The Dual Simplex Method

Combinatorial Problem Solving (CPS)

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Basic Idea

- **Abuse of terminology:**
  Henceforth sometimes by “optimal” we will mean “satisfying the optimality conditions”
  If not explicit, the context will disambiguate

- The algorithm as explained so far is known as **primal simplex:**
  starting with feasible basis,
  find optimal basis (= satisfying optimality conds.) while keeping feasibility

- There is an alternative algorithm known as **dual simplex:**
  starting with optimal basis (= satisfying optimality conds.),
  find feasible basis while keeping optimality
Basic Idea

\[
\begin{align*}
\min & \quad -x_1 - x_2 \\
2x_1 + x_2 & \geq 3 \\
2x_1 + x_2 & \leq 6 \\
x_1 + 2x_2 & \leq 6 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*}
\Rightarrow
\begin{align*}
\min & \quad -x_1 - x_2 \\
2x_1 + x_2 & \geq 3 \\
-2x_1 - x_2 & \geq -6 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*}
\Rightarrow
\begin{align*}
\min & \quad -x_1 - x_2 \\
2x_1 + x_2 - x_3 & = 3 \\
-2x_1 - x_2 - x_4 & = -6 \\
x_1 - 2x_2 - x_5 & = -6 \\
x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{align*}
\]

\[
\begin{align*}
\min & \quad -6 + x_2 + x_5 \\
x_1 & = 6 - 2x_2 - x_5 \\
x_3 & = 9 - 3x_2 - 2x_5 \\
x_4 & = -6 + 3x_2 + 2x_5
\end{align*}
\]

Basis \((x_1, x_3, x_4)\) is optimal
(\(=\) satisfies optimality conditions)
but is not feasible!
Basic Idea

\[ 2x_1 + x_2 \leq 6 \]

\[ x_1 \geq 0 \]

\[ x_2 \geq 0 \]

\[ x_1 + 2x_2 \leq 6 \]

\[ 2x_1 + x_2 \geq 3 \]

\[ \min -x_1 - x_2 \]

\[ (6, 0) \]
Basic Idea

- Let us make a violating basic variable non-negative ...
  - Increase $x_4$ by making it non-basic: then it will be 0

- ... while preserving optimality (= optimality conditions are satisfied)
  - If $x_5$ replaces $x_4$ in the basis, then $x_5 = 3 + \frac{1}{2}(x_4 - 3x_2)$, $-x_1 - x_2 = -3 + \frac{1}{2}(x_4 - x_2)$
  - If $x_2$ replaces $x_4$ in the basis, then $x_2 = 2 + \frac{1}{3}(x_4 - 2x_5)$, $-x_1 - x_2 = -4 + \frac{1}{3}(x_4 + x_5)$
Basic Idea

Let us make a violating basic variable non-negative ...

- Increase $x_4$ by making it non-basic: then it will be 0

... while preserving optimality (= optimality conditions are satisfied)

- If $x_5$ replaces $x_4$ in the basis,
  then $x_5 = 3 + \frac{1}{2}(x_4 - 3x_2), -x_1 - x_2 = -3 + \frac{1}{2}(x_4 - x_2)$

- If $x_2$ replaces $x_4$ in the basis,
  then $x_2 = 2 + \frac{1}{3}(x_4 - 2x_5), -x_1 - x_2 = -4 + \frac{1}{3}(x_4 + x_5)$

- To preserve optimality, we must swap $x_2$ and $x_4$
Basic Idea

\[
\begin{align*}
\min & -6 + x_2 + x_5 \\
x_1 &= 6 - 2x_2 - x_5 \\
x_3 &= 9 - 3x_2 - 2x_5 \\
x_4 &= -6 + 3x_2 + 2x_5
\end{align*}
\]

\[
\begin{align*}
\min & -4 + \frac{1}{3}x_4 + \frac{1}{3}x_5 \\
x_1 &= 2 - \frac{2}{3}x_4 + \frac{1}{3}x_5 \\
x_2 &= 2 + \frac{1}{3}x_4 - \frac{2}{3}x_5 \\
x_3 &= 3 - x_4
\end{align*}
\]
Basic Idea

\[
\begin{align*}
\min & -6 + x_2 + x_5 \\
x_1 & = 6 - 2x_2 - x_5 \\
x_3 & = 9 - 3x_2 - 2x_5 \\
x_4 & = -6 + 3x_2 + 2x_5
\end{align*}
\implies
\begin{align*}
\min & -4 + \frac{1}{3}x_4 + \frac{1}{3}x_5 \\
x_1 & = 2 - \frac{2}{3}x_4 + \frac{1}{3}x_5 \\
x_2 & = 2 + \frac{1}{3}x_4 - \frac{2}{3}x_5 \\
x_3 & = 3 - x_4
\end{align*}
\]

- Current basis is **feasible** and **optimal**!
Basic Idea

\[\min -x_1 - x_2\]

\[2x_1 + x_2 \leq 6\]

\[2x_1 + x_2 \geq 3\]

\[x_1 \geq 0\]

\[x_2 \geq 0\]

\[(2, 2)\]

\[(6, 0)\]
Outline of the Dual Simplex

1. Initialization: Pick an optimal basis (= satisfies optimality conditions).

2. Dual Pricing: If all values in basic solution are $\geq 0$, then return OPTIMAL. Else pick a basic variable with value $< 0$.

