Basics on Linear Programming

Combinatorial Problem Solving (CPS)

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A linear program is an optimization problem of the form

\[
\begin{align*}
\text{min} & \quad c^T x \\
A_1 x & \leq b_1 \\
A_2 x & = b_2 \\
A_3 x & \geq b_3 \\
x & \in \mathbb{R}^n
\end{align*}
\]

- \(x\) is the vector of variables
- \(c^T x\) is the cost or objective function
- \(A_1 x \leq b_1, A_2 x = b_2\) and \(A_3 x \geq b_3\) are the constraints
Linear Programs (LP’s)

- A linear program is an optimization problem of the form

\[
\begin{align*}
    \min & \quad c^T x \\
    \text{subject to} & \quad A_1 x \leq b_1 \\
    & \quad A_2 x = b_2 \\
    & \quad A_3 x \geq b_3 \\
    & \quad x \in \mathbb{R}^n
\end{align*}
\]

\[c \in \mathbb{R}^n, \; b_i \in \mathbb{R}^{m_i}, \; A_i \in \mathbb{R}^{m_i \times n}, \; i = 1, 2, 3\]

- \(x\) is the vector of variables
- \(c^T x\) is the cost or objective function
- \(A_1 x \leq b_1, \; A_2 x = b_2\) and \(A_3 x \geq b_3\) are the constraints
- Example:

\[
\begin{align*}
    \min & \quad x + y + z \\
    x + y &= 3 \\
    0 &\leq x \leq 2 \\
    0 &\leq y \leq 2
\end{align*}
\]
Notes on the Definition of LP

- Solving minimization or maximization is equivalent:

\[
\max \{ f(x) \mid x \in S \} = -\min \{ -f(x) \mid x \in S \}
\]

- Satisfiability problems are a particular case: take arbitrary cost function, e.g., \( c = 0 \)
Equivalent Forms of LP’s

- This form is not the most convenient for algorithms

- We will assume problems to be in **canonical form**: 

\[
\min c^T x \\
Ax = b \\
x \geq 0
\]

\[
c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, n \geq m, \text{rank}(A) = m
\]

- Often **variables** are identified with **columns** of the matrix, and **constraints** are identified with **rows**
Methods for Solving LP’s

- Simplex algorithms
- Interior-point algorithms
Methods for Solving LP’s

- Simplex algorithms
- Interior-point algorithms
Basic Definitions (1)

\[
\begin{align*}
\min c^T x \\
Ax &= b \\
x &\geq 0
\end{align*}
\]

- Any vector \( x \) such that \( Ax = b \) is called a \textit{solution}
- A solution \( x \) satisfying \( x \geq 0 \) is called a \textit{feasible solution}
- An LP with feasible solutions is called \textit{feasible}; otherwise it is called \textit{infeasible}
- A feasible solution \( x^* \) is called \textit{optimal} if \( c^T x^* \leq c^T x \) for all feasible solution \( x \)
- A feasible LP with no optimal solution is \textit{unbounded}
Basic Definitions (2)

\[
\begin{align*}
\text{max } & x + 2y \\
\text{s.t. } & x + y + s_1 = 3 \\
& x + s_2 = 2 \\
& y + s_3 = 2 \\
& x, y, s_1, s_2, s_3 \geq 0
\end{align*}
\]

- \((x, y, s_1, s_2, s_3) = (-1, -1, 5, 3, 3)\) is a solution but not feasible
- \((x, y, s_1, s_2, s_3) = (1, 1, 1, 1, 1)\) is a feasible solution
Basic Definitions (3)

\[
\begin{align*}
\text{max } x + \beta y \\
x + y + s_1 &= \alpha \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

- If $\alpha = -1$ the LP is not feasible
- If $\alpha = 3, \beta = 2$ then $\text{(x, y, s_1, s_2, s_3)} = (1, 2, 0, 1, 0)$ is the (only) optimal solution
Basic Definitions (3)

\[
\begin{align*}
\max \quad & x + \beta y \\
\text{subject to} \quad & x + y + s_1 = \alpha \\
& x + s_2 = 2 \\
& y + s_3 = 2 \\
& x, y, s_1, s_2, s_3 \geq 0
\end{align*}
\]

