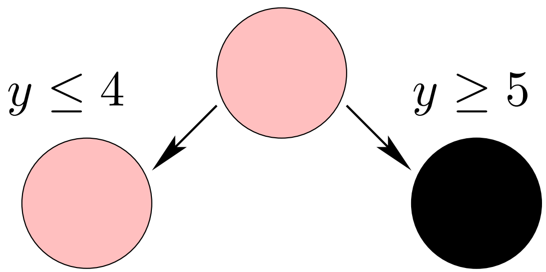


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
y >= 5
End
====================================================================

CPLEX> optimize
Bound infeasibility column 'x'.
Presolve time = 0.00 sec. (0.00 ticks)
Presolve - Infeasible.
Solution time =    0.00 sec.
Deterministic time = 0.00 ticks  (1.67 ticks/sec)
```

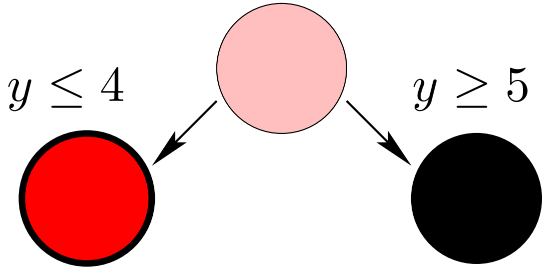
Example

$$\begin{aligned}&\min\ -x-y\\&-2x+2y\geq 1\\&-8x+10y\leq 13\\&x,y\geq 0\\&x,y\in\mathbb{Z}\end{aligned}$$



Example

$$\min\ -x - y$$
$$-2x + 2y \geq 1$$
$$-8x + 10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$

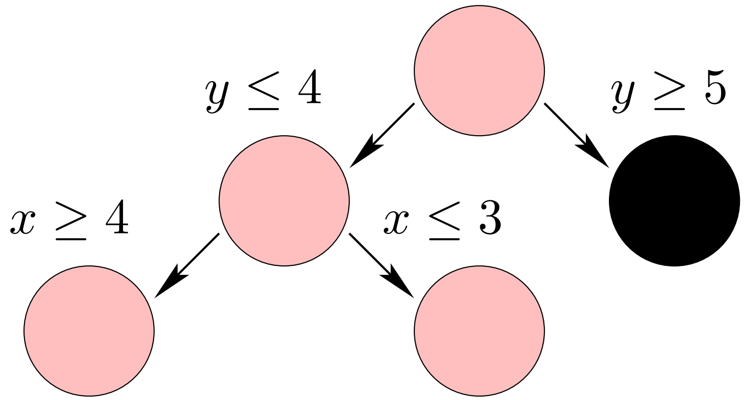


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
y <= 4
End
=====================================================================

CPLEX> optimize
Dual simplex - Optimal:  Objective = - 7.5000000000e+00
Solution time =    0.00 sec.  Iterations = 0 (0)
Deterministic time = 0.00 ticks  (2.68 ticks/sec)

CPLEX> display solution variables x
Variable Name           Solution Value
x                             3.500000

CPLEX> display solution variables y
Variable Name           Solution Value
y                             4.000000
```

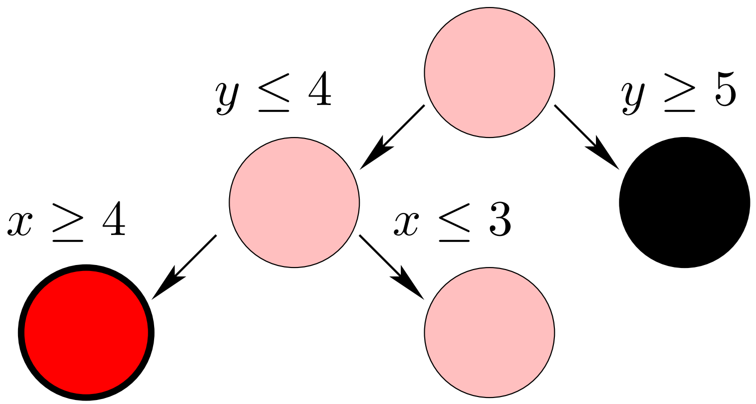
Example

$$\begin{aligned}
&\min\ -x-y\\
&-2x+2y \geq 1\\
&-8x+10y \leq 13\\
&x, y \geq 0\\
&x, y \in \mathbb{Z}
\end{aligned}$$



Example

$$\min \; -x - y$$
$$-2x + 2y \geq 1$$
$$-8x + 10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$



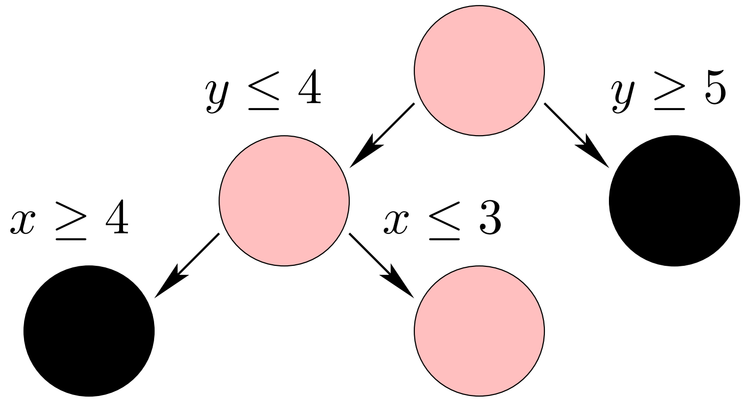
Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
x >= 4
y <= 4
End
=====================================================================
```

```
CPLEX> optimize
Row 'c1' infeasible, all entries at implied bounds.
Presolve time = 0.00 sec. (0.00 ticks)
Presolve - Infeasible.
Solution time =    0.00 sec.
Deterministic time = 0.00 ticks  (1.11 ticks/sec)
```

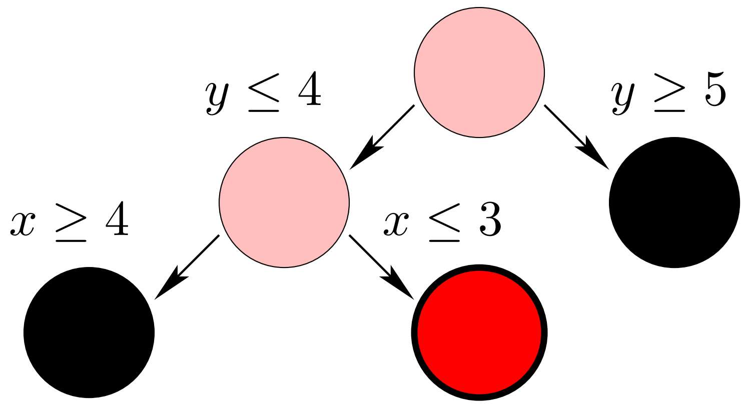
Example

$$\begin{aligned}
&\min\ -x-y\\
&-2x+2y\geq 1\\
&-8x+10y\leq 13\\
&x,y\geq 0\\
&x,y\in\mathbb{Z}
\end{aligned}$$



Example

$$
\begin{aligned}
&\min\ -x-y\\
&-2x+2y\geq 1\\
&-8x+10y\leq 13\\
&x,y\geq 0\\
&x,y\in\mathbb{Z}
\end{aligned}
$$

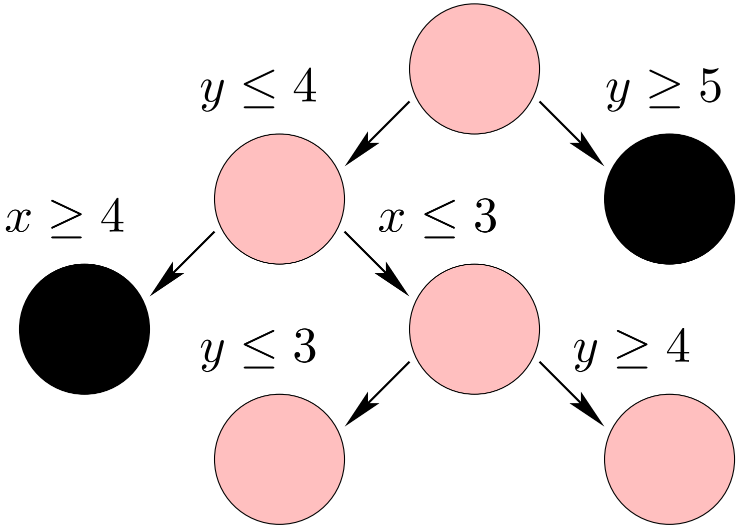


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
x <= 3
y <= 4
End
====================================================================

CPLEX> optimize
Dual simplex - Optimal:  Objective = - 6.7000000000e+00
Solution time =    0.00 sec.  Iterations = 0 (0)
Deterministic time = 0.00 ticks  (2.71 ticks/sec)

CPLEX> display solution variables x
Variable Name           Solution Value
x                             3.000000

CPLEX> display solution variables y
Variable Name           Solution Value
y                             3.700000
```

