The Dual Simplex Method

Combinatorial Problem Solving (CPS)

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Basic Idea

- **Abuse of terminology:**

  Henceforth sometimes by “optimal” we will mean “satisfying the optimality conditions”

  If not explicit, the context will disambiguate

- The algorithm as explained so far is known as **primal simplex**:

  starting with feasible basis,

  find optimal basis (= satisfying optimality conds.) while keeping feasibility

- There is an alternative algorithm known as **dual simplex**:

  starting with optimal basis (= satisfying optimality conds.),

  find feasible basis while keeping optimality
Basic Idea

\[
\begin{aligned}
\min & \quad -x - y \\
2x + y & \geq 3 \\
2x + y & \leq 6 \\
x + 2y & \leq 6 \\
x & \geq 0 \\
y & \geq 0
\end{aligned}
\]

\[
\begin{aligned}
\min & \quad -x - y \\
2x + y - s_1 & = 3 \\
2x + y + s_2 & = 6 \\
x + 2y + s_3 & = 6 \\
x, y, s_1, s_2, s_3 & \geq 0
\end{aligned}
\]

\[
\begin{aligned}
\min & \quad -x - y \\
x = 6 - 2y - s_3 \\
s_1 = 9 - 3y - 2s_3 \\
s_2 = -6 + 3y + 2s_3
\end{aligned}
\]

Basis \((x, s_1, s_2)\) is optimal
\((= \text{satisfies optimality conditions})\)
but is not feasible!
Basic Idea

\[
\begin{align*}
-2x - y &\leq 6 \\
2x + y &\geq 3 \\
x \geq 0 \\
y \geq 0 \\
(6, 0)
\end{align*}
\]
Basic Idea

Let us make a violating basic variable non-negative ...

- Increase $s_2$ by making it non-basic: then it will be 0

... while preserving optimality (= optimality conditions are satisfied)

- If $y$ replaces $s_2$ in the basis,
  then $y = \frac{1}{3}(s_2 + 6 - 2s_3), -x - y = -4 + \frac{1}{3}(s_2 + s_3)$

- If $s_3$ replaces $s_2$ in the basis,
  then $s_3 = \frac{1}{2}(s_2 + 6 - 3y), -x - y = -3 + \frac{1}{2}(s_2 - y)$
Basic Idea

- Let us make a violating basic variable non-negative ...
  - Increase $s_2$ by making it non-basic: then it will be 0

- ... while preserving optimality (= optimality conditions are satisfied)
  - If $y$ replaces $s_2$ in the basis, then $y = \frac{1}{3}(s_2 + 6 - 2s_3), -x - y = -4 + \frac{1}{3}(s_2 + s_3)$
  - If $s_3$ replaces $s_2$ in the basis, then $s_3 = \frac{1}{2}(s_2 + 6 - 3y), -x - y = -3 + \frac{1}{2}(s_2 - y)$
  - To preserve optimality, $y$ must replace $s_2$
Basic Idea

\[
\begin{aligned}
\min & \quad -6 + y + s_3 \\
x &= 6 - 2y - s_3 \\
s_1 &= 9 - 3y - 2s_3 \\
s_2 &= -6 + 3y + 2s_3
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\min & \quad -4 + \frac{1}{3}s_2 + \frac{1}{3}s_3 \\
x &= 2 - \frac{2}{3}s_2 + \frac{1}{3}s_3 \\
y &= 2 + \frac{1}{3}s_2 - \frac{2}{3}s_3 \\
s_1 &= 3 - s_2
\end{aligned}
\]
Basic Idea

\[
\begin{aligned}
&\min -6 + y + s_3 \\
x &= 6 - 2y - s_3 \\
s_1 &= 9 - 3y - 2s_3 \\
s_2 &= -6 + 3y + 2s_3
\end{aligned}
\quad \implies \quad
\begin{aligned}
&\min -4 + \frac{1}{3}s_2 + \frac{1}{3}s_3 \\
x &= 2 - \frac{2}{3}s_2 + \frac{1}{3}s_3 \\
y &= 2 + \frac{1}{3}s_2 - \frac{2}{3}s_3 \\
s_1 &= 3 - s_2
\end{aligned}
\]

- Current basis is feasible and optimal!
Basic Idea

\[\begin{align*}
2x + y &\leq 6 \\
2x + y &\geq 3 \\
x &\geq 0 \\
y &\geq 0
\end{align*}\]

\[\begin{align*}
\min -x - y
\end{align*}\]
Outline of the Dual Simplex

1. Initialization: Pick an optimal basis.

2. Dual Pricing: If all basic values are $\geq 0$, then return OPTIMAL. Else pick a basic variable with value $< 0$.

3. Dual Ratio test: Find non-basic variable for swapping while preserving optimality, i.e., non-negativity constraints on reduced costs.
   