3. Dual Ratio Test: Find non-basic variable for swapping that preserves optimality, i.e., non-negativity constraints on reduced costs.
   - If it does not exist, then return INFEASIBLE.
   - Else swap chosen non-basic variable with violating basic variable.

4. Update: Update the tableau and go to 2.
Duality

■ To understand better how the dual simplex works: theory of duality

■ We can get lower bounds on LP optimum value by adding constraints in a convenient way

\[
\begin{align*}
\min & \quad -x_1 - x_2 \\
2x_1 + x_2 & \geq 3 \\
-2x_1 - x_2 & \geq -6 \\
-x_1 - 2x_2 & \geq -6 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*}
\]

\[
\begin{align*}
-x_1 - 2x_2 & \geq -6 \\
x_2 & \geq 0
\end{align*}
\]

\[
-x_1 - x_2 \geq -6
\]
Duality

- In general we can get lower bounds on LP optimum value by linearly combining constraints with convenient multipliers.

\[
\begin{align*}
\min & \quad -x_1 - x_2 \\
2x_1 + x_2 & \geq 3 \\
-2x_1 - x_2 & \geq -6 \\
-x_1 - 2x_2 & \geq -6 \\
x_1 & \geq 0 \\
x_2 & \geq 0
\end{align*}
\]

\[
\begin{align*}
1 \cdot (2x_1 + x_2) & \geq 3 \\
2 \cdot (-2x_1 - x_2) & \geq -6 \\
1 \cdot (x_1) & \geq 0
\end{align*}
\]

\[
\begin{align*}
2x_1 + x_2 & \geq 3 \\
-4x_1 - 2x_2 & \geq -12 \\
x_1 & \geq 0
\end{align*}
\]

\[
\begin{align*}
-x_1 - x_2 & \geq -9
\end{align*}
\]

- There may be different choices, each giving a different lower bound.
Let $\mu_1, \ldots, \mu_5 \geq 0$:

\[
\begin{align*}
\min \ & -x_1 - x_2 \\
\ & 2x_1 + x_2 \geq 3 \\
\ & -2x_1 - x_2 \geq -6 \\
\ & -x_1 - 2x_2 \geq -6 \\
\ & x_1 \geq 0 \\
\ & x_2 \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\mu_1 \cdot ( & 2x_1 + x_2 \geq 3 ) \\
\mu_2 \cdot ( & -2x_1 - x_2 \geq -6 ) \\
\mu_3 \cdot ( & -x_1 - 2x_2 \geq -6 ) \\
\mu_4 \cdot ( & x_1 \geq 0 ) \\
\mu_5 \cdot ( & x_2 \geq 0 ) \\
\end{align*}
\]

\[
\begin{align*}
2\mu_1 x_1 + \mu_1 x_2 & \geq 3\mu_1 \\
-2\mu_2 x_1 - \mu_2 x_2 & \geq -6\mu_2 \\
-\mu_3 x_1 - 2\mu_3 x_2 & \geq -6\mu_3 \\
\mu_4 x_1 & \geq 0 \\
\mu_5 x_2 & \geq 0 \\
\end{align*}
\]

\[
(2\mu_1 - 2\mu_2 - \mu_3 + \mu_4) x_1 + (\mu_1 - \mu_2 - 2\mu_3 + \mu_5) x_2 \geq 3\mu_1 - 6\mu_2 - 6\mu_3
\]

If $2\mu_1 - 2\mu_2 - \mu_3 + \mu_4 = -1$, $\mu_1 - \mu_2 - 2\mu_3 + \mu_5 = -1$, $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$, $\mu_4 \geq 0$, $\mu_5 \geq 0$, then $3\mu_1 - 6\mu_2 - 6\mu_3$ is a lower bound.
Duality

- We can skip the multipliers of the non-negativity constraints

- We have:

\[
\begin{align*}
\min & \quad -x_1 - x_2 \\
2x_1 + x_2 & \geq 3 \\
-2x_1 - x_2 & \geq -6 \\
-x_1 - 2x_2 & \geq -6 \\
x_1 & \geq 0 \\
x_2 & \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
\mu_1 \cdot (2x_1 + x_2) & \geq 3 \\
\mu_2 \cdot (-2x_1 - x_2) & \geq -6 \\
\mu_3 \cdot (-x_1 - 2x_2) & \geq -6 \\
\end{align*}
\]

\[
(2\mu_1 - 2\mu_2 - \mu_3) x_1 + (\mu_1 - \mu_2 - 2\mu_3) x_2 \geq 3\mu_1 - 6\mu_2 - 6\mu_3
\]