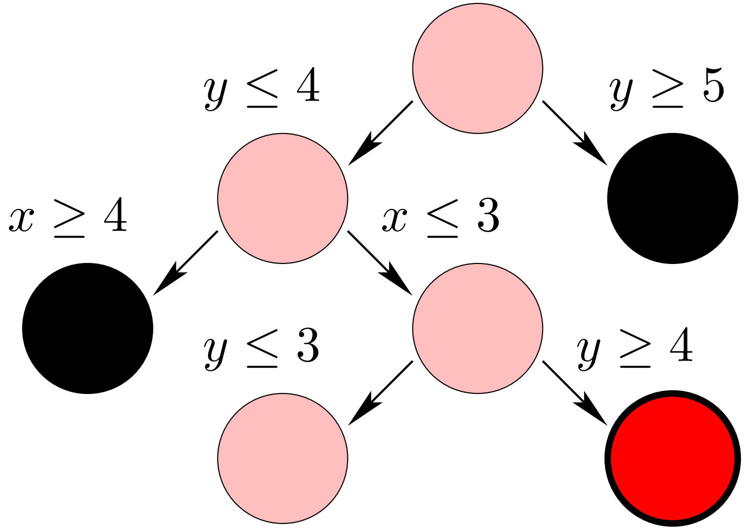
Example

$$\min\ -x-y$$
$$-2x+2y \geq 1$$
$$-8x+10y \leq 13$$
$$x,y \geq 0$$
$$x,y \in \mathbb{Z}$$



Example

$$\min \ -x - y$$
$$-2x + 2y \geq 1$$
$$-8x + 10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$

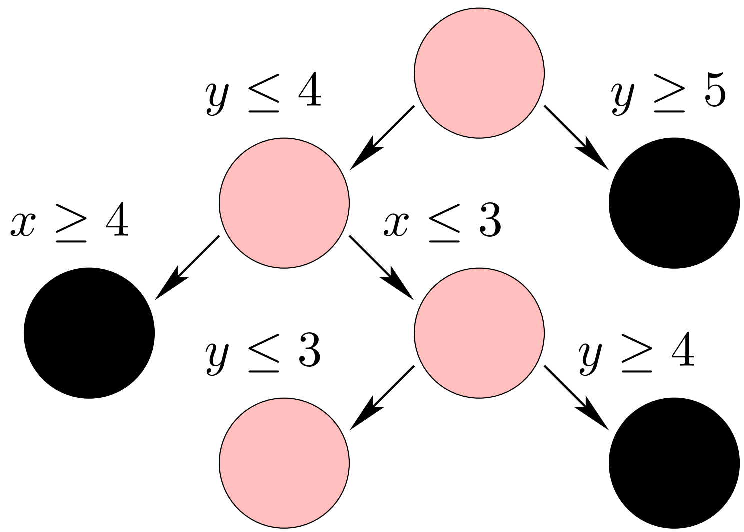


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
x <= 3
y = 4
End
=====================================================================

CPLEX> optimize
Bound infeasibility column 'x'.
Presolve time = 0.00 sec. (0.00 ticks)
Presolve - Infeasible.
Solution time =    0.00 sec.
Deterministic time = 0.00 ticks  (1.12 ticks/sec)
```

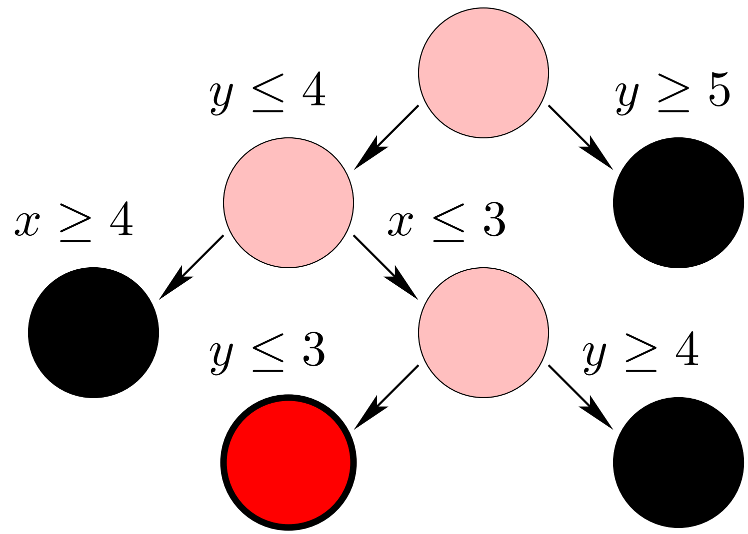
Example

$$\begin{aligned}
&\min\ -x - y \\
&-2x + 2y \geq 1 \\
&-8x + 10y \leq 13 \\
&x, y \geq 0 \\
&x, y \in \mathbb{Z}
\end{aligned}$$



Example

$$\begin{aligned}
&\min\ -x-y\\
&-2x+2y\geq 1\\
&-8x+10y\leq 13\\
&x,y\geq 0\\
&x,y\in\mathbb{Z}
\end{aligned}$$

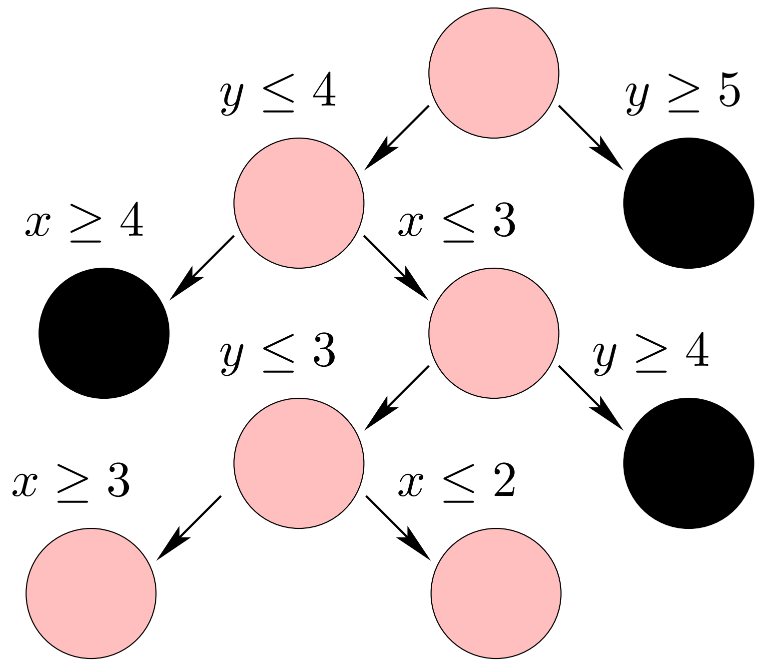


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
x <= 3
y <= 3
End
=====================================================================

CPLEX> optimize
Dual simplex - Optimal:  Objective = - 5.5000000000e+00
Solution time =    0.00 sec.  Iterations = 0 (0)
Deterministic time = 0.00 ticks  (2.71 ticks/sec)

CPLEX> display solution variables x
Variable Name           Solution Value
x                             2.500000

CPLEX> display solution variables y
Variable Name           Solution Value
y                             3.000000
```