   If it does not exist, then return INFEASIBLE. Else swap chosen non-basic variable with violating basic variable.

4. Update: Update the tableau and go to 2.
Duality

- To understand better how the dual simplex works: theory of duality
- We can get lower bounds on LP optimum value by adding constraints in a convenient way

\[
\begin{align*}
\min & \quad -x - y \\
2x + y & \geq 3 \\
2x + y & \leq 6 \\
x + 2y & \leq 6 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\Rightarrow
\begin{align*}
\min & \quad -x - y \\
2x + y & \geq 3 \\
-2x - y & \geq -6 \\
-x - 2y & \geq -6 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

\[
\begin{align*}
-x - 2y & \geq -6 \\
y & \geq 0
\end{align*}
\]

\[
-x - y \geq -6
\]
Duality

- In general we can get **lower bounds** on LP optimum value by linearly combining **constraints** with convenient **multipliers**

\[
\begin{align*}
\min \ -x - y \\
2x + y &\geq 3 \\
-2x - y &\geq -6 \\
-x - 2y &\geq -6 \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]

\[
1 \cdot (2x + y \geq 3)
\]

\[
2 \cdot (-2x - y \geq -6)
\]

\[
1 \cdot (x \geq 0)
\]

\[
\begin{align*}
2x + y &\geq 3 \\
-4x - 2y &\geq -12 \\
x &\geq 0
\end{align*}
\]

\[
-x - y \geq -9
\]

- There may be different choices, each giving a different lower bound
Duality

In general:

\[
\begin{align*}
\min &\quad -x - y \\
\text{s.t.} &\quad 2x + y \geq 3 \\
&\quad -2x - y \geq -6 \\
&\quad -x - 2y \geq -6 \\
&\quad x \geq 0 \\
&\quad y \geq 0
\end{align*}
\]

\[
\begin{align*}
\mu_1 \cdot (2x + y) &\geq 3 \\
\mu_2 \cdot (-2x - y) &\geq -6 \\
\mu_3 \cdot (-x - 2y) &\geq -6 \\
\mu_4 \cdot x &\geq 0 \\
\mu_5 \cdot y &\geq 0
\end{align*}
\]

If \( \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0, \mu_5 \geq 0 \),
\[
2 \mu_1 - 2 \mu_2 - \mu_3 + \mu_4 = -1 \quad \text{and} \quad \mu_1 - \mu_2 - 2 \mu_3 + \mu_5 = -1
\]
then \( 3 \mu_1 - 6 \mu_2 - 6 \mu_3 \) is a lower bound.
Duality

- We can skip the multipliers of the non-negativity constraints

- We have:

\[
\begin{align*}
\min & \quad -x - y \\
2x + y & \geq 3 \\
-2x - y & \geq -6 \\
-x - 2y & \geq -6 \\
x & \geq 0 \\
y & \geq 0
\end{align*}
\]

\[
\begin{align*}
\mu_1 \cdot (2x + y) & \geq 3 \\
\mu_2 \cdot (-2x - y) & \geq -6 \\
\mu_3 \cdot (-x - 2y) & \geq -6
\end{align*}
\]

\[
(2\mu_1 - 2\mu_2 - \mu_3) x + (\mu_1 - \mu_2 - 2\mu_3) y \geq 3\mu_1 - 6\mu_2 - 6\mu_3
\]

- In the coefficient of \( x \) we can “complete” \( 2\mu_1 - 2\mu_2 - \mu_3 \) to reach \(-1\) by adding a suitable multiple of \( x \geq 0 \) (the multiplier will be the slack)

- If \( \mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \)

\[
2\mu_1 - 2\mu_2 - \mu_3 \leq -1 \text{ and } \mu_1 - \mu_2 - 2\mu_3 \leq -1
\]

then \( 3\mu_1 - 6\mu_2 - 6\mu_3 \) is a lower bound
Duality

- Best possible lower bound with this “trick” can be found by solving

\[
\begin{align*}
\max & \quad 3\mu_1 - 6\mu_2 - 6\mu_3 \\
2\mu_1 - 2\mu_2 - \mu_3 & \leq -1 \\
\mu_1 - \mu_2 - 2\mu_3 & \leq -1 \\
\mu_1, \mu_2, \mu_3 & \geq 0
\end{align*}
\]

- How far will it be from the optimum?
Duality

- Best possible lower bound with this “trick” can be found by solving

\[
\begin{aligned}
\text{max} & \quad 3\mu_1 - 6\mu_2 - 6\mu_3 \\
2\mu_1 - 2\mu_2 - \mu_3 & \leq -1 \\
\mu_1 - \mu_2 - 2\mu_3 & \leq -1 \\
\mu_1, \mu_2, \mu_3 & \geq 0
\end{aligned}
\]

- How far will it be from the optimum?