- Imagine \(2\mu_1 - 2\mu_2 - \mu_3 \leq -1\).
  In the coefficient of \(x_1\) we can “complete” \(2\mu_1 - 2\mu_2 - \mu_3\) to reach \(-1\) by adding a suitable multiple of \(x_1 \geq 0\) (the multiplier will be the slack)

- If \(2\mu_1 - 2\mu_2 - \mu_3 \leq -1\), \(\mu_1 - \mu_2 - 2\mu_3 \leq -1\), \(\mu_1 \geq 0\), \(\mu_2 \geq 0\), \(\mu_3 \geq 0\), then \(3\mu_1 - 6\mu_2 - 6\mu_3\) is a lower bound
Duality

- Best possible lower bound with this “trick” can be found by solving

\[
\begin{align*}
\max & \quad 3\mu_1 - 6\mu_2 - 6\mu_3 \\
2\mu_1 - 2\mu_2 - \mu_3 & \leq -1 \\
\mu_1 - \mu_2 - 2\mu_3 & \leq -1 \\
\mu_1, \mu_2, \mu_3 & \geq 0
\end{align*}
\]

- How far will it be from the optimum?
Duality

- Best possible lower bound with this “trick” can be found by solving

\[
\begin{align*}
\max & \quad 3\mu_1 - 6\mu_2 - 6\mu_3 \\
& \quad 2\mu_1 - 2\mu_2 - \mu_3 \leq -1 \\
& \quad \mu_1 - \mu_2 - 2\mu_3 \leq -1 \\
& \quad \mu_1, \mu_2, \mu_3 \geq 0
\end{align*}
\]

- How far will it be from the optimum?

- A best solution is given by \((\mu_1, \mu_2, \mu_3) = (0, \frac{1}{3}, \frac{1}{3})\)

\[
\begin{align*}
0 \cdot (2x_1 + x_2 & \geq 3) \\
\frac{1}{3} \cdot (-2x_1 - x_2 & \geq -6) \\
\frac{1}{3} \cdot (-x_1 - 2x_2 & \geq -6)
\end{align*}
\]

\[\begin{align*}
-x_1 - x_2 & \geq -4
\end{align*}\]

Matches the optimum!
Dual Problem

- If we multiply $Ax \geq b$ by multipliers $y^T \geq 0$ we get $y^T Ax \geq y^T b$
- If $y^T A \leq c^T$ then we get a lower bound $y^T b$ for the cost function $c^T x$
- Given an LP (called primal problem)

\[
\begin{align*}
\min & \quad c^T x \\
Ax & \geq b \\
x & \geq 0
\end{align*}
\]

its dual problem is the LP

\[
\begin{align*}
\max & \quad y^T b \\
y^T A & \leq c^T \\
y^T & \geq 0
\end{align*}
\]

or equivalently

\[
\begin{align*}
\max & \quad b^T y \\
A^T y & \leq c \\
y & \geq 0
\end{align*}
\]

- Primal variables associated with columns of $A$
- Dual variables (multipliers) associated with rows of $A$
- Objective and right-hand side vectors swap their roles
Prop. The dual of the dual is the primal.

Proof:

\[ \text{max } b^T y \quad \rightarrow \quad \text{min } (\mathbf{-}b)^T y \]
\[ A^T y \leq c \quad \Rightarrow \quad \mathbf{-}A^T y \geq \mathbf{-}c \]
\[ y \geq 0 \quad \rightarrow \quad y \geq 0 \]

\[ \mathbf{-}\text{max } \mathbf{-}c^T x \quad \rightarrow \quad \text{min } c^T x \]
\[ (-A^T)^T x \leq \mathbf{-}b \quad \Rightarrow \quad Ax \geq b \]
\[ x \geq 0 \quad \rightarrow \quad x \geq 0 \]

We say the primal and the dual form a primal-dual pair
Dual Problem

Prop. \[ \begin{align*}
\min \ c^T x \\
Ax &= b \\
x &\geq 0
\end{align*} \]
and
\[ \begin{align*}
\max \ b^T y \\
A^T y &\leq c
\end{align*} \]
form a primal-dual pair

Proof:

\[ \begin{align*}
\min \ c^T x \\
Ax &= b \\
x &\geq 0 \\
\rightarrow
\min \ c^T x \\
Ax &\geq b \\
-Ax &\geq -b \\
x &\geq 0
\end{align*} \]

\[ \begin{align*}
\max \ b^T y_1 - b^T y_2 \\
A^T y_1 - A^T y_2 &\leq c \\
y_1, y_2 &\geq 0 \\
\rightarrow
\max \ b^T y \\
A^T y &\leq c
\end{align*} \]
Duality Theorems

