Basics on Linear Programming

Combinatorial Problem Solving (CPS)

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A linear program is an optimization problem of the form

\[
\begin{align*}
\text{min } & \quad c^T x \\
A_1 x & \leq b_1 \\
A_2 x & = b_2 \\
A_3 x & \geq b_3 \\
x & \in \mathbb{R}^n
\end{align*}
\]

- \( x \) is the vector of variables
- \( c^T x \) is the cost or objective function
- \( A_1 x \leq b_1, A_2 x = b_2 \text{ and } A_3 x \geq b_3 \) are the constraints
Linear Programs (LP’s)

- A linear program is an optimization problem of the form

\[
\begin{align*}
\text{min } c^T x \\
A_1 x &\leq b_1 \\
A_2 x &= b_2 \\
A_3 x &\geq b_3 \\
x &\in \mathbb{R}^n
\end{align*}
\]

- \( c \in \mathbb{R}^n, b_i \in \mathbb{R}^{m_i}, A_i \in \mathbb{R}^{m_i \times n}, i = 1, 2, 3 \)

- \( x \) is the vector of variables

- \( c^T x \) is the cost or objective function

- \( A_1 x \leq b_1, A_2 x = b_2 \) and \( A_3 x \geq b_3 \) are the constraints

- Example:

\[
\begin{align*}
\text{min } x + y + z \\
x + y &= 3 \\
0 &\leq x \leq 2 \\
0 &\leq y \leq 2
\end{align*}
\]
Notes on the Definition of LP

- Solving minimization or maximization is equivalent:

\[
\max\{ f(x) \mid x \in S \} = -\min\{ -f(x) \mid x \in S \}
\]

- Satisfiability problems are a particular case:
take arbitrary cost function, e.g., \( c = 0 \)
Equivalent Forms of LP’s (1)

■ This form is not the most convenient for algorithms
  WLOG we can transform such a problem as follows

1. Split \( = \) constraints into \( \geq \) and \( \leq \) constraints

\[
\begin{align*}
\text{min } c^T x & \\
A_1x & \leq b_1 \\
A_2x & = b_2 \\
A_3x & \geq b_3
\end{align*}
\quad \Rightarrow \quad
\begin{align*}
\text{min } c^T x & \\
A_1x & \leq b_1 \\
A_2x & \leq b_2 \\
A_2x & \geq b_2 \\
A_3x & \geq b_3
\end{align*}
\]

Now all constraints are \( \leq \) or \( \geq \)
Equivalent Forms of LP’s (2)

Example of step 1.:

\[
\begin{align*}
\text{min } & \quad x + y + z \\
& x + y = 3 \\
& 0 \leq x \leq 2 \\
& 0 \leq y \leq 2
\end{align*}
\]

\[
\begin{align*}
\text{min } & \quad x + y + z \\
& x + y \leq 3 \\
& x + y \geq 3 \\
& 0 \leq x \leq 2 \\
& 0 \leq y \leq 2
\end{align*}
\]
Equivalent Forms of LP’s (3)

2. Transform $\geq$ constraints into $\leq$ constraints by multiplying by -1

$$\begin{align*}
\min c^T x & \quad \Rightarrow \quad \min c^T x \\
A_1 x & \leq b_1 \\
A_2 x & \geq b_2
\end{align*}$$

$$\begin{align*}
\min c^T x & \quad \Rightarrow \quad \min c^T x \\
A_1 x & \leq b_1 \\
-A_2 x & \leq -b_2
\end{align*}$$

Now all constraints are $\leq$
Equivalent Forms of LP’s (4)

Example of step 2.:

\[
\begin{align*}
\min \ x + y + z \\
x + y &\leq 3 \\
x + y &\geq 3 \\
0 &\leq x \leq 2 \\
0 &\leq y \leq 2 
\end{align*}
\implies
\begin{align*}
\min \ x + y + z \\
x + y &\leq 3 \\
-x - y &\leq -3 \\
0 &\leq x \leq 2 \\
0 &\leq y \leq 2
\end{align*}
\]
3. Replace variables $x$ by $y - z$, where $y, z$ are vectors of fresh variables, and add constraints $y \geq 0, z \geq 0$

\[
\begin{align*}
\min c^T x & \quad \implies \quad \min c^T y - c^T z \\
Ax \leq b & \quad \implies \quad Ay - Az \leq b \\
y, z \geq 0 &
\end{align*}
\]