- A best solution is given by \((\mu_1, \mu_2, \mu_3) = (0, \frac{1}{3}, \frac{1}{3})\)

\[
\begin{aligned}
0 \cdot (2x + y & \geq 3) \\
\frac{1}{3} \cdot (-2x - y & \geq -6) \\
\frac{1}{3} \cdot (-x - 2y & \geq -6)
\end{aligned}
\]

Matches the optimum!

\[
-x - y \geq -4
\]
Dual Problem

- Given an LP (called **primal**)

\[
\begin{align*}
\text{min} \quad & c^T x \\
Ax & \geq b \\
x & \geq 0
\end{align*}
\]

its **dual** is the LP

\[
\begin{align*}
\text{max} \quad & b^T y \\
A^T y & \leq c \\
y & \geq 0
\end{align*}
\]

- Primal variables associated with columns of \( A \)
- Dual variables (**multipliers**) associated with rows of \( A \)
- Objective and right-hand side vectors swap their roles
Prop. The dual of the dual is the primal.

Proof:

\[
\begin{align*}
\max b^T y & \quad - \min ( -b )^T y \\
A^T y \leq c & \quad - A^T y \geq - c \\
y \geq 0 & \quad y \geq 0
\end{align*}
\]

\[
\begin{align*}
- \max - c^T x & \quad \min c^T x \\
(-A^T)^T x \leq - b & \quad A x \geq b \\
x \geq 0 & \quad x \geq 0
\end{align*}
\]

We say the primal and the dual form a primal-dual pair.
Dual Problem

Prop. \[ \begin{align*}
\min & \quad c^T x \\
Ax &= b \\
x &\geq 0
\end{align*} \quad \text{and} \quad \begin{align*}
\max & \quad b^T y \\
A^T y &\leq c
\end{align*} \]

form a primal-dual pair

Proof:

\[ \begin{align*}
\min & \quad c^T x \\
Ax &= b \\
x &\geq 0 \\
\quad \Rightarrow \quad & \quad \min c^T x \\
Ax &\geq b \\
-Ax &\geq -b \\
x &\geq 0
\end{align*} \]

\[ \begin{align*}
\max & \quad b^T y_1 - b^T y_2 \\
A^T y_1 - A^T y_2 &\leq c \\
y_1, y_2 &\geq 0 \\
\quad \Rightarrow \quad & \quad \max b^T y \\
y_1 - y_2 &:= y \\
A^T y &\leq c
\end{align*} \]
Duality Theorems

- **Th. (Weak Duality)** Let $(P, D)$ be a primal-dual pair

\[
\begin{align*}
(P) & \quad \min c^T x \\
& \quad Ax = b \\
& \quad x \geq 0 \\
(D) & \quad \max b^T y \\
& \quad A^T y \leq c
\end{align*}
\]

If $x$ is feasible solution to $P$ and $y$ is feasible solution to $D$ then $b^T y \leq c^T x$

**Proof:**

\[c - A^T y \geq 0, \text{ i.e., } c^T - y^T A \geq 0, \text{ and } x \geq 0 \text{ imply } c^T x - y^T Ax \geq 0.\]

So $c^T x \geq y^T Ax$, and

\[b^T y = y^T b = y^T Ax \leq c^T x\]
Duality Theorems

- Feasible solutions to $D$ give lower bounds on $P$
- Feasible solutions to $P$ give upper bounds on $D$
- Will the two optimum values be always equal?
Duality Theorems

- Feasible solutions to $D$ give lower bounds on $P$
- Feasible solutions to $P$ give upper bounds on $D$
- Will the two optimum values be always equal?

**Th. (Strong Duality)** Let $(P, D)$ be a primal-dual pair

\[
\begin{align*}
(P) \quad & \min c^T x \\
& Ax = b \\
& x \geq 0 \\
(D) \quad & \max b^T y \\
& A^T y \leq c
\end{align*}
\]

If any of $P$ or $D$ has a feasible solution and a finite optimum then the same holds for the other problem and the two optimum values are equal.
Proof (Th. of Strong Duality):

By symmetry it is sufficient to prove only one direction. Wlog. let us assume $P$ is feasible with finite optimum.
Proof (Th. of Strong Duality):

By symmetry it is sufficient to prove only one direction. Wlog. let us assume $P$ is feasible with finite optimum. After executing the Simplex algorithm to $P$ we find $B$ optimal feasible basis. Then:

- $c_B^T B^{-1} a_j \leq c_j$ for all $j \in \mathcal{R}$ (optimality conds hold)
- $c_B^T B^{-1} a_j = c_j$ for all $j \in \mathcal{B}$

So $\pi^T := c_B^T B^{-1}$ is dual feasible: $\pi^T A \leq c^T$, i.e. $A^T \pi \leq c$. 
Duality Theorems

■ Proof (Th. of Strong Duality):

By symmetry it is sufficient to prove only one direction. Wlog. let us assume $P$ is feasible with finite optimum.