- **Th. (Weak Duality)** Let \((P, D)\) be a primal-dual pair

\[
\begin{align*}
(P) & \quad \min_{x} c^T x \\
& \quad Ax = b \\
& \quad x \geq 0 \\
(D) & \quad \max_{y} b^T y \\
& \quad A^T y \leq c
\end{align*}
\]

If \(x\) is feasible solution to \(P\) and \(y\) is feasible solution to \(D\) then \(b^T y \leq c^T x\)

**Proof:**

\[c - A^T y \geq 0, \text{ i.e., } c^T - y^T A \geq 0, \text{ and } x \geq 0 \text{ imply } c^T x - y^T A x \geq 0.\]

So \(c^T x \geq y^T A x\), and

\[c^T x \geq y^T A x = y^T b = b^T y\]
Duality Theorems

- Feasible solutions to $D$ give lower bounds on $P$
- Feasible solutions to $P$ give upper bounds on $D$
- Will the two optimum values be always equal?
Duality Theorems

- Feasible solutions to $D$ give lower bounds on $P$
- Feasible solutions to $P$ give upper bounds on $D$
- Will the two optimum values be always equal?

**Th. (Strong Duality)** Let $(P, D)$ be a primal-dual pair

\[
\begin{align*}
\text{(P)} & \quad \min c^T x \\
\quad & \quad Ax = b \\
\quad & \quad x \geq 0
\end{align*}
\quad \text{and} \quad
\begin{align*}
\text{(D)} & \quad \max b^T y \\
\quad & \quad A^T y \leq c
\end{align*}
\]

If any of $P$ or $D$ has a feasible solution and a finite optimum then the same holds for the other problem and the two optimum values are equal.
Duality Theorems

- Proof (Th. of Strong Duality):
  By duality it is sufficient to prove only one direction. Wlog. let us assume $P$ is feasible with finite optimum.
Duality Theorems

- **Proof (Th. of Strong Duality):**

  By duality it is sufficient to prove only one direction. Wlog. let us assume $P$ is feasible with finite optimum.

  After executing the Simplex algorithm to $P$ we find $B$ optimal feasible basis. Then:

  - $c_B^T B^{-1} a_j \leq c_j$ for all $j \in \mathcal{R}$ (optimality conds hold)
  - $c_B^T B^{-1} a_j = c_j$ for all $j \in \mathcal{B}$

  Hence $c_B^T B^{-1} A \leq c^T$.

  So $\pi^T := c_B^T B^{-1}$ is a dual feasible solution: $\pi^T A \leq c^T$, i.e., $A^T \pi \leq c$
Duality Theorems

- **Proof (Th. of Strong Duality):**

  By duality it is sufficient to prove only one direction. Wlog. let us assume $P$ is feasible with finite optimum.

  After executing the Simplex algorithm to $P$ we find $B$ optimal feasible basis. Then:

  - $c_B^T B^{-1} a_j \leq c_j$ for all $j \in R$ (optimality conds hold)
  - $c_B^T B^{-1} a_j = c_j$ for all $j \in B$

  Hence $c_B^T B^{-1} A \leq c^T$.

  So $\pi^T := c_B^T B^{-1}$ is a dual feasible solution: $\pi^T A \leq c^T$, i.e., $A^T \pi \leq c$

Moreover, $c_B^T \beta = c_B^T B^{-1} b = \pi^T b = b^T \pi$

By the theorem of weak duality, $\pi$ is optimum for $D$
Duality Theorems

- **Proof (Th. of Strong Duality):**

  By duality it is sufficient to prove only one direction. Wlog. let us assume $P$ is feasible with finite optimum.

  After executing the Simplex algorithm to $P$ we find $B$ optimal feasible basis. Then:

  - $c_B^T B^{-1} a_j \leq c_j$ for all $j \in \mathcal{R}$ (optimality conds hold)
  - $c_B^T B^{-1} a_j = c_j$ for all $j \in \mathcal{B}$

  Hence $c_B^T B^{-1} A \leq c^T$.

  So $\pi^T := c_B^T B^{-1}$ is a dual feasible solution: $\pi^T A \leq c^T$, i.e., $A^T \pi \leq c$

  Moreover, $c_B^T \beta = c_B^T B^{-1} b = \pi^T b = b^T \pi$

  By the theorem of weak duality, $\pi$ is optimum for $D$

- **If** $B$ is an optimal feasible basis for $P$, then simplex multipliers $\pi^T := c_B^T B^{-1}$ are optimal feasible solution for $D$

- **We can solve the dual by applying the simplex algorithm on the primal**

  **We can solve the primal by applying the simplex algorithm on the dual**
Duality Theorems

■ Prop. Let \((P, D)\) be a primal-dual pair

\[
\begin{align*}
\text{(P)} & \quad \min \ c^T x \\
& \quad \text{s.t.} \quad Ax = b, \quad x \geq 0 \\
\text{(D)} & \quad \max \ b^T y \\
& \quad A^T y \leq c
\end{align*}
\]

(1) If \(P\) has a feasible solution but is unbounded, then \(D\) is infeasible
(2) If \(D\) has a feasible solution but is unbounded, then \(P\) is infeasible

Proof:

Let us prove (1) by contradiction.
If \(y\) were a feasible solution to \(D\), by the weak duality theorem, objective of \(P\) would be lower bounded!