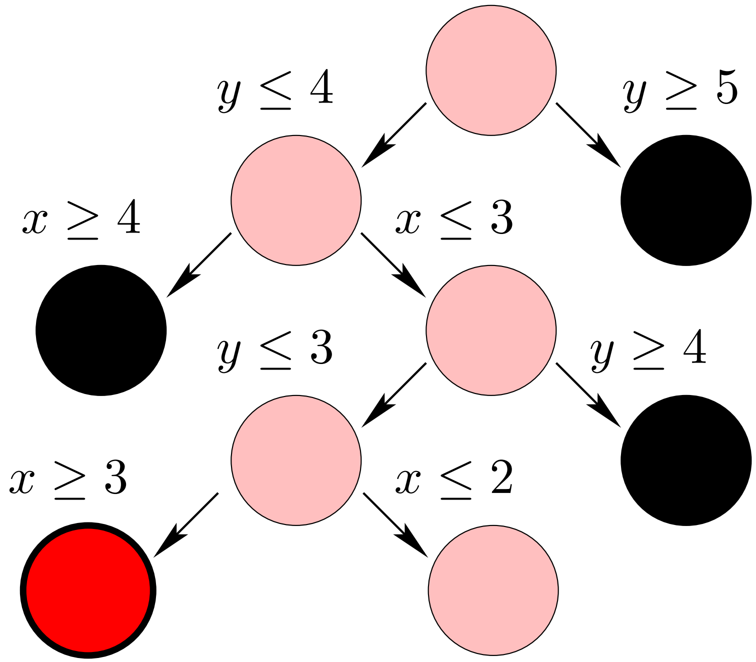
Example

$$\begin{aligned}
&\min\ -x - y \\
&-2x + 2y \geq 1 \\
&-8x + 10y \leq 13 \\
&x, y \geq 0 \\
&x, y \in \mathbb{Z}
\end{aligned}$$



Example

$$
\begin{aligned}
&\min\ -x-y\\
&-2x+2y\geq 1\\
&-8x+10y\leq 13\\
&x,y\geq 0\\
&x,y\in\mathbb{Z}
\end{aligned}
$$

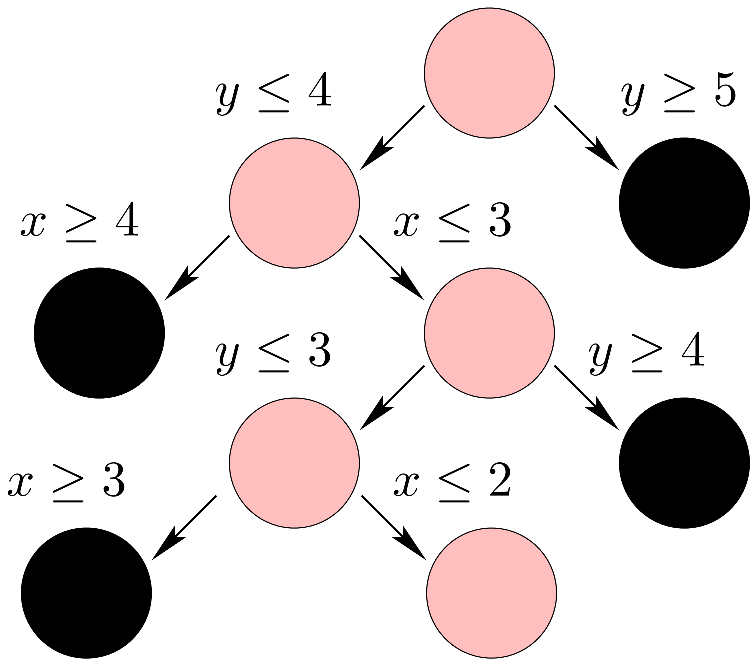


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
x = 3
y <= 3
End
=====================================================================

CPLEX> optimize
Bound infeasibility column 'y'.
Presolve time = 0.00 sec. (0.00 ticks)
Presolve - Infeasible.
Solution time =    0.00 sec.
Deterministic time = 0.00 ticks  (1.11 ticks/sec)
```

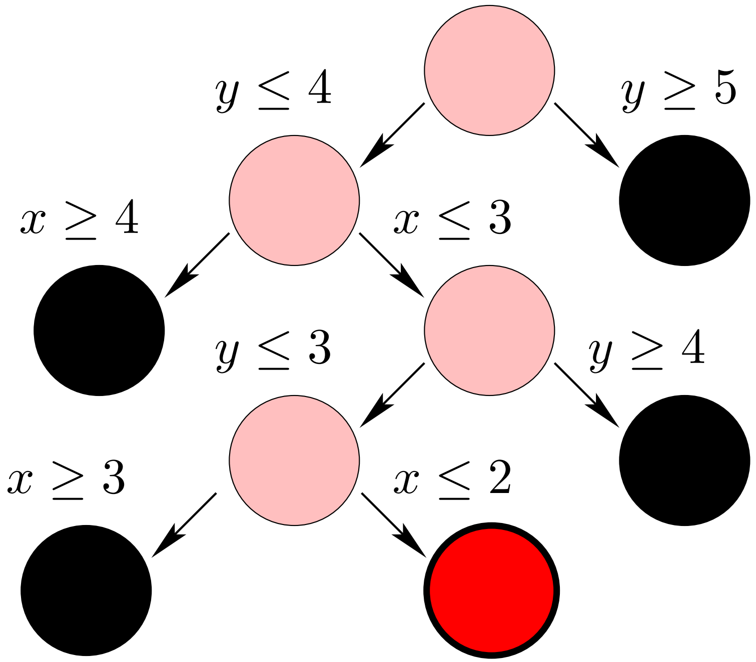
Example

$$\begin{aligned}&\min\ -x-y\\&-2x+2y\geq 1\\&-8x+10y\leq 13\\&x,y\geq 0\\&x,y\in\mathbb{Z}\end{aligned}$$



Example

$$
\begin{aligned}
&\min\ -x-y\\
&-2x+2y\geq 1\\
&-8x+10y\leq 13\\
&x,y\geq 0\\
&x,y\in\mathbb{Z}
\end{aligned}
$$

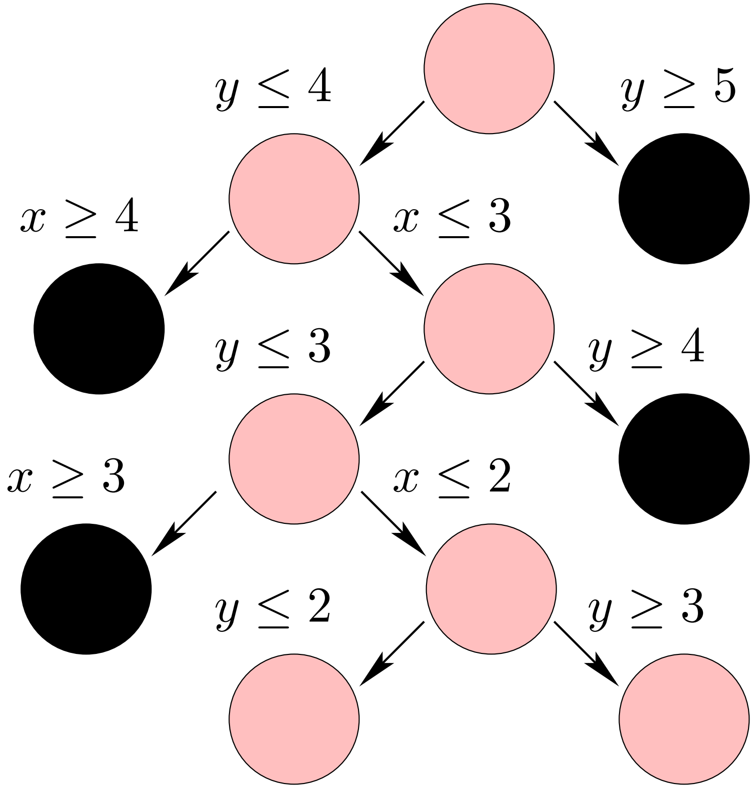


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
x <= 2
y <= 3
End
=====================================================================

CPLEX> optimize
Dual simplex - Optimal:  Objective = - 4.9000000000e+00
Solution time =    0.00 sec.  Iterations = 0 (0)
Deterministic time = 0.00 ticks  (2.71 ticks/sec)

CPLEX> display solution variables x
Variable Name           Solution Value
x                             2.000000
CPLEX> display solution variables y
Variable Name           Solution Value
y                             2.900000
```