After executing the Simplex algorithm to $P$ we find $B$ optimal feasible basis. Then:

◆ $c_B^T B^{-1} a_j \leq c_j$ for all $j \in \mathcal{R}$ (optimality conds hold)
◆ $c_B^T B^{-1} a_j = c_j$ for all $j \in \mathcal{B}$

So $\pi^T := c_B^T B^{-1}$ is dual feasible: $\pi^T A \leq c^T$, i.e. $A^T \pi \leq c$.

Moreover, $c_B^T \beta = c_B^T B^{-1} b = \pi^T b = b^T \pi$

By the theorem of weak duality, $\pi$ is optimum for $D$
Duality Theorems

- **Proof (Th. of Strong Duality):**
  By symmetry it is sufficient to prove only one direction. Wlog. let us assume \( P \) is feasible with finite optimum.
  After executing the Simplex algorithm to \( P \) we find \( B \) optimal feasible basis. Then:

  ◆ \( c_B^T B^{-1} a_j \leq c_j \) for all \( j \in \mathcal{R} \)  (optimality conds hold)

  ◆ \( c_B^T B^{-1} a_j = c_j \) for all \( j \in \mathcal{B} \)

  So \( \pi^T := c_B^T B^{-1} \) is dual feasible: \( \pi^T A \leq c \), i.e. \( A^T \pi \leq c \).

  Moreover, \( c_B^T \beta = c_B^T B^{-1} b = \pi^T b = b^T \pi \)

  By the theorem of weak duality, \( \pi \) is optimum for \( D \)

- If \( B \) is an optimal feasible basis for \( P \),
  then simplex multipliers \( \pi^T := c_B^T B^{-1} \) are optimal feasible solution for \( D \)

- We can solve the dual by applying the simplex algorithm on the primal

- We can solve the primal by applying the simplex algorithm on the dual
Prop. Let \((P, D)\) be a primal-dual pair

\[
\begin{align*}
(P) \quad & \min c^T x \\
& Ax = b \\
& x \geq 0 \\
(D) \quad & \max b^T y \\
& A^T y \leq c
\end{align*}
\]

(1) If \(P\) has a feasible solution but is unbounded, then \(D\) is infeasible

(2) If \(D\) has a feasible solution but is unbounded, then \(P\) is infeasible

Proof:

Let us prove (1) by contradiction.

If \(y\) were a feasible solution to \(D\), by the weak duality theorem, objective of \(P\) would be lower bounded!

(2) is proved by duality.
Duality Theorems

Prop. Let \((P, D)\) be a primal-dual pair

\[
\begin{align*}
(P) & : \min c^T x \\
& \text{subject to } Ax = b \\
& \quad \quad x \geq 0 \\
(D) & : \max b^T y \\
& \text{subject to } A^T y \leq c 
\end{align*}
\]

(1) If \(P\) has a feasible solution but is unbounded, then \(D\) is infeasible

(2) If \(D\) has a feasible solution but is unbounded, then \(P\) is infeasible

And the converse?
Does infeasibility of one imply unboundedness of the other?
Duality Theorems

Prop. Let \((P, D)\) be a primal-dual pair

\[
\begin{align*}
(P) & \quad \min c^T x \\
& \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
(D) & \quad \max b^T y \\
& \quad A^T y \leq c
\end{align*}
\]

(1) If \(P\) has a feasible solution but is unbounded, then \(D\) is infeasible

(2) If \(D\) has a feasible solution but is unbounded, then \(P\) is infeasible

And the converse? Does infeasibility of one imply unboundedness of the other?

\[
\begin{align*}
\min & \quad 3x_1 + 5x_2 \\
& \quad x_1 + 2x_2 = 3 \\
& \quad 2x_1 + 4x_2 = 1 \\
& \quad x_1, x_2 \text{ free}
\end{align*}
\]

\[
\begin{align*}
\max & \quad 3y_1 + y_2 \\
& \quad y_1 + 2y_2 = 3 \\
& \quad 2y_1 + 4y_2 = 5 \\
& \quad y_1, y_2 \text{ free}
\end{align*}
\]
### Duality Theorems