(2) is proved by duality.
Prop. Let \((P, D)\) be a primal-dual pair

\[
\begin{align*}
(P) & \quad \min_{x} c^T x \\
& \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]
and
\[
\begin{align*}
(D) & \quad \max_{y} b^T y \\
& \quad A^T y \leq c
\end{align*}
\]

(1) If \(P\) has a feasible solution but is unbounded, then \(D\) is infeasible

(2) If \(D\) has a feasible solution but is unbounded, then \(P\) is infeasible

And the converse?
Does infeasibility of one imply unboundedness of the other?
Duality Theorems

Prop. Let \((P, D)\) be a primal-dual pair

\[
\begin{align*}
\min \ c^T x \\
(P) \quad Ax = b \\
\quad x \geq 0 \\
\end{align*}
\]

and

\[
\begin{align*}
\max \ b^T y \\
(D) \quad A^T y \leq c
\end{align*}
\]

(1) If \(P\) has a feasible solution but is unbounded, then \(D\) is infeasible
(2) If \(D\) has a feasible solution but is unbounded, then \(P\) is infeasible

And the converse?
Does infeasibility of one imply unboundedness of the other?

\[
\begin{align*}
\min \ 3x_1 + 5x_2 \\
x_1 + 2x_2 &= 3 \\
2x_1 + 4x_2 &= 1 \\
x_1, x_2 \text{ free}
\end{align*}
\]

\[
\begin{align*}
\max \ 3y_1 + y_2 \\
y_1 + 2y_2 &= 3 \\
2y_1 + 4y_2 &= 5 \\
y_1, y_2 \text{ free}
\end{align*}
\]
Duality Theorems

Primal unbounded $\implies$ Dual infeasible
Dual unbounded $\implies$ Primal infeasible
Primal infeasible $\implies$ Dual \{ infeasible, unbounded \}
Dual infeasible $\implies$ Primal \{ infeasible, unbounded \}

Consider a primal-dual pair of the form

\[
\begin{align*}
\min & \quad c^T x \\
Ax &= b & \text{and} & \max & \quad b^T y \\
x &\geq 0 & \quad A^T y \leq c & \iff & \max & \quad b^T y \\
& & & \quad A^T y + w = c & w &\geq 0
\end{align*}
\]

Karush-Kuhn-Tucker (KKT) optimality conditions are

\begin{itemize}
    \item \(Ax = b\)
    \item \(A^T y + w = c\)
    \item \(x, w \geq 0\)
    \item \(x^T w = 0\) (complementary slackness)
\end{itemize}

They are necessary and sufficient conditions for optimality of the pair of primal-dual solutions \((x, y, w)\)

Used, e.g., as a test of quality in LP solvers

\[
\begin{align*}
\min & \quad c^T x \\
(P) & \quad Ax = b \\
& \quad x \geq 0 \\
\max & \quad b^T y \\
(D) & \quad A^T y + w = c \\
& \quad w \geq 0
\end{align*}
\]

\[(KKT)\]
\[
\begin{align*}
\bullet & \quad Ax = b \\
\bullet & \quad A^T y + w = c \\
\bullet & \quad x, w \geq 0 \\
\bullet & \quad x^T w = 0
\end{align*}
\]

\textbf{Th.} \((x, y, w)\) is solution to \(KKT\) iff \\
\(x\) optimal solution to \(P\) and \((y, w)\) optimal solution to \(D\)

\textbf{Proof:}

\(\Rightarrow\) By \(0 = x^T w = x^T (c - A^T y) = c^T x - b^T y\), and Weak Duality

\(\Leftarrow\) \(x\) is feasible solution to \(P\), \((y, w)\) is feasible solution to \(D\).

By Strong Duality \(x^T w = x^T (c - A^T y) = c^T x - b^T y = 0\)

as both solutions are optimal
Relating Bases

Consider a primal-dual pair of the form

\[
\begin{align*}
\text{(P)} \quad & \min z = c^T x \\
& Ax = b \quad \max Z = b^T y \\
& x \geq 0 \quad w \geq 0
\end{align*}
\]

Let us denote by \(a_1, \ldots, a_n\) the columns of \(A\), i.e., \(A = (a_1, \ldots, a_n)\).