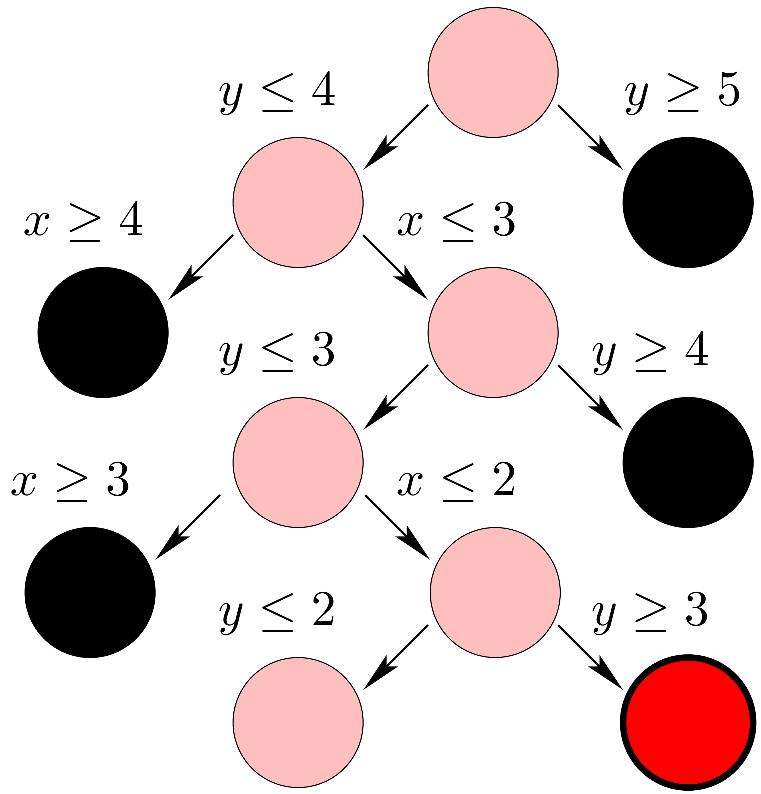
Example

$$\min\ -x - y$$
$$-2x + 2y \geq 1$$
$$-8x + 10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$



Example

$$\min\ -x-y$$
$$-2x+2y\geq 1$$
$$-8x+10y\leq 13$$
$$x,y\geq 0$$
$$x,y\in\mathbb{Z}$$

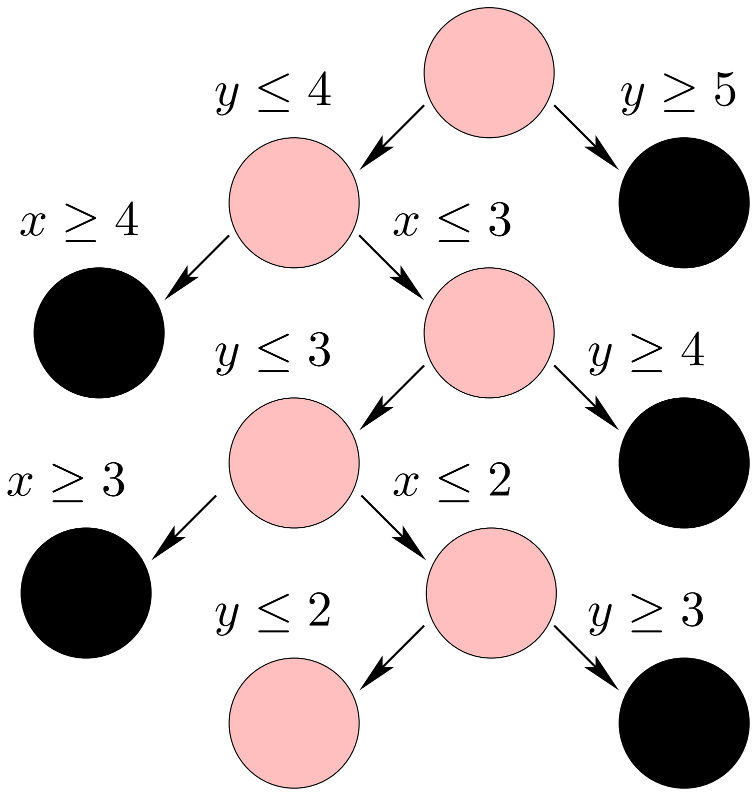


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
x <= 2
y = 3
End
=====================================================================

CPLEX> optimize
Bound infeasibility column 'x'.
Presolve time = 0.00 sec. (0.00 ticks)
Presolve - Infeasible.
Solution time =    0.00 sec.
Deterministic time = 0.00 ticks  (1.12 ticks/sec)
```

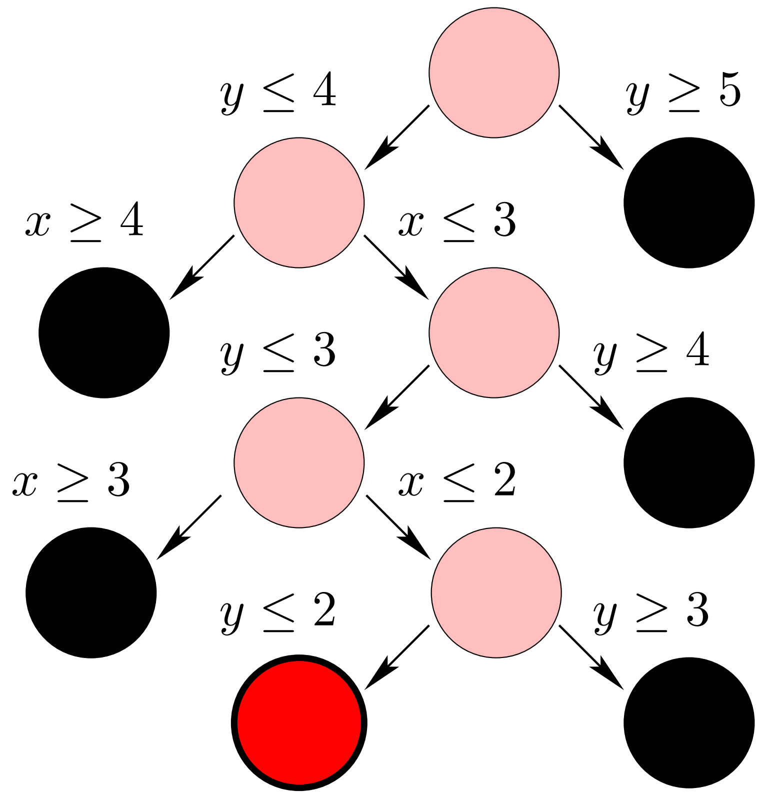
Example

$$\begin{aligned}
&\min\ -x-y\\
&-2x+2y\geq 1\\
&-8x+10y\leq 13\\
&x,y\geq 0\\
&x,y\in\mathbb{Z}
\end{aligned}$$



Example

$$\min\ -x-y$$
$$-2x+2y \geq 1$$
$$-8x+10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$

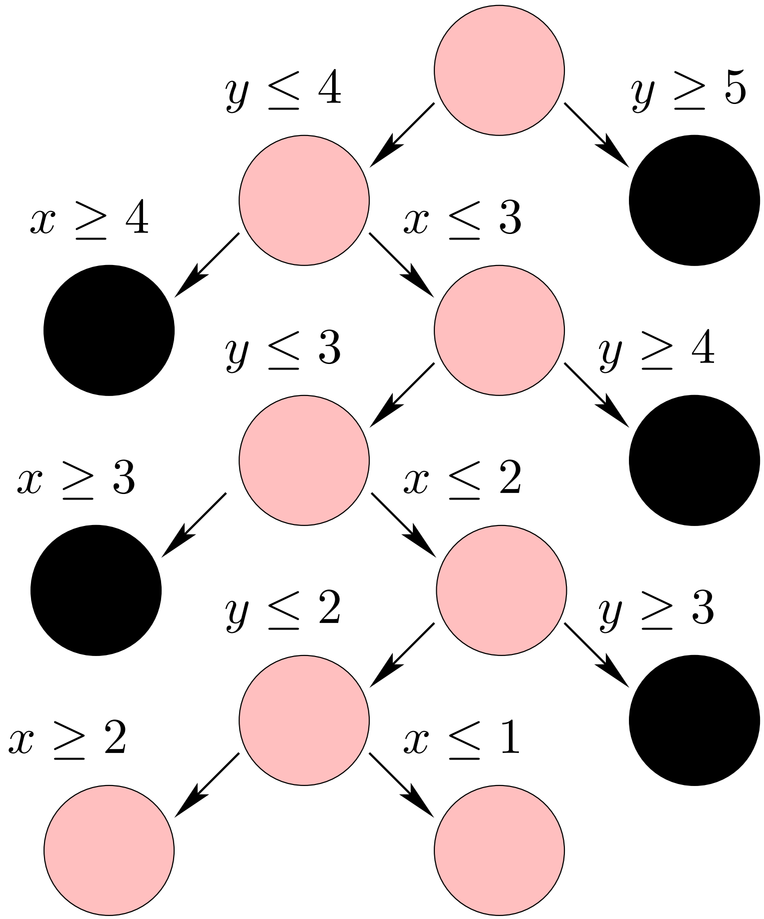


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
x <= 2
y <= 2
End
=====================================================================

CPLEX> optimize
Dual simplex - Optimal:  Objective = - 3.5000000000e+00
Solution time =    0.00 sec.  Iterations = 0 (0)
Deterministic time = 0.00 ticks  (2.71 ticks/sec)

CPLEX> display solution variables x
Variable Name           Solution Value
x                             1.500000

CPLEX> display solution variables y
Variable Name           Solution Value
y                             2.000000
```