<table>
<thead>
<tr>
<th>Condition</th>
<th>Implication</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primal unbounded</td>
<td>$\implies$ Dual infeasible</td>
</tr>
<tr>
<td>Dual unbounded</td>
<td>$\implies$ Primal infeasible</td>
</tr>
<tr>
<td>Primal infeasible</td>
<td>$\implies$ Dual { infeasible, unbounded }</td>
</tr>
<tr>
<td>Dual infeasible</td>
<td>$\implies$ Primal { infeasible, unbounded }</td>
</tr>
</tbody>
</table>
Consider a primal-dual pair of the form

\[
\begin{align*}
\min & \quad c^T x \\
\text{subject to} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\quad \text{and} \quad
\begin{align*}
\max & \quad b^T y \\
\text{subject to} & \quad A^T y \leq c \\
& \quad \text{and} \quad w \geq 0
\end{align*}
\]

Karush-Kuhn-Tucker (KKT) optimality conditions are

- \( Ax = b \)
- \( A^T y + w = c \) (complementary slackness)
- \( x, w \geq 0 \)
- \( x^T w = 0 \)

They are necessary and sufficient conditions for optimality of the pair of primal-dual solutions \((x, (y, w))\)

Used, e.g., as a test of quality in LP solvers

\[
\begin{align*}
\min & \quad c^T x \\
\text{(P)} & \quad Ax = b \
\quad x \geq 0 \\
\max & \quad b^T y \\
\text{(D)} & \quad A^T y + w = c \\
\quad w \geq 0
\end{align*}
\]

\[\begin{align*}
\text{(KKT)}
\quad & \bullet Ax = b \\
\quad & \bullet A^T y + w = c \\
\quad & \bullet x, w \geq 0 \\
\quad & \bullet x^T w = 0
\end{align*}\]

**Th.** \((x, (y, w))\) is solution to **KKT** iff
\(x\) optimal solution to **P** and \((y, w)\) optimal solution to **D**

**Proof:**

\[0 = x^T w = x^T (c - A^T y) = c^T x - b^T y, \text{ and Weak Duality}\]

\[x \text{ is feasible solution to **P**, } (y, w) \text{ is feasible solution to **D}.\]

By Strong Duality \(x^T w = x^T (c - A^T y) = c^T x - b^T y = 0\)
as both solutions are optimal
Consider a primal-dual pair of the form

\[ \begin{align*}
\min_z \ z &= c^T x \\
(P) \ Ax &= b \\
x &\geq 0
\end{align*} \]

\[ \begin{align*}
\max_Z \ Z &= b^T y \\
(D) \ A^T y + w &= c \\
w &\geq 0
\end{align*} \]

Let us denote by $a_1, \ldots, a_n$ the columns of $A$, i.e., $A = (a_1, \ldots, a_n)$.

Let $B$ be a basis of $P$. Let us see how we can get a basis of $D$.

Assume that the basic variables are the first $m$: $B = (a_1, \ldots, a_m)$. Then $R = (a_{m+1}, \ldots, a_n)$.

If slacks $w$ are split into $w_B^T = (w_1, \ldots, w_m)$, $w_R^T = (w_{m+1}, \ldots, w_n)$, then

\[
A^T y + w = \begin{pmatrix}
    a^T_1 y \\
    \vdots \\
    a^T_m y \\
    a^T_{m+1} y \\
    \vdots \\
    a^T_n y
\end{pmatrix} + \begin{pmatrix}
    w_1 \\
    \vdots \\
    w_m \\
    w_{m+1} \\
    \vdots \\
    w_n
\end{pmatrix} = \begin{pmatrix}
    B^T y + w_B \\
    R^T y + w_R
\end{pmatrix}
\]
Relating Bases

■ Hence we have

\[ A^T y + w = \left( \begin{array}{c} B^T y + w_B \\ R^T y + w_R \end{array} \right) \]

■ Then the matrix of the system in the dual problem \( D \) is

\[
\begin{pmatrix}
B^T & I & 0 \\
R^T & 0 & I
\end{pmatrix}
\begin{pmatrix}
y \\
w_B \\
w_R
\end{pmatrix}
\]

■ Now let us consider the submatrix of vars \( y \) and vars \( w_R \):

\[ \hat{B} = \begin{pmatrix}
B^T \\
R^T
\end{pmatrix}
\begin{pmatrix}
0 \\
I
\end{pmatrix} \]

■ Note \( \hat{B} \) is a square \( n \times n \) matrix
Relating Bases

- Dual variables $\hat{B} = (y, w_\mathcal{R})$ determine a basis of $D$:

$$\hat{B} = \left( \begin{array}{c|c} B^T & 0 \\ \hline R^T & I \end{array} \right)$$

$$\hat{B}^{-1} = \left( \begin{array}{c|c} B^{-T} & 0 \\ \hline -R^T B^{-T} & I \end{array} \right)$$
Relating Bases

- Dual variables $\hat{\mathcal{B}} = (y, w_\mathcal{R})$ determine a basis of $D$:

$$
\hat{B} = \begin{pmatrix}
B^T & 0 \\
R^T & I
\end{pmatrix}
$$

$$
\hat{B}^{-1} = \begin{pmatrix}
B^{-T} & 0 \\
-R^T B^{-T} & I
\end{pmatrix}
$$

- In the next slides we answer the following questions:

1. If basis $\hat{\mathcal{B}}$ of the dual $D$ is feasible, what can we say about basis $\mathcal{B}$ of the primal $P$?