Let \(B\) be a basis of \(P\). Let us see how we can get a basis of \(D\).
Assume that the basic variables are the first \(m\): \(B = (a_1, \ldots, a_m)\).
Then \(R = (a_{m+1}, \ldots, a_n)\).
If slacks \(w\) are split into \(w_B^T = (w_1, \ldots, w_m), \ w_R^T = (w_{m+1}, \ldots, w_n)\), then

\[
A^T y + w = \begin{pmatrix}
a_1^T y \\
\vdots \\
a_m^T y \\
\vspace{0.5cm}
a_{m+1}^T y \\
\vdots \\
a_n^T y
\end{pmatrix} + \begin{pmatrix}
w_1 \\
\vdots \\
w_m \\
\vspace{0.5cm}
w_{m+1} \\
\vdots \\
w_n
\end{pmatrix} = \begin{pmatrix}
B^T y + w_B \\
R^T y + w_R
\end{pmatrix}
\]
Relating Bases

■ Hence we have

\[ A^T y + w = \left( \begin{array}{c} B^T y + w_B \\ R^T y + w_R \end{array} \right) \]

■ Then the matrix of the system in the dual problem \( D \) is

\[
\begin{pmatrix}
B^T & I & 0 \\
R^T & 0 & I
\end{pmatrix}
\begin{pmatrix}
y \\
w_B \\
w_R
\end{pmatrix}
\]

■ Now let us consider the submatrix of vars \( y \) and vars \( w_{\mathcal{R}} \):

\[ \hat{B} = \begin{pmatrix} B^T & 0 \\ R^T & I \end{pmatrix} \]

■ Note \( \hat{B} \) is a square \( n \times n \) matrix
Relating Bases

- Dual variables \( \hat{B} = (y, w_R) \) determine a basis of \( D \):

\[
\hat{B} = \left( \begin{array}{c|c}
B^T & 0 \\
R^T & I 
\end{array} \right)
\]

\[
\hat{B}^{-1} = \left( \begin{array}{c|c}
B^{-T} & 0 \\
-R^T B^{-T} & I 
\end{array} \right)
\]
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- In the next slides we answer the following questions:

1. If basis \( \hat{\mathcal{B}} \) of the dual \( D \) is feasible, what can we say about basis \( \mathcal{B} \) of the primal \( P \)?

2. If basis \( \hat{\mathcal{B}} \) of the dual \( D \) is optimal (satisfies the optimality conds.), what can we say about basis \( \mathcal{B} \) of the primal \( P \)?

3. If we apply the simplex algorithm to the dual \( D \) using basis \( \hat{\mathcal{B}} \), how does that translate into the primal \( P \) and its basis \( \mathcal{B} \)?
Relating Bases

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- In the next slides we answer the following questions:

1. If basis $\hat{B}$ of the dual $D$ is feasible, what can we say about basis $B$ of the primal $P$?

2. If basis $\hat{B}$ of the dual $D$ is optimal (satisfies the optimality conditions), what can we say about basis $B$ of the primal $P$?

3. If we apply the simplex algorithm to the dual $D$ using basis $\hat{B}$, how does that translate into the primal $P$ and its basis $B$?

- Recall that each variable $w_j$ in $D$ is associated to a variable $x_j$ in $P$.

- Note that $w_j$ is $\hat{B}$-basic iff $x_j$ is not $B$-basic.
Dual Feasibility = Primal Optimality

- If \( \hat{B} \) is feasible for dual \( D \), what about \( B \) in primal \( P \)?
- Let us compute the basic solution of basis \( \hat{B} \) in the dual problem \( D \)

\[
\begin{pmatrix}
\frac{y}{w_{\mathcal{R}}} \\
w_{\mathcal{R}}
\end{pmatrix} = \hat{B}^{-1} c = \begin{pmatrix}
\frac{B^{-T} - R^T B^{-T}}{I} & 0 \\
-R^T B^{-T} & I
\end{pmatrix} \begin{pmatrix} c_B \\ c_{\mathcal{R}} \end{pmatrix} = \begin{pmatrix}
\frac{B^{-T} c_B}{-R^T B^{-T} c_B + c_{\mathcal{R}}} \\
\frac{0}{-R^T B^{-T} c_B + c_{\mathcal{R}}} \end{pmatrix}
\]

- Recall that there is no restriction on the sign of \( y_1, \ldots, y_m \)
- Variables \( w_j \) have to be non-negative. But

\[
-R^T B^{-T} c_B + c_{\mathcal{R}} \geq 0 \quad \text{iff} \quad c^T_{\mathcal{R}} - c_B B^{-1} R \geq 0
\]
Dual Feasibility = Primal Optimality

- If $\hat{B}$ is feasible for dual $D$, what about $B$ in primal $P$?
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c_{\mathcal{R}} \\
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\[-R^T B^{-T} c_B + c_{\mathcal{R}} \geq 0 \quad \text{iff} \quad c_{\mathcal{R}}^T - c_B^T B^{-1} R \geq 0 \quad \text{iff} \quad d_{\mathcal{R}}^T \geq 0\]
Dual Feasibility = Primal Optimality