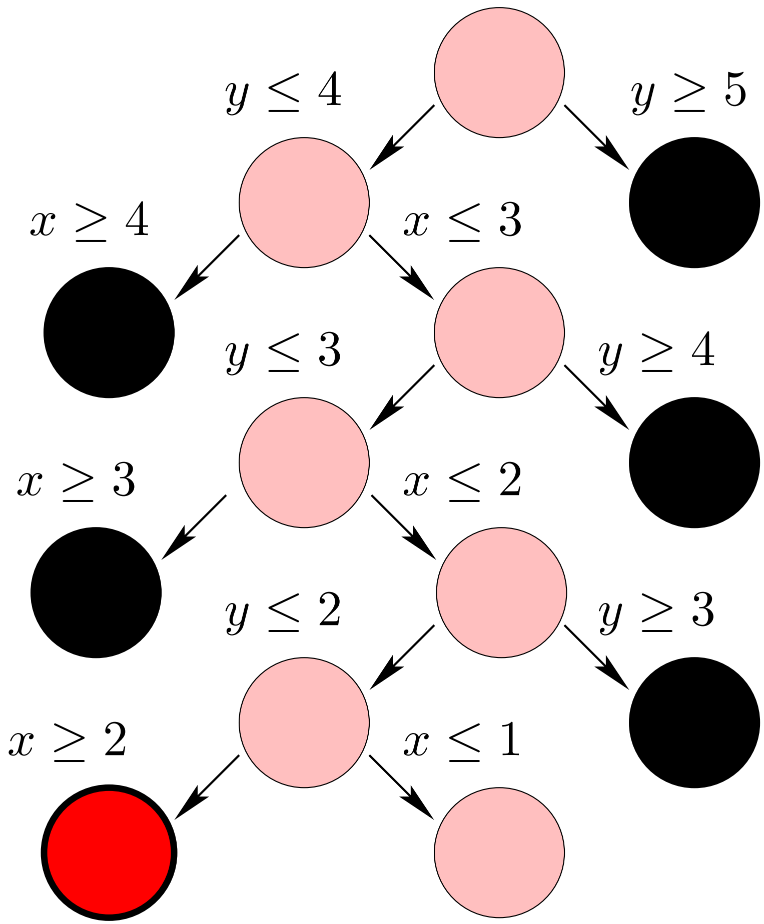
Example

$$\min\ -x-y$$
$$-2x+2y \geq 1$$
$$-8x+10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$



Example

$$\min\ -x-y$$
$$-2x+2y \geq 1$$
$$-8x+10y \leq 13$$
$$x,y \geq 0$$
$$x,y \in \mathbb{Z}$$

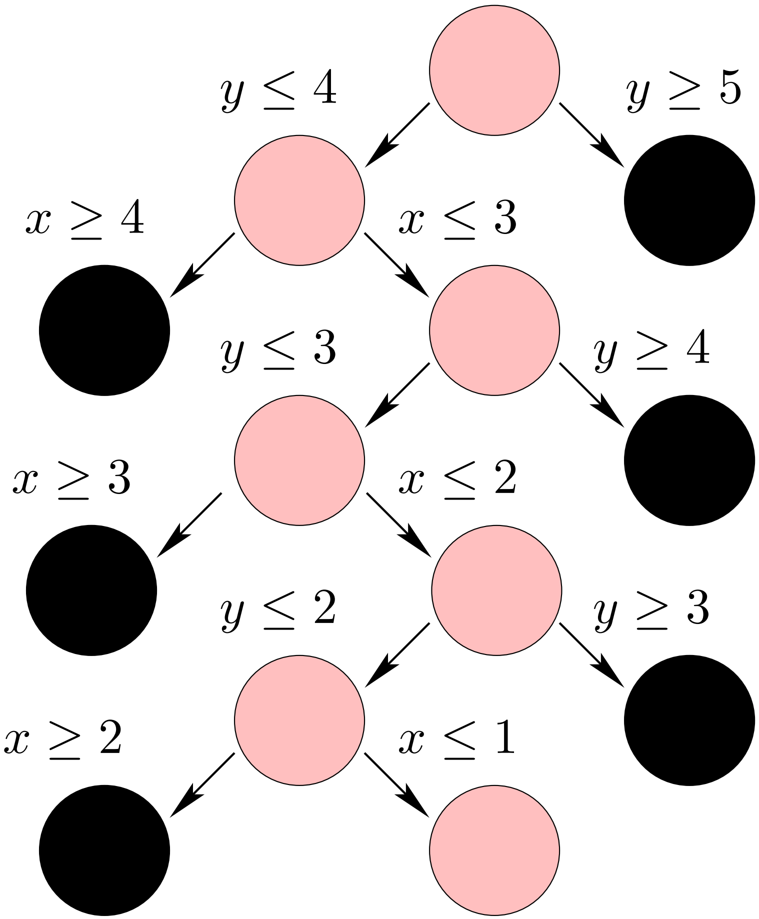


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
x = 2
y <= 2
End
=====================================================================

CPLEX> optimize
Bound infeasibility column 'y'.
Presolve time = 0.00 sec. (0.00 ticks)
Presolve - Infeasible.
Solution time =    0.00 sec.
Deterministic time = 0.00 ticks  (1.11 ticks/sec)
```

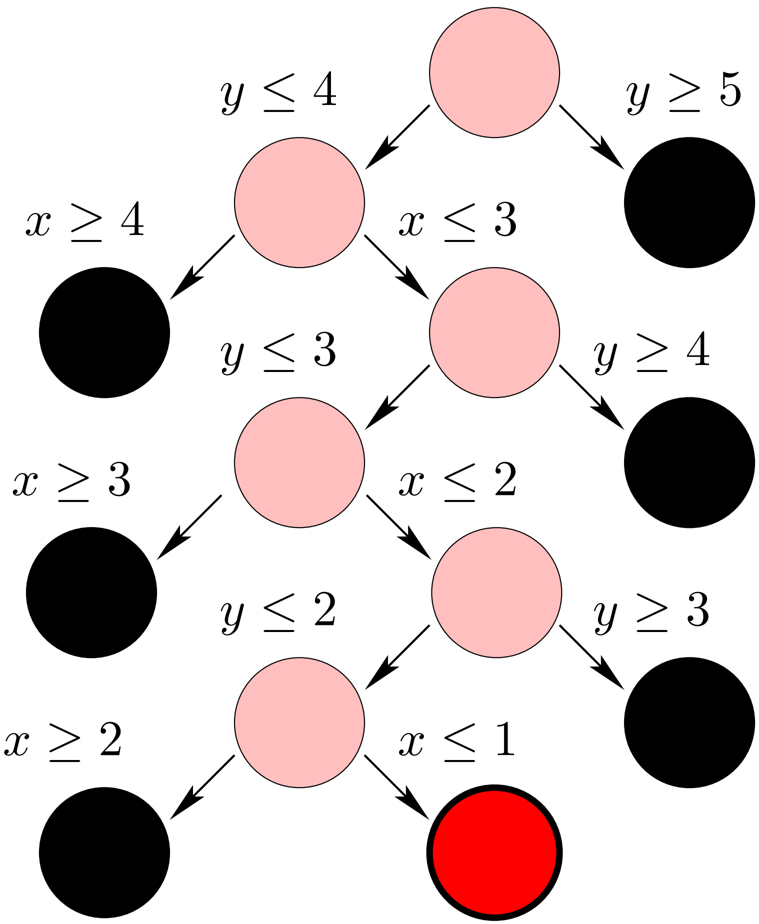
Example

$$\min \; -x - y$$
$$-2x + 2y \geq 1$$
$$-8x + 10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$



Example

$$\min\ -x - y$$
$$-2x + 2y \geq 1$$
$$-8x + 10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$

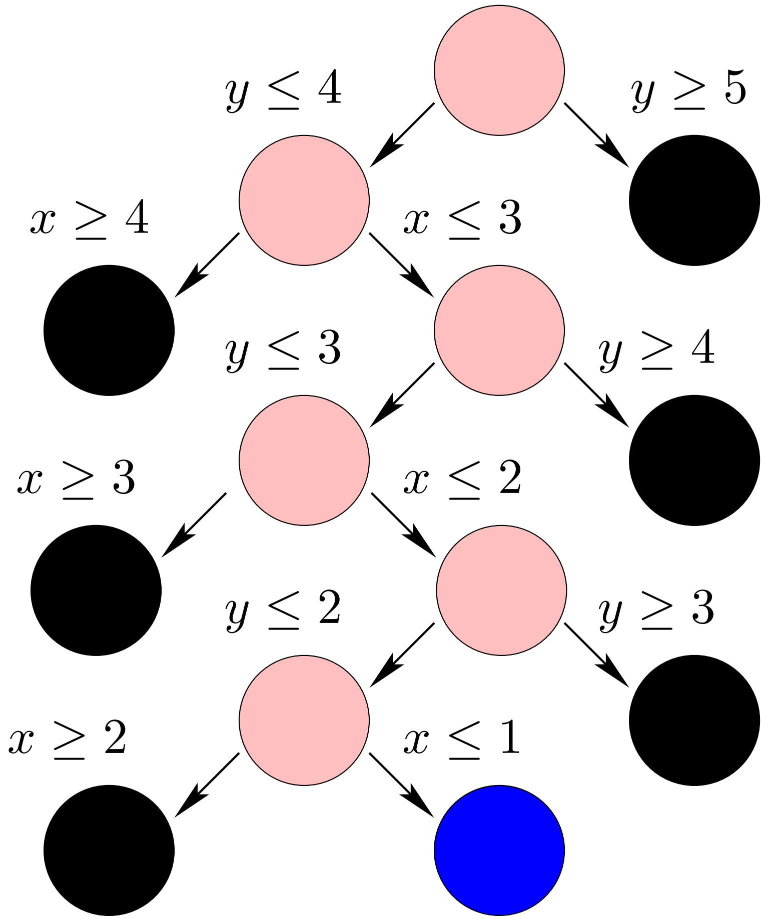


Example