2. If basis $\hat{\mathcal{B}}$ of the dual $D$ is optimal (satisfies the optimality conds.), what can we say about basis $\mathcal{B}$ of the primal $P$?

3. If we apply the simplex algorithm to the dual $D$ using basis $\hat{\mathcal{B}}$, how does that translate into the primal $P$ and its basis $\mathcal{B}$?
Relating Bases

- Dual variables $\hat{B} = (y, w_\mathcal{R})$ determine a basis of $D$:

$$\hat{B} = \begin{pmatrix} B^T & 0 \\ R^T & I \end{pmatrix}$$

$$\hat{B}^{-1} = \begin{pmatrix} B^{-T} & 0 \\ -R^T B^{-T} & I \end{pmatrix}$$

- In the next slides we answer the following questions:
  1. If basis $\hat{B}$ of the dual $D$ is feasible, what can we say about basis $B$ of the primal $P$?
  2. If basis $\hat{B}$ of the dual $D$ is optimal (satisfies the optimality conds.), what can we say about basis $B$ of the primal $P$?
  3. If we apply the simplex algorithm to the dual $D$ using basis $\hat{B}$, how does that translate into the primal $P$ and its basis $B$?

- Recall that each variable $w_j$ in $D$ is associated to a variable $x_j$ in $P$.
- Note that $w_j$ is $\hat{B}$-basic iff $x_j$ is not $B$-basic
Dual Feasibility = Primal Optimality

- If $\hat{B}$ is feasible for dual $D$, what about $B$ in primal $P$?

$$\hat{B}^{-1}c = \begin{pmatrix} B^{-T} & 0 \\ -R^TB^{-T} & I \end{pmatrix} \begin{pmatrix} c_B \\ c_R \end{pmatrix} = \begin{pmatrix} B^{-T}c_B \\ -R^TB^{-T}c_B + c_R \end{pmatrix}$$

- There is no restriction on the sign of $y_1, \ldots, y_m$
- Variables $w_j$ have to be non-negative. But

$$-R^TB^{-T}c_B + c_R \geq 0 \iff c_R^T - c_B^TB^{-1}R \geq 0$$
Dual Feasibility $=$ Primal Optimality

- If $\hat{B}$ is feasible for dual $D$, what about $B$ in primal $P$?

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$$-R^T B^{-T}c_B + c_R \geq 0 \iff c_R^T c_B B^{-1} R \geq 0 \iff d_R^T \geq 0$$
Dual Feasibility = Primal Optimality

- If \( \hat{B} \) is feasible for dual \( D \), what about \( B \) in primal \( P \)?

\[
\hat{B}^{-1}c = \begin{pmatrix}
  B^{-T} & 0 \\
  -R^T B^{-T} & I
\end{pmatrix}
\begin{pmatrix}
  c_B \\
  c_R
\end{pmatrix}
= \begin{pmatrix}
  B^{-T}c_B \\
  -R^T B^{-T}c_B + c_R
\end{pmatrix}
\]

- There is no restriction on the sign of \( y_1, \ldots, y_m \)
- Variables \( w_j \) have to be non-negative. But

\[
-R^T B^{-T}c_B + c_R \geq 0 \iff c_R^T - c_B^T B^{-1}R \geq 0 \iff d_R^T \geq 0
\]

- \( \hat{B} \) is dual feasible iff \( d_j \geq 0 \) for all \( j \in R \)
- Dual feasibility is primal optimality!
Dual Optimality = Primal Feasibility

- If \( \hat{B} \) satisfies the optimality conds. for dual \( D \), what about \( B \) in primal \( P \)?
- Non \( \hat{B} \)-basic vars: \( w_B \) with costs \((0)\)
- \( \hat{B} \)-basic vars: \((y \mid w_R)\) with costs \((b^T \mid 0)\)
- Matrix of non \( \hat{B} \)-basic vars: \( \begin{pmatrix} I \\ 0 \end{pmatrix} \)
- Optimality condition: \( 0 \geq \) reduced costs (maximization!)