- If $\hat{B}$ is feasible for dual $D$, what about $B$ in primal $P$?
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$$\begin{pmatrix} y \\ w_R \end{pmatrix} = \hat{B}^{-1} c = \begin{pmatrix} B^{-T} & 0 \\ -R^T B^{-T} & I \end{pmatrix} \begin{pmatrix} c_B \\ c_R \end{pmatrix} = \begin{pmatrix} B^{-T} c_B \\ -R^T B^{-T} c_B + c_R \end{pmatrix}$$

- Recall that there is no restriction on the sign of $y_1, \ldots, y_m$
- Variables $w_j$ have to be non-negative. But

$$-R^T B^{-T} c_B + c_R \geq 0 \iff c_R^T - c_B^T B^{-1} R \geq 0 \iff d_R^T \geq 0$$

- $\hat{B}$ is dual feasible iff $d_j \geq 0$ for all $j \in \mathcal{R}$
- Dual feasibility is primal optimality!
Dual Optimality = Primal Feasibility

- If $\hat{B}$ satisfies the optimality conds. for dual $D$, what about $B$ in primal $P$?
- Let us formulate the optimality conds. of basis $\hat{B}$ in the dual problem $D$.
- Non $\hat{B}$-basic vars: $w_B$ with costs $(0)$
- $\hat{B}$-basic vars: $(y \mid w_R)$ with costs $(b^T \mid 0)$
- Matrix of non $\hat{B}$-basic vars: $\begin{pmatrix} I \\ 0 \end{pmatrix}$
- Optimality condition: $0 \geq$ reduced costs (maximization!)

\[
0 \geq (0) - (b^T \mid 0) \begin{pmatrix} B^{-T} \\ -R^T B^{-T} \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} =
\]
\[
= (0) - (b^T B^{-T} \mid 0) \begin{pmatrix} I \\ 0 \end{pmatrix} = -b^T B^{-T} = -\beta^T \text{ where } \beta = B^{-1}b
\]
Dual Optimality = Primal Feasibility

- If \( \hat{B} \) satisfies the optimality conds. for dual \( D \), what about \( B \) in primal \( P \)?
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= (0) - (b^T B^{-T} \mid 0) \begin{pmatrix} I \\ 0 \end{pmatrix} = -b^T B^{-T} = -\beta^T \text{ where } \beta = B^{-1}b
\]

- In the dual, for all \( 1 \leq p \leq m \) var \( w_{kp} \) satisfies optimality cond. iff \( \beta_p \geq 0 \)
- Dual optimality is primal feasibility!
Improving a Non-Optimal Solution

- Next we apply the simplex algorithm to basis $\hat{B}$ in the dual problem $D$ and translate it to the primal problem $P$

- Let $p$ (where $1 \leq p \leq m$) be such that $\beta_p < 0$.
  I.e., the reduced cost of non-basic dual variable $w_{kp}$ is positive.
  So by giving $w_{kp}$ a larger value we can improve the dual objective value.
  If $w_{kp}$ takes value $t \geq 0$:

$$
\begin{pmatrix}
y(t) \\
w_R(t)
\end{pmatrix} = \hat{B}^{-1}c - \hat{B}^{-1}te_p =
$$

$$
= \begin{pmatrix} B^{-T}c_B \\ d_R \end{pmatrix} - \begin{pmatrix} B^{-T} \\ -R^TB^{-T} \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} \begin{pmatrix} te_p \\ 0 \end{pmatrix} = \begin{pmatrix} B^{-T}c_B - tB^{-T}e_p \\ d_R + tR^TB^{-T}e_p \end{pmatrix}
$$

- Dual objective value improvement is

$$\Delta Z = b^Ty(t) - b^Ty(0) = -tb^TB^{-T}e_p = -t\beta^Te_p = -t\beta_p$$
Improving a Non-Optimal Solution

- Of all basic dual variables, only $w_R$ variables need to be $\geq 0$
- For $j \in R$

\[ w_j(t) = d_j + ta_j^T B^{-T} e_p = d_j + te_p^T B^{-1} a_j = d_j + te_p^T \alpha_j = d_j + t\alpha_j^p \]

where $\alpha_j^p$ is the $p$-th component of $\alpha_j = B^{-1}a_j$. Hence:

\[ w_j(t) \geq 0 \iff d_j + t\alpha_j^p \geq 0 \iff d_j \geq t(-\alpha_j^p) \]

- If $\alpha_j^p \geq 0$ the constraint is satisfied for all $t \geq 0$
- If $\alpha_j^p < 0$ we need $\frac{d_j}{-\alpha_j^p} \geq t$

- Best improvement achieved with

\[ \Theta_D := \min\{\frac{d_j}{-\alpha_j^p} \mid \alpha_j^p < 0\} \]

- Variable $w_q$ is blocking when $\Theta_D = \frac{d_q}{-\alpha_q^p}$
Improving a Non-Optimal Solution

1. If $\Theta_D = +\infty$ (there is no $j \in R$ such that $\alpha_j^p < 0$):
   
   Value of dual objective can be increased infinitely.