```
Min - x - y
Subject To
-2 x +  2 y >= 1
-8 x + 10 y <= 13
Bounds
x <= 1
y <= 2
End
=====================================================================

CPLEX> optimize
Dual simplex - Optimal:  Objective = - 3.0000000000e+00
Solution time =    0.00 sec.  Iterations = 0 (0)
Deterministic time = 0.00 ticks  (2.40 ticks/sec)

CPLEX> display solution variables x
Variable Name           Solution Value
x                             1.000000

CPLEX> display solution variables y
Variable Name           Solution Value
y                             2.000000
```

Example

$$\min \; -x - y$$
$$-2x + 2y \geq 1$$
$$-8x + 10y \leq 13$$
$$x, y \geq 0$$
$$x, y \in \mathbb{Z}$$



Pruning in Branch & Bound

- We have already seen that if relaxation is infeasible,
  the problem can be pruned

- Now assume an (integral) solution has been previously found

- If solution has cost Z then any pending problem P_j whose relaxation has optimal value $\geq Z$ can be ignored, since

  $$\text{cost}(P_j) \geq \text{cost}(\mathrm{LP}(P_j)) \geq Z$$

  The optimum will not be in any descendant of P_j!

- This cost-based pruning of the search tree has a huge impact on the efficiency of Branch & Bound

Branch & Bound: Algorithm