\[
0 \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b^T \\ 0 \end{pmatrix} \begin{pmatrix} B^{-T} \\ -R^T B^{-T} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b^T B^{-T} \\ 0 \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} = -b^T B^{-T} = -(B^{-1}b)^T
\]
Dual Optimality = Primal Feasibility

- If $\hat{B}$ satisfies the optimality conds. for dual $D$, what about $B$ in primal $P$?
- Non $\hat{B}$-basic vars: $w_B$ with costs $(0)$
- $\hat{B}$-basic vars: $(y \mid w_R)$ with costs $(b^T \mid 0)$
- Matrix of non $\hat{B}$-basic vars: $\begin{pmatrix} I \\ 0 \end{pmatrix}$
- Optimality condition: $0 \geq$ reduced costs (maximization!)

\[
0 \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b^T \\ 0 \end{pmatrix} \begin{pmatrix} B^{-T} & 0 \\ -R^T B^{-T} & I \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} =
\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b^T B^{-T} \\ 0 \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} = -b^T B^{-T} = -(B^{-1}b)^T = -\beta^T \quad \text{iff} \quad \beta \geq 0
\]
Dual Optimality = Primal Feasibility

- If \( \hat{B} \) satisfies the optimality conds. for dual \( D \), what about \( B \) in primal \( P \)?
- Non \( \hat{B} \)-basic vars: \( w_B \) with costs \((0)\)
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0 \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b^T \mid 0 \end{pmatrix} \begin{pmatrix} \frac{B^{-T}}{-R^T B^{-T}} & 0 \\ I \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} b^T B^{-T} \mid 0 \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} = -b^T B^{-T} = -(B^{-1}b)^T = -\beta^T \quad \text{iff} \quad \beta \geq 0
\]

- In the dual problem,
  for all \( 1 \leq p \leq m \), var \( w_{kp} \) cannot improve objective function iff \( \beta_p \geq 0 \)
- Dual optimality is primal feasibility!
Improving a Non-Optimal Solution

- Next we apply the simplex algorithm to basis $\hat{B}$ in the dual problem $D$ and translate it to the primal problem $P$.

- Let $p$ (where $1 \leq p \leq m$) be such that $\beta_p < 0$. I.e., the reduced cost of non-basic dual variable $w_{kp}$ is positive.
So by giving $w_{kp}$ a larger value we can improve the dual objective value. If $w_{kp}$ takes value $t \geq 0$:

$$
\begin{pmatrix}
    y(t) \\
    w_\mathcal{R}(t)
\end{pmatrix} = \hat{B}^{-1}c - \hat{B}^{-1}te_p =
$$

$$
= \begin{pmatrix}
    B^{-T}c_B \\
    d_\mathcal{R}
\end{pmatrix} - \begin{pmatrix}
    B^{-T} \\
    -R^TB^{-T}
\end{pmatrix} \begin{pmatrix}
    0 \\
    I
\end{pmatrix} \begin{pmatrix}
    te_p \\
    0
\end{pmatrix} = \begin{pmatrix}
    B^{-T}c_B - tB^{-T}e_p \\
    d_\mathcal{R} + tR^TB^{-T}e_p
\end{pmatrix}
$$

- Dual objective value improvement is

$$
\Delta Z = b^Ty(t) - b^Ty(0) = -tb^TB^{-T}e_p = -t\beta^Te_p = -t\beta_p = t(-\beta_p)
$$
Improving a Non-Optimal Solution

- Of all basic dual variables, only $w_\mathcal{R}$ variables need to be $\geq 0$

- For $j \in \mathcal{R}$

$$w_j(t) = d_j + t\alpha^T_j B^{-T}e_p = d_j + t\alpha^p_j$$

where $\alpha^p_j$ is the $p$-th component of $\alpha_j$. Hence:

$$w_j(t) \geq 0 \iff d_j + t\alpha^p_j \geq 0$$

- If $\alpha^p_j \geq 0$ the constraint is satisfied for all $t \geq 0$

- If $\alpha^p_j < 0$ we need $\frac{d_j}{-\alpha^p_j} \geq t$

- **Best improvement** achieved with

$$\Theta_D := \min\{\frac{d_j}{-\alpha^p_j} \mid \alpha^p_j < 0\}$$

- Variable $w_q$ is **blocking** when $\Theta_D = \frac{d_q}{-\alpha^p_q}$
Improving a Non-Optimal Solution

1. If $\Theta_D = +\infty$ (there is no $j \in \mathcal{R}$ such that $\alpha_j^p < 0$):
   
   Value of dual objective can be increased infinitely.

   Dual LP is unbounded.
   Primal LP is infeasible.