   Dual LP is unbounded.

   Primal LP is infeasible.

2. If $\Theta_D < +\infty$ and $w_q$ is blocking:
   
   When setting $w_{kp} = \Theta_D$,
   non-negativity constraints of basic vars of dual are respected

   We can make a basis change:
   
   - In dual: $w_{kp}$ enters $\hat{B}$ and $w_q$ leaves
   - In primal: $x_{kp}$ leaves $B$ and $x_q$ enters
Update

- We do **not** actually need to form the dual LP: it is **enough** to have a representation of the primal LP

- New basic indices: \( \bar{B} = (k_1, \ldots, k_{p-1}, q, k_{p+1} \ldots, k_m) \)

- New dual objective value: \( \bar{Z} = Z - \Theta_D \beta_p \)

- New dual basic sol: \( \bar{y} = y - \Theta_D \rho_p \) where \( \rho_p = B^{-T}e_p \)
  \( \bar{d}_j = d_j + \Theta_D \alpha^p_j \) if \( j \in R \), \( \bar{d}_{kp} = \Theta_D \)

- New primal basic sol: \( \bar{\beta}_p = \Theta_P \), \( \bar{\beta}_i = \beta_i - \Theta_P \alpha^i_q \) if \( i \neq p \)
  where \( \Theta_P = \frac{\beta_p}{\alpha^p_q} \)

- New basis inverse: \( \bar{B}^{-1} = EB^{-1} \)
  where \( E = (e_1, \ldots, e_{p-1}, \eta, e_{p+1}, \ldots, e_m) \) and
  \( \eta^T = \left( \left( -\frac{\alpha^1_q}{\alpha^p_q} \right), \ldots, \left( -\frac{\alpha^{p-1}_q}{\alpha^p_q} \right), \frac{1}{\alpha^p_q} \left( -\frac{\alpha^{p+1}_q}{\alpha^p_q} \right), \ldots, \left( -\frac{\alpha^m_q}{\alpha^p_q} \right) \right)^T \)
Algorithmic Description

1. **Initialization:** Find an initial dual feasible basis $B$
   Compute $B^{-1}$, $\beta = B^{-1}b$,
   $y^T = c_B^T B^{-1}$, $d_R^T = c_R^T - y^T R$, $Z = b^T y$

2. **Dual Pricing:**
   If for all $i \in B, \beta_i \geq 0$ then return **OPTIMAL**
   Else let $p$ be such that $\beta_p < 0$.
   Compute $\rho_p^T = c_p^T B^{-1}$ and $\alpha_j^p = \rho_p^T a_j$ for $j \in R$

3. **Dual Ratio Test:** Compute $J = \{j \mid j \in R, \alpha_j^p < 0\}$.
   If $J = \emptyset$ then return **INFEASIBLE**
   Else compute $\Theta_D = \min_{j \in J} \left( \frac{d_j}{\alpha_j^p} \right)$ and $q$ st. $\Theta_D = \frac{d_q}{\alpha_q^p}$
Algorithmic Description

4. Update:
\[
\begin{align*}
\bar{B} &= B - \{k_p\} \cup \{q\} \\
\bar{Z} &= Z - \Theta_D \beta_p
\end{align*}
\]

Dual solution
\[
\begin{align*}
\bar{y} &= y - \Theta_D \rho_p \\
\bar{d}_j &= d_j + \Theta_D \alpha^p_j \text{ if } j \in \mathcal{R}, \quad \bar{d}_{k_p} = \Theta_D
\end{align*}
\]

Primal solution
Compute \( \alpha_q = B^{-1} a_q \) and \( \Theta_P = \frac{\beta_p}{\alpha_q} \)
\[
\begin{align*}
\bar{\beta}_p &= \Theta_P, \quad \bar{\beta}_i &= \beta_i - \Theta_P \alpha^i_q \text{ if } i \neq p \\
\bar{B}^{-1} &= EB^{-1}
\end{align*}
\]

Go to 2.
<table>
<thead>
<tr>
<th>PRIMAL</th>
<th>DUAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>■ Can handle bounds efficiently</td>
<td>■ Can handle bounds efficiently (not explained here)</td>
</tr>
<tr>
<td>■ Many years of research and implementation</td>
<td>■ Developments in the 90’s made it an alternative</td>
</tr>
<tr>
<td>■ There are classes of LP’s for which it is the best</td>
<td>■ Nowadays on average it gives better performance</td>
</tr>
<tr>
<td>■ Not suitable for solving LP’s with integer variables</td>
<td>■ Suitable for solving LP’s with integer variables</td>
</tr>
</tbody>
</table>