Now all constraints are $\leq$ and all variables have to be $\geq 0$
Equivalent Forms of LP’s (6)

Actually only needed for variables which are not already non-negative. (in the example, only \( z \))

Example of step 3.:

\[
\begin{align*}
\text{min } x + y + z \\
x + y & \leq 3 \\
-x - y & \leq -3 \\
0 & \leq x \leq 2 \\
0 & \leq y \leq 2 \\
\end{align*}
\]

\[
\begin{align*}
\text{min } x + y + u - v \\
x + y & \leq 3 \\
-x - y & \leq -3 \\
0 & \leq x \leq 2 \\
0 & \leq y \leq 2 \\
u, v & \geq 0
\end{align*}
\]
4. Add a **slack** variable to each $\leq$ constraint to convert it into $=$

$$\begin{align*}
\min c^T x \\
Ax &\leq b \\
x &\geq 0
\end{align*} \quad \implies \quad \begin{align*}
\min c^T x \\
Ax + s & = b \\
x, s &\geq 0
\end{align*}$$

Now all constraints are $=$ and all variables have to be $\geq 0$
Equivalent Forms of LP’s (8)

Example of step 4.:

\[
\begin{align*}
\min \ & x + y + u - v \\
\ & x + y \leq 3 \\
\ & -x - y \leq -3 \\
\ & 0 \leq x \leq 2 \\
\ & 0 \leq y \leq 2 \\
\ & u, v \geq 0
\end{align*}
\]

\[
\begin{align*}
\min \ & x + y + u - v \\
\ & x + y + s_1 = 3 \\
\ & -x - y + s_2 = -3 \\
\ & x + s_3 = 2 \\
\ & y + s_4 = 2 \\
\ & x, y, u, v, s_1, s_2, s_3, s_4 \geq 0
\end{align*}
\]
Equivalent Forms of LP’s (9)

Altogether:

\[
\begin{align*}
\text{min } & \quad x + y + z \\
& \quad x + y = 3 \\
& \quad 0 \leq x \leq 2 \\
& \quad 0 \leq y \leq 2 \\
\end{align*}
\]

\[
\begin{align*}
\text{min } & \quad x + y + u - v \\
x + y + s_1 &= 3 \\
-x - y + s_2 &= -3 \\
x + s_3 &= 2 \\
y + s_4 &= 2 \\
x, y, u, v, s_1, s_2, s_3, s_4 &\geq 0
\end{align*}
\]
In the end we get a problem in \textbf{standard form}:

\[
\begin{align*}
\min c^T x \\
Ax &= b \\
x &\geq 0
\end{align*}
\]

\[c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, n \geq m, \text{rank}(A) = m\]

These transformations are not strictly necessary (they increase no. of constraints and variables), but are convenient in a first formulation of the algorithms.

Often \textbf{variables} are identified with \textbf{columns} of the matrix, and \textbf{constraints} are identified with \textbf{rows}. 
Methods for Solving LP’s

- Simplex algorithms
- Interior-point algorithms
Methods for Solving LP’s

- Simplex algorithms
- Interior-point algorithms
Basic Definitions (1)

\[
\begin{align*}
\min c^T x \\
Ax &= b \\
x &\geq 0
\end{align*}
\]

- Any vector \( x \) such that \( Ax = b \) is called a solution
- A solution \( x \) satisfying \( x \geq 0 \) is called a feasible solution
- An LP with feasible solutions is called feasible; otherwise it is called infeasible
- A feasible solution \( x^* \) is called optimal if \( c^T x^* \leq c^T x \) for all feasible solution \( x \)
- A feasible LP with no optimal solution is unbounded
Basic Definitions (2)

\[
\begin{align*}
\text{max } x + 2y \\
x + y + s_1 &= 3 \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