```
S := {P_0}                                                /* set of pending problems */
Z := +∞                                                   /* best cost found so far */
while S ≠ ∅ do
    remove P from S
    solve LP(P)
    if LP(P) is feasible then                             /* if unfeasible P can be pruned */
        let β be optimal basic solution of LP(P)
        if β satisfies integrality constraints then
            if cost(β) < Z then store β; update Z
        else
            if cost(LP(P)) ≥ Z then continue              /* P can be pruned */
            let x_j be integer variable such that β_j ∉ ℤ
            S := S ∪ { P ∧ x_j ≤ ⌊β_j⌋,  P ∧ x_j ≥ ⌈β_j⌉ }
return Z
```

Heuristics in Branch & Bound

- Possible choices in Branch & Bound
  - Choice of the pending problem
    - Depth-first search
    - Breadth-first search
    - Best-first search: assuming a relaxation is solved when it is added to the set of pending problems, select the one with best cost value

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    - most important in the model (e.g., 0-1 variable)
    - biggest in a variable ordering
    - the one with the largest/smallest cost coefficient
- No known strategy is best for all problems!

Remarks on Branch & Bound

- If integer variables are not bounded, Branch & Bound may not terminate:

$$
\begin{aligned}
&\min\ 0 \\
&1 \leq 3x - 3y \leq 2 \\
&x, y \in \mathbb{Z}
\end{aligned}
$$

is infeasible but Branch & Bound loops forever looking for solutions!

Remarks on Branch & Bound

- After solving the relaxation of P,
  we have to solve the relaxations of $P \wedge x_j \leq \lfloor \beta_j \rfloor$ and $P \wedge x_j \geq \lceil \beta_j \rceil$
- These problems are similar. Do we have to start from scratch?
  Can be reuse somehow the computation for P?
- Idea: start from the optimal solution of the parent problem

Remarks on Branch & Bound

- Let us assume that P is of the form

  $$\begin{array}{l} \min\ c^T x \\ Ax = b \\ x \geq 0, \qquad x_i \in \mathbb{Z}\ \ \forall i \in \mathcal{I} \end{array}$$

- Let B be an optimal basis of the relaxation
- Let x_j be integer variable which at optimal solution is assigned $\beta_j \notin \mathbb{Z}$
- Note that x_j must be basic
- Let us consider the problem $P_1 = P \wedge x_j \leq \lfloor \beta_j \rfloor$
- We add a fresh slack variable s and a new equation: $P \wedge x_j + s = \lfloor \beta_j \rfloor$
- Since s is fresh we have $(x_{\mathcal{B}}, s)$ defines a basis for the relaxation of P_1

Remarks on Branch & Bound

$$\begin{array}{lcl} \min\ -x-y & & \min\ -x-y \\ -2x+2y \geq 1 & & -2x+2y-s_1=1 \\ -8x+10y \leq 13 & \Rightarrow & -8x+10y+s_2=13 \\ x,y \geq 0 & & x,y\geq 0 \\ x,y \in \mathbb{Z} & & x,y \in \mathbb{Z} \end{array}$$

- Optimal basis of the linear relaxation is $\mathcal{B} = (x, y)$ with tableau

$$\begin{cases} \min -\frac{17}{2} + \frac{9}{2}s_1 + s_2 \\ x = 4 - \frac{5}{2}s_1 - \frac{1}{2}s_2 \\ y = \frac{9}{2} - 2s_1 - \frac{1}{2}s_2 \end{cases}$$

- For the subproblem with $y \leq 4$ we add equation $y + s = 4$
  $\mathcal{B} = (x, y, s)$ is a basis for this subproblem with tableau

$$\begin{cases} \min -\frac{17}{2} + \frac{9}{2}s_1 + s_2 \\ x = 4 - \frac{5}{2}s_1 - \frac{1}{2}s_2 \\ y = \frac{9}{2} - 2s_1 - \frac{1}{2}s_2 \\ s = 4 - y = -\frac{1}{2} + 2s_1 + \frac{1}{2}s_2 \end{cases}$$

Remarks on Branch & Bound

- $(x_{\mathcal{B}}, s)$ defines a basis for the relaxation of P_1

- This basis is not feasible:
  the value in the basic solution assigned to s is $\lfloor \beta_j \rfloor - \beta_j < 0$.

  We would need a Phase I to apply the primal simplex method!

- But since s is a slack the reduced costs have not changed:
  $(x_{\mathcal{B}}, s)$ satisfies the optimality conditions!

- Dual simplex method can be used:
  basis $(x_{\mathcal{B}}, s)$ is already dual feasible, no need of (dual) Phase I

- In practice often the dual simplex only needs very few iterations to obtain the optimal solution to the new problem

Cutting Planes

- Let us consider a MIP of the form

$$\min_{x \in S} c^T x \quad \text{where } S = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} Ax = b \\ x \geq 0 \\ x_i \in \mathbb{Z} \quad \forall i \in \mathcal{I} \end{array} \right\}$$

and its linear relaxation

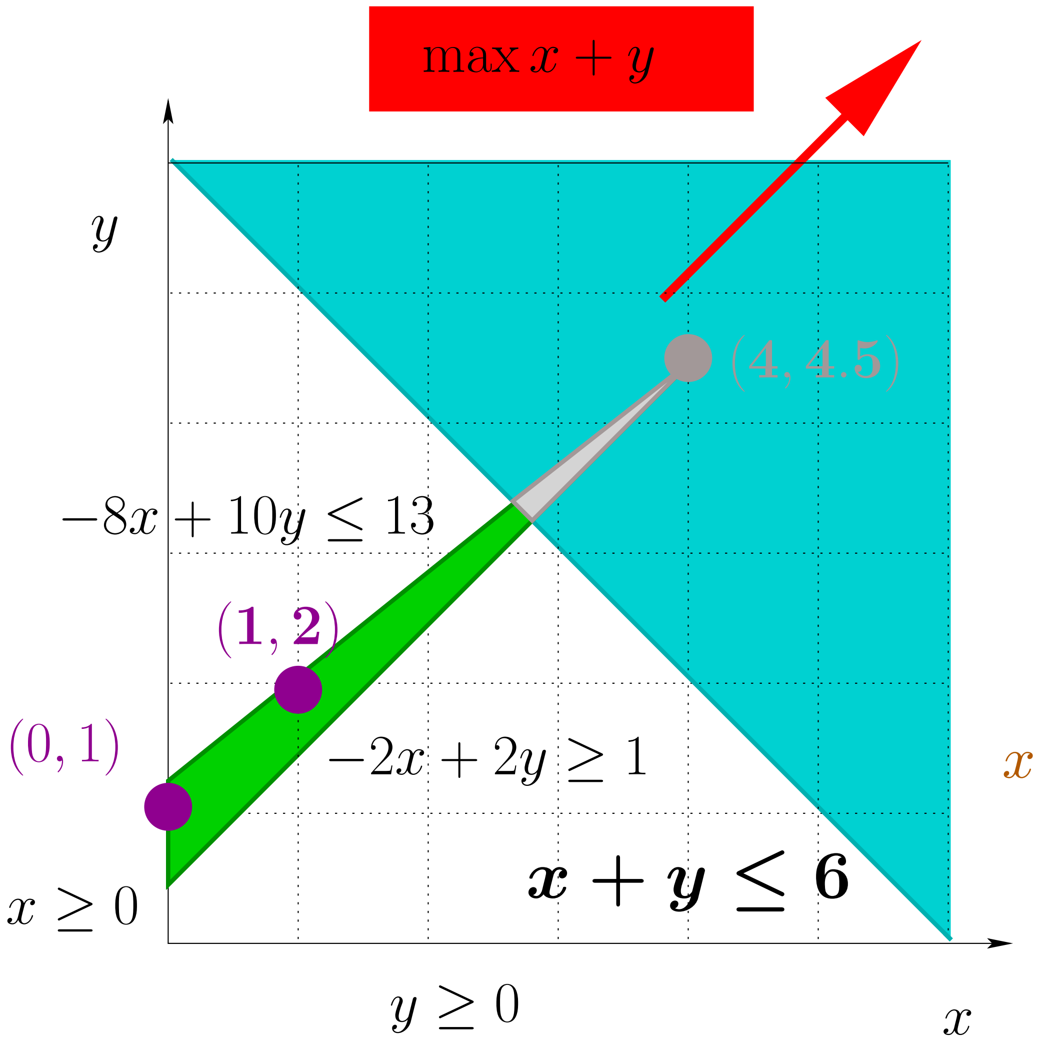
$$\min_{x \in P} c^T x \quad \text{where } P = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \right\}$$

- Let β be such that $\beta \in P$ but $\beta \notin S$.

  A cut for β is a linear inequality $p^T x \leq q$ such that

  - $p^T \sigma \leq q$ for any $\sigma \in S$ (feasible solutions of the MIP respect the cut)
  - and $p^T \beta > q$ (β does not respect the cut)

Cutting Planes



$$\begin{array}{rcl} \max\ x + y & & \\ -2x + 2y & \geq & 1 \\ -8x + 10y & \leq & 13 \\ x, y \geq 0 & & \\ x, y \in \mathbb{Z} & & \end{array}$$

$x + y \leq 6$ is a cut for $(4, 4.5)$

Using Cuts for Solving MIP's

- Let $p^T x \le q$ be a cut (of some spurious feasible solution of the relaxation). Then the MIP

$$\begin{array}{l} \min\ c^T x \\ x \in S' \end{array} \quad \text{where } S' = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} Ax = b \\ p^T x \le q \\ x \ge 0 \\ x_i \in \mathbb{Z} \quad \forall i \in \mathcal{I} \end{array} \right\}$$

  has the same set of feasible solutions S
  but its LP relaxation is strictly more constrained

- Instead of splitting into subproblems (Branch & Bound), one can add the cut and solve the relaxation of the new problem

- In practice cuts are used together with Branch & Bound: If after adding some cuts no integer solution is found, then branch

  This technique is called Branch & Cut

Gomory Cuts

- There are several techniques for deriving cuts
- Some are problem-specific (e.g., for the travelling salesman problem)
- Here we will see a generic technique: Gomory cuts
- Let us consider a basis B and let β be the associated basic solution. Note that for all $j \in \mathcal{R}$ we have $\beta_j = 0$
- Let x_i be a basic variable such that $i \in \mathcal{I}$ and $\beta_i \notin \mathbb{Z}$
- E.g., this happens in the optimal basis of the relaxation when the basic solution does not meet the integrality constraints
- Let the row of the tableau corresponding to x_i be of the form

$$x_i = \beta_i + \sum_{j \in \mathcal{R}} \alpha_{ij} x_j$$

Gomory Cuts

- Let $x \in S$. Then $x_i \in \mathbb{Z}$ and

$$x_i = \beta_i + \sum_{j \in \mathcal{R}} \alpha_{ij} x_j$$

$$x_i - \beta_i = \sum_{j \in \mathcal{R}} \alpha_{ij} x_j$$

- Let $\delta = \beta_i - \lfloor \beta_i \rfloor$. Then $0 < \delta < 1$
- Hence

$$\begin{aligned} x_i - \lfloor \beta_i \rfloor &= x_i - \beta_i + \beta_i - \lfloor \beta_i \rfloor \\ &= x_i - \beta_i + \delta \\ &= \delta + x_i - \beta_i \\ &= \delta + \sum_{j \in \mathcal{R}} \alpha_{ij} x_j \end{aligned}$$

Gomory Cuts

$$\delta = \beta_i - \lfloor \beta_i \rfloor \qquad x_i - \lfloor \beta_i \rfloor = \delta + \textstyle\sum_{j \in \mathcal{R}} \alpha_{ij} x_j$$

- Let us define

$$\mathcal{R}^+ = \{j \in \mathcal{R} \mid \alpha_{ij} \geq 0\} \qquad \mathcal{R}^- = \{j \in \mathcal{R} \mid \alpha_{ij} < 0\}$$

- Assume $\sum_{j \in \mathcal{R}} \alpha_{ij} x_j \geq 0$.

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  Then $\delta + \sum_{j \in \mathcal{R}} \alpha_{ij} x_j > 0$ and $x_i - \lfloor \beta_i \rfloor \in \mathbb{Z}$ imply

$$\delta + \sum_{j \in \mathcal{R}} \alpha_{ij} x_j \geq 1$$

$$\sum_{j \in \mathcal{R}^+} \alpha_{ij} x_j \geq \sum_{j \in \mathcal{R}} \alpha_{ij} x_j \geq 1 - \delta$$

$$\sum_{j \in \mathcal{R}^+} \frac{\alpha_{ij}}{1 - \delta} x_j \geq 1$$

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  Moreover $\sum_{j \in \mathcal{R}^-} \left(\frac{-\alpha_{ij}}{\delta} \right) x_j \geq 0$

Gomory Cuts

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Gomory Cuts