- If \( \alpha = -1 \) the LP is not feasible
- If \( \alpha = 3, \beta = 2 \) then
  \( (x, y, s_1, s_2, s_3) = (1, 2, 0, 1, 0) \) is the (only) optimal solution
- There may be more than one optimal solution:
  - If \( \alpha = 3 \) and \( \beta = 1 \) then
    \( \{(1, 2, 0, 1, 0), (2, 1, 0, 0, 1), \left(\frac{3}{2}, \frac{3}{2}, 0, \frac{1}{2}, \frac{1}{2}\right)\} \) are optimal
Basic Definitions (4)

\[ \begin{align*}
    x + y &\leq 3 \\
    x + 2y &\leq 2 \\
    y &\leq 2 \\
    x &\geq 0 \\
    y &\geq 0 \\
    (1, 2) &\in \text{Region} \\
    (2, 1) &\in \text{Region} \\
\end{align*} \]
Basic Definitions (5)

\[
\begin{align*}
\min x + 2y \\
x + y &\leq 3 \\
0 &\leq x \leq 2 \\
y &\leq 2
\end{align*}
\]

Unbounded LP
Basic Definitions (6)

\[ \text{max } x + 2y \]

\[ x + y \leq 3 \]

\[ 0 \leq x \leq 2 \]

\[ y \leq 2 \]

LP is bounded, but set of feasible solutions is not
Let us denote by $a_1, \ldots, a_n$ the columns of $A$.

Recall that $n \geq m$, $\operatorname{rank}(A) = m$.

- A matrix of $m$ columns $(a_{k_1}, \ldots, a_{k_m})$ is a basis if the columns are linearly independent.

- Note that a basis is a square matrix!

- If $(a_{k_1}, \ldots, a_{k_m})$ is a basis, then the variables $(x_{k_1}, \ldots, x_{k_m})$ are called basic.

- We usually denote

  by $\mathcal{B}$ the list of indices $(k_1, \ldots, k_m)$, and

  by $\mathcal{R}$ the list of indices $(1, 2, \ldots, n) - \mathcal{B}$; and

  by $B$ the matrix $(a_i \mid i \in \mathcal{B})$, and

  by $R$ the matrix $(a_i \mid i \in \mathcal{R})$

  $x_B$ the basic variables, $x_\mathcal{R}$ the non-basic ones.
Bases (2)

max \(x + 2y\)
\[
\begin{align*}
x + y + s_1 &= 3 \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

\[
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

- \((x, s_1, s_2)\) do not form a basis:
  \[
  \begin{pmatrix}
  1 & 1 & 0 \\
  1 & 0 & 1 \\
  0 & 0 & 0 \\
  \end{pmatrix}
  \]
does not have linearly independent columns

- \((s_1, s_2, s_3)\) form a basis, where \(x_B = (s_1, s_2, s_3)\), \(x_R = (x, y)\)
  \[
  B = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  \end{pmatrix} \quad R = \begin{pmatrix}
  1 & 1 \\
  1 & 0 \\
  0 & 1 \\
  \end{pmatrix}
  \]
Bases (3)

- If $B$ is a basis, then the following holds

$$Bx_B + Rx_R = b$$

Hence:

$$x_B = B^{-1}b - B^{-1}Rx_R$$

Non-basic variables determine values of basic ones

- If non-basic variables are set to 0, we get the solution

$$x_R = 0, x_B = B^{-1}b$$

Such a solution is called a **basic** solution

- If a basic solution satisfies $x_B \geq 0$ then it is called a **basic feasible solution**, and the basis is feasible
Bases (4)

Consider basis \((s_1, s_2, s_3)\)

\[
\begin{align*}
\text{max} \quad & x + 2y \\
& x + y + s_1 = 3 \\
& x + s_2 = 2 \\
& y + s_3 = 2 \\
& x, y, s_1, s_2, s_3 \geq 0
\end{align*}
\]

\[
B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

Equations \(x_B = B^{-1}b - B^{-1}Rx_R\) are

\[
\begin{align*}
s_1 &= 3 - x - y \\
s_2 &= 2 - x \\
s_3 &= 2 - y
\end{align*}
\]

Basic solution is

\[
\sigma_B = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \quad \sigma_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

So basis \((s_1, s_2, s_3)\) is feasible.
Bases (5)

Basis \((x, y, s_1)\) is not feasible

\[
\begin{align*}
\text{max } x + 2y \\
x + y + s_1 &= 3 \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

\[
\begin{align*}
B &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & R &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
\begin{align*}
x &= 2 - s_2 \\
y &= 2 - s_3 \\
s_1 &= -1 + s_2 + s_3
\end{align*}
\]

\[
\begin{align*}
\sigma_B &= \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} & \sigma_R &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
A basis is called **degenerate** when at least one component of its basic solution $x_B$ is null.