- \((x, y, s_1, s_2, s_3) = (-1, -1, 5, 3, 3)\) is a solution but not feasible
- \((x, y, s_1, s_2, s_3) = (1, 1, 1, 1, 1)\) is a feasible solution
Basic Definitions (3)

\[ \begin{align*}
\text{max } x + \beta y \\
x + y + s_1 &= \alpha \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0
\end{align*} \]

- If \( \alpha = -1 \) the LP is not feasible
- If \( \alpha = 3, \beta = 2 \) then 
  \((x, y, s_1, s_2, s_3) = (1, 2, 0, 1, 0)\) is the (only) optimal solution
Basic Definitions (3)

\[
\text{max } x + \beta y \\
x + y + s_1 = \alpha \\
x + s_2 = 2 \\
y + s_3 = 2 \\
x, y, s_1, s_2, s_3 \geq 0
\]

- If \(\alpha = -1\) the LP is not feasible
- If \(\alpha = 3, \beta = 2\) then \((x, y, s_1, s_2, s_3) = (1, 2, 0, 1, 0)\) is the (only) optimal solution
- There may be more than one optimal solution: If \(\alpha = 3\) and \(\beta = 1\) then \(\{(1, 2, 0, 1, 0), (2, 1, 0, 0, 1), (\frac{3}{2}, \frac{3}{2}, 0, \frac{1}{2}, \frac{1}{2})\}\) are optimal
Basic Definitions (4)

\[
\begin{align*}
\max & \quad x + y \\
\text{s.t.} & \quad x + 2y & \leq 2 \\
& \quad x + y & \leq 3 \\
& \quad x, y & \geq 0
\end{align*}
\]

Graph showing the feasible region and the objective function. The feasible region is shaded, and the points (1, 2) and (2, 1) are marked.
Basic Definitions (5)

\[ \text{min } x + 2y \]
\[ x + y \leq 3 \]
\[ 0 \leq x \leq 2 \]
\[ y \leq 2 \]

Unbounded LP
Basic Definitions (6)

\[
\text{max } x + 2y \\
x + y \leq 3 \\
0 \leq x \leq 2 \\
y \leq 2
\]

LP is bounded, but set of feasible solutions is not
Bases (1)

Let us denote by $a_1, \ldots, a_n$ the columns of $A$.

Recall that $n \geq m$, $\text{rank}(A) = m$.

- A matrix of $m$ columns $(a_{k_1}, \ldots, a_{k_m})$ is a basis if the columns are linearly independent.

- Note that a basis is a square matrix!

- If $(a_{k_1}, \ldots, a_{k_m})$ is a basis, then the variables $(x_{k_1}, \ldots, x_{k_m})$ are called basic.

- We usually denote

  by $B$ the list of indices $(k_1, \ldots, k_m)$, and
  by $R$ the list of indices $(1, 2, \ldots, n) - B$; and
  
  by $B$ the matrix $(a_i \mid i \in B)$, and
  by $R$ the matrix $(a_i \mid i \in R)$

  $x_B$ the basic variables, $x_R$ the non-basic ones.
Bases (2)

\[
\begin{align*}
\text{max } x + 2y \\
x + y + s_1 &= 3 \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

\[
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1
\end{pmatrix}
\]

- \((x, s_1, s_2)\) do not form a basis:

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]
does not have linearly independent columns

- \((s_1, s_2, s_3)\) form a basis, where \(x_B = (s_1, s_2, s_3), x_R = (x, y)\)

\[
B = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
R = \begin{pmatrix}
1 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]
If $B$ is a basis, then the following holds

$$Bx_B + Rx_R = b$$

Hence:

$$x_B = B^{-1}b - B^{-1}Rx_R$$

Non-basic variables determine values of basic ones

If non-basic variables are set to 0, we get the solution

$$x_R = 0, x_B = B^{-1}b$$

Such a solution is called a basic solution

If a basic solution satisfies $x_B \geq 0$ then it is called a basic feasible solution, and the basis is feasible
Consider basis \((s_1, s_2, s_3)\)

\[
\begin{align*}
\text{max } x + 2y \\
x + y + s_1 &= 3 \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0
\end{align*}
\]

\[
B = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad R = \begin{pmatrix}
1 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]