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$$\delta + \sum_{j \in \mathcal{R}} \alpha_{ij} x_j \leq 0$$

$$\sum_{j \in \mathcal{R}^-} \alpha_{ij} x_j \leq \sum_{j \in \mathcal{R}} \alpha_{ij} x_j \leq -\delta$$

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Moreover $\sum_{j \in \mathcal{R}^+} \frac{\alpha_{ij}}{1-\delta} x_j \geq 0$

Gomory Cuts

- In any case
$$\sum_{j\in\mathcal{R}^-}\left(\frac{-\alpha_{ij}}{\delta}\right)x_j + \sum_{j\in\mathcal{R}^+}\frac{\alpha_{ij}}{1-\delta}x_j \geq 1$$
 for any $x \in S$.

 However, when $x = \beta$ this inequality is not satisfied (set $x_j = 0$ for $j \in \mathcal{R}$)

- In the example:
$$\begin{cases} \min -\frac{17}{2} + \frac{9}{2}s_1 + s_2 \\ x = 4 - \frac{5}{2}s_1 - \frac{1}{2}s_2 \\ y = \frac{9}{2} - 2s_1 - \frac{1}{2}s_2 \end{cases}$$
 y violates the integrality condition,
 we have $\delta = \frac{1}{2}$, $\sum_{j\in\mathcal{R}}\alpha_{ij}x_j = -2s_1 - \frac{1}{2}s_2$

 The cut is $4s_1 + s_2 \geq 1$, which projected on x, y is $y \leq 4$.

Ensuring All Vertices Are Integer

- Consider an IP of the form

$$
\begin{aligned}
&\min\ c^T x\\
&Ax = b\\
&x \geq 0\\
&x \in \mathbb{Z}
\end{aligned}
$$

- Let us assume A, b have coefficients in $\mathbb{Z}$

- Are there any sufficient conditions to ensure that the simplex algorithm will directly provide an integer solution, without branch & bound/cut?

Ensuring All Vertices Are Integer

- Let us assume A, b have coefficients in $\mathbb{Z}$
- We will see sufficient conditions to ensure that all vertices of the relaxation are integer
- For instance, when the matrix A is totally unimodular: the determinant of every square submatrix is 0 or ± 1

Ensuring All Vertices Are Integer

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- For instance, when the matrix A is totally unimodular: the determinant of every square submatrix is 0 or ± 1

  In that case all bases have inverses with integer coefficients

  Recall Cramer's rule: if B is an invertible matrix, then

$$B^{-1} = \frac{1}{\det(B)} \operatorname{adj}(B)$$

  where $\operatorname{adj}(B)$ is the adjugate matrix of B

  Recall also that

$$\operatorname{adj}(B) = ((-1)^{i+j} \det(M_{ji}))_{1 \leq i,j \leq n},$$

  where M_{ij} is matrix B after removing the i-th row and the j-th column

Ensuring All Vertices Are Integer

- Sufficient condition for total unimodularity of a matrix A:
(Hoffman & Gale's Theorem)

1. Each element of A is 0 or ± 1
2. No more than two non-zeros appear in each columm
3. Rows can be partitioned in two subsets R_1 and R_2 s.t.

    (a) If a column contains two non-zeros of the same sign, the row of one of them belongs to one subset, and the row of the other, to the other subset

    (b) If a column contains two non-zeros of different signs, the rows of both of them belong to the same subset

Ensuring All Vertices Are Integer

- Let us consider an assignment problem
- $n = \#$ of workers $= \#$ of tasks
- Each worker must be assigned to exactly one task
- Each task is to be performed by exactly one worker
- $c_{ij} =$ cost when worker i performs task j

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- c_{ij} = cost when worker i performs task j

$$x_{ij} = \begin{cases} 1 & \text{if worker } i \text{ performs task } j \\ 0 & \text{otherwise} \end{cases}$$

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\begin{array}{ll} \sum_{j=1}^{n} x_{ij} = 1 & \forall i \in \{1, \dots, n\} \\ \sum_{i=1}^{n} x_{ij} = 1 & \forall j \in \{1, \dots, n\} \\ x_{ij} \in \{0, 1\} & \forall i, j \in \{1, \dots, n\} \end{array}$$

- This problem satisfies Hoffman & Gale's conditions

Ensuring All Vertices Are Integer

- Several kinds of IP's satisfy Hoffman & Gale's conditions:
  - Assignment
  - Transportation
  - Maximum flow
  - Shortest path
  - ...
- Usually ad-hoc graph algorithms are more efficient for these problems than the simplex method as presented here

Ensuring All Vertices Are Integer

- Several kinds of IP's satisfy Hoffman & Gale's conditions:
  - Assignment
  - Transportation
  - Maximum flow
  - Shortest path
  - ...
- Usually ad-hoc graph algorithms are more efficient for these problems than the simplex method as presented here
- But:
  - The simplex method can be specialized: network simplex method
  - Simplex techniques can be applied
    if the problem is not a purely network one but has extra constraints

Modeling with 0-1 Variables

- Sometimes we want to have an indicator variable of a contraint:
  a $0/1$ variable equal to 1 iff the constraint is true (= reification in CP)

- E.g., let us to encode $\delta = 1 \leftrightarrow a^T x \leq b$, where δ is a $0/1$ var

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- Assume $a^T x \in \mathbb{Z}$ for all feasible solution x

  Let U be an upper bound of $a^T x - b$ for all feasible solutions

  Let L be a lower bound of $a^T x - b$ for all feasible solutions

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1. $\delta = 1 \rightarrow a^T x \leq b$

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2. $\delta = 1 \leftarrow a^T x \leq b$

   $\delta = 0 \rightarrow a^T x > b$

   $\delta = 0 \rightarrow a^T x \geq b + 1$

   can be encoded with $a^T x - b \geq (L - 1)\delta + 1$

Modeling with 0-1 Variables

- Sometimes it is convenient to think constraints from a logical perspective, and then translate them into linear inequalities

- If x_1, ..., x_n, y_1, ..., y_m are $0/1$ ($=$ Boolean) variables then

$$x_1 \vee \ldots \vee x_n \vee \neg y_1 \vee \ldots \vee \neg y_m$$

is equivalent to

$$x_1 + \ldots + x_n + (1 - y_1) + \ldots + (1 - y_m) \geq 1 \ .$$

Example (Logical Constraints)

Let X_i represent "Ingredient i is in the blend", $i \in \{A, B, C\}$.
Express the sentence

"If ingredient A is in the blend,
then ingredient B or C (or both) must also be in the blend"

with linear constraints.

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with linear constraints.

- We need to express $X_A \to (X_B \vee X_C)$.
- Equivalently, $\neg X_A \vee X_B \vee X_C$.
- $\neg X_A \vee X_B \vee X_C$ is equivalent to $(1 - x_A) + x_B + x_C \geq 1$.
- So $x_B + x_C \geq x_A$

Example (Fixed Setup Charge)

Let x be the quantity of a product with unit production cost c_1.

If the product is manufactured at all, there is a setup cost c_0

$$\text{Cost of producing } x \text{ units } = \begin{cases} 0 & \text{if} \quad x = 0 \\ c_0 + c_1 x & \text{if} \quad x > 0 \end{cases}$$

Want to minimize costs. Model as a MIP?

(for simplicity, additional constraints are not specified and can be omitted)

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Want to minimize costs. Model as a MIP?

(for simplicity, additional constraints are not specified and can be omitted)

Let δ be $0/1$ var such that $x > 0 \rightarrow \delta = 1$ (i.e., $\delta = 0 \rightarrow x \leq 0$):
add constraint $x - U\delta \leq 0$, where U is the upper bound on x

Then the cost is $c_0\delta + c_1 x$.

No need to express $x > 0 \leftarrow \delta = 1$, i.e. $x = 0 \rightarrow \delta = 0$

Minimization will make $\delta = 0$ if possible (i.e., if $x = 0$)

Example (Capacity Expansion)

Let $a^T x$ be the consumption of a limited resource in a production process

Want to relax the constraint $a^T x \leq b$ by increasing capacity b.

Capacity can be expanded to b_i

$$b = b_0 < b_1 < b_2 < \cdots < b_t$$

with costs, respectively,

$$0 = c_0 < c_1 < c_2 < \cdots < c_t$$

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Want to minimize costs. Model as a MIP?
(for simplicity, additional constraints are not specified and can be omitted)

Let $0/1$ variables δ_i mean "capacity expanded to b_i". Then:

- $\sum_{i=0}^{t} \delta_i = 1$
- $a^T x \leq \sum_{i=0}^{t} b_i \delta_i$
- Cost function: $\sum_{i=0}^{t} c_i \delta_i$