For example:

\[
\begin{align*}
\text{max } & x + 2y \\
x + y + s_1 &= 4 \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

\[
B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
\begin{align*}
x &= 2 + s_3 - s_1 \\
y &= 2 - s_3 \\
s_2 &= s_1 - s_3
\end{align*}
\]

\[
\sigma_B = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}
\]
Geometry of LP’s (1)

- Set of feasible solutions of an LP is a *convex polyhedron*
- Basic feasible solutions are *vertices* of the convex polyhedron
Geometry of LP’s (2)

\[
\begin{align*}
\text{max } x + 2y \\
x + y + s_1 &= 3 \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

- \( x_{B_1} = (y, s_1, s_2) \)
- \( x_{B_2} = (x, y, s_2) \)
- \( x_{B_3} = (x, y, s_3) \)
- \( x_{B_4} = (x, s_1, s_3) \)
- \( x_{B_5} = (s_1, s_2, s_3) \)
Theorem (Minkowski-Weyl)

Let $P$ be a feasible LP.

There exist basic feasible solutions $v_1, ..., v_r \in \mathbb{R}^n$ and vectors $r_1, ..., r_s \in \mathbb{R}^n$ such that a point $x$ is a feasible solution to $P$ iff

$$x = \sum_{i=1}^{r} \lambda_i v_i + \sum_{j=1}^{s} \mu_j r_j$$

for certain $\lambda_i, \mu_j$ such that $\sum_{i=1}^{r} \lambda_i = 1$ and $\lambda_i, \mu_j \geq 0$. 
Theorem (Fundamental Theorem of Linear Programming)

Let $P$ be an LP.

Then exactly one of the following holds:

1. $P$ is infeasible
2. $P$ is unbounded
3. $P$ has an optimal basic feasible solution

It is sufficient to investigate basic feasible solutions!
Possible Outcomes of an LP (2)

Proof: Assume $P$ feasible and with optimal solution $x^*$.
Let us see we can find a basic feasible solution as good as $x^*$.

By Minkowski-Weyl theorem, we can write

$$x^* = \sum_{i=1}^{r} \lambda_i^* v_i + \sum_{j=1}^{s} \mu_j^* r_j$$

where $\sum_{i=1}^{r} \lambda_i^* = 1$ and $\lambda_i^*, \mu_j^* \geq 0$. Then

$$c^T x^* = \sum_{i=1}^{r} \lambda_i^* c^T v_i + \sum_{j=1}^{s} \mu_j^* c^T r_j$$
Possible Outcomes of an LP (2)

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$$c^T x^* = \sum_{i=1}^{r} \lambda_i^* c^T v_i + \sum_{j=1}^{s} \mu_j^* c^T r_j$$

- If there is $j$ such that $c^T r_j < 0$ then objective value can be decreased by taking $\mu_j^*$ larger. **Contradiction!**
Possible Outcomes of an LP (2)

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$$c^T x^* = \sum_{i=1}^{r} \lambda_i^* c^T v_i + \sum_{j=1}^{s} \mu_j^* c^T r_j$$

- If there is $j$ such that $c^T r_j < 0$ then objective value can be decreased by taking $\mu_j^*$ larger. Contradiction!

- Otherwise $c^T r_j \geq 0$ for all $j$. Assume $c^T x^* < c^T v_i$ for all $i$.

$$c^T x^* \geq \sum_{i=1}^{r} \lambda_i^* c^T v_i > \sum_{i=1}^{r} \lambda_i^* c^T x^* = c^T x^* \sum_{i=1}^{r} \lambda_i^* = c^T x^*$$
Possible Outcomes of an LP (2)

Proof: Assume \( P \) feasible and with optimal solution \( x^* \).

Let us see we can find a basic feasible solution as good as \( x^* \).

By Minkowski-Weyl theorem, we can write

\[
x^* = \sum_{i=1}^{r} \lambda_i^* v_i + \sum_{j=1}^{s} \mu_j^* r_j
\]

where \( \sum_{i=1}^{r} \lambda_i^* = 1 \) and \( \lambda_i^*, \mu_j^* \geq 0 \). Then

\[
c^T x^* = \sum_{i=1}^{r} \lambda_i^* c^T v_i + \sum_{j=1}^{s} \mu_j^* c^T r_j
\]

- If there is \( j \) such that \( c^T r_j < 0 \) then objective value can be decreased by taking \( \mu_j^* \) larger. **Contradiction!**

- Otherwise \( c^T r_j \geq 0 \) for all \( j \). Assume \( c^T x^* < c^T v_i \) for all \( i \).

\[
c^T x^* \geq \sum_{i=1}^{r} \lambda_i^* c^T v_i > \sum_{i=1}^{r} \lambda_i^* c^T x^* = c^T x^* \sum_{i=1}^{r} \lambda_i^* = c^T x^*
\]

**Contradiction!** Thus there is \( i \) such that \( c^T x^* \geq c^T v_i \); in fact, \( c^T x^* = c^T v_i \) by the optimality of \( x^* \).