2. If $\Theta_D < +\infty$ and $w_q$ is blocking:

   When setting $w_{kp} = \Theta_D$,
   non-negativity constraints of basic vars of dual are respected

   In particular, $w_q(\Theta_D) = d_q + \Theta_D \alpha_q^p = d_q + \left(\frac{d_q}{-\alpha_q^p}\right)\alpha_q^p = 0$

   We can make a basis change:
   
   - In dual: $w_{kp}$ enters $\hat{\mathcal{B}}$ and $w_q$ leaves
   - In primal: $x_{kp}$ leaves $\mathcal{B}$ and $x_q$ enters
Update

- We do **not** actually need to form the dual LP: it is **enough** to have a representation of the primal LP

- **New basic indices:** \( \overline{B} = (k_1, \ldots, k_{p-1}, q, k_{p+1} \ldots, k_m) \)

- **New dual objective value:** \( \overline{Z} = Z - \Theta_D \beta_p \)

- **New dual basic sol:** \( \overline{y} = y - \Theta_D \rho_p \)
  \( \overline{d}_j = d_j + \Theta_D \alpha_j^p \) if \( j \in \mathcal{R}, \overline{d}_{k_p} = \Theta_D \)

- **New primal basic sol:** \( \overline{\beta}_p = \Theta_P, \quad \overline{\beta}_i = \beta_i - \Theta_P \alpha_i^q \) if \( i \neq p \)
  where \( \Theta_P = \frac{\beta_p}{\alpha_q^p} \)

- **New basis inverse:** \( \overline{B}^{-1} = E \overline{B}^{-1} \)
  where \( E = (e_1, \ldots, e_{p-1}, \eta, e_{p+1}, \ldots, e_m) \) and
  \( \eta^T = \left( \left( \frac{-\alpha_1^p}{\alpha_q^p} \right), \ldots, \left( \frac{-\alpha_{p-1}^p}{\alpha_q^p} \right), \frac{1}{\alpha_q^p} \left( \frac{-\alpha_{p+1}^p}{\alpha_q^p} \right), \ldots, \left( \frac{-\alpha_m^p}{\alpha_q^p} \right) \right)^T \)
Algorithmic Description

1. **Initialization:** Find an initial dual feasible basis $B$
   
   Compute $B^{-1}$, $\beta = B^{-1}b$,
   
   $y^T = c_B^T B^{-1}$, $d_R^T = c_R^T - y^T R$, $Z = b^T y$

2. **Dual Pricing:**
   
   If for all $i \in B$, $\beta_i \geq 0$ then return **OPTIMAL**
   
   Else let $p$ be such that $\beta_p < 0$.
   
   Compute $\rho_p^T = e_p^T B^{-1}$ and $\alpha^p_j = \rho_p^T a_j$ for $j \in R$

3. **Dual Ratio test:** Compute $\mathcal{J} = \{j \mid j \in R, \alpha^p_j < 0\}$.
   
   If $\mathcal{J} = \emptyset$ then return **INFEASIBLE**
   
   Else compute $\Theta_D = \min_{j \in \mathcal{J}} \left( \frac{d_j}{-\alpha^p_j} \right)$ and $q$ st. $\Theta_D = \frac{d_q}{-\alpha^q_p}$
Algorithmic Description

4. Update:
\[
\bar{B} = B - \{k_p\} \cup \{q\} \\
\bar{Z} = Z - \Theta_D \beta_p
\]

**Dual solution**

\[
\bar{y} = y - \Theta_D \rho_p \\
\bar{d}_j = d_j + \Theta_D \alpha^p_j \quad \text{if } j \in \mathcal{R}, \quad \bar{d}_{kp} = \Theta_D
\]

**Primal solution**

Compute \( \alpha_q = B^{-1} a_q \) and \( \Theta_P = \frac{\beta_p}{\alpha^q_p} \)

\[
\bar{\beta}_p = \Theta_P, \quad \bar{\beta}_i = \beta_i - \Theta_P \alpha^i_q \quad \text{if } i \neq p
\]

\[\bar{B}^{-1} = EB^{-1}\]

Go to 2.
## Primal vs. Dual Simplex

<table>
<thead>
<tr>
<th>PRIMAL</th>
<th>DUAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Can handle <strong>bounds efficiently</strong></td>
<td>Can handle <strong>bounds efficiently</strong> (not explained here)</td>
</tr>
<tr>
<td>Many years of research and implementation</td>
<td>Developments in the 90’s made it an alternative</td>
</tr>
<tr>
<td>There are classes of LP’s for which it is the best</td>
<td>Nowadays on average it gives <strong>better performance</strong></td>
</tr>
<tr>
<td><strong>Not suitable</strong> for solving LP’s with <strong>integer</strong> variables</td>
<td><strong>Suitable</strong> for solving LP’s with <strong>integer</strong> variables</td>
</tr>
</tbody>
</table>