Equations \(x_B = B^{-1}b - B^{-1}Rx_R\) are

\[
\begin{align*}
s_1 &= 3 - x - y \\
s_2 &= 2 - x \\
s_3 &= 2 - y
\end{align*}
\]

Basic solution is

\[
\sigma_B = \begin{pmatrix}
3 \\
2 \\
2
\end{pmatrix} \quad \sigma_R = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

So basis \((s_1, s_2, s_3)\) is feasible
Bases (5)

Basis \((x, y, s_1)\) is not feasible

\[
\begin{align*}
\text{max } & \quad x + 2y \\
& \quad x + y + s_1 = 3 \\
& \quad x + s_2 = 2 \\
& \quad y + s_3 = 2 \\
& \quad x, y, s_1, s_2, s_3 \geq 0 \\

\end{align*}
\]

\[
B = \begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} \quad R = \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
\sigma_B = \begin{pmatrix}
2 \\
2 \\
-1
\end{pmatrix} \quad \sigma_R = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[
\begin{align*}
x &= 2 - s_2 \\
y &= 2 - s_3 \\
s_1 &= -1 + s_2 + s_3
\end{align*}
\]
Bases (6)

A basis is called **degenerate**
when at least one component of its basic solution $x_B$ is null

For example:

$$\begin{align*}
\text{max } x + 2y \\
x + y + s_1 &= 4 \\
x + s_2 &= 2 \\
y + s_3 &= 2 \\
x, y, s_1, s_2, s_3 &\geq 0
\end{align*}$$

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\{ \\
x = 2 + s_3 - s_1 \\
y = 2 - s_3 \\
s_2 = s_1 - s_3 \\
\}$$

$$\sigma_B = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$
Geometry of LP’s (1)

- Set of feasible solutions of an LP is a **convex polyhedron**
- Basic feasible solutions are **vertices** of the convex polyhedron
Geometry of LP’s (2)

max \( x + 2y \)
\[ x + y + s_1 = 3 \]
\[ x + s_2 = 2 \]
\[ y + s_3 = 2 \]
\[ x, y, s_1, s_2, s_3 \geq 0 \]

- \( x_{B_1} = (y, s_1, s_2) \)
- \( x_{B_2} = (x, y, s_2) \)
- \( x_{B_3} = (x, y, s_3) \)
- \( x_{B_4} = (x, s_1, s_3) \)
- \( x_{B_5} = (s_1, s_2, s_3) \)
Geometry of LP’s (3)

- **Theorem (Minkowski-Weyl)**

Let $P$ be an LP.

A point $x$ is a feasible solution to $P$ iff there exist basic feasible solutions $v_1, \ldots, v_r \in \mathbb{R}^n$ and vectors $r_1, \ldots, r_s \in \mathbb{R}^n$ such that

$$x = \sum_{i=1}^{r} \lambda_i v_i + \sum_{j=1}^{s} \mu_j r_j$$

for certain $\lambda_i, \mu_j$ such that $\sum_{i=1}^{r} \lambda_i = 1$ and $\lambda_i, \mu_j \geq 0$. 
Possible Outcomes of an LP (1)

Theorem (Fundamental Theorem of Linear Programming)

Let $P$ be an LP.

Then exactly one of the following holds:

1. $P$ is infeasible
2. $P$ is unbounded
3. $P$ has an optimal basic feasible solution

It is sufficient to investigate basic feasible solutions!
Possible Outcomes of an LP (2)

**Proof:** Assume $P$ feasible and with optimal solution $x^*$. Let us see we can find a basic feasible solution as good as $x^*$.

By Minkowski-Weyl theorem, we can write

$$x^* = \sum_{i=1}^{R} \lambda_i^* v_i + \sum_{j=1}^{S} \mu_j^* r_j$$

where $\sum_{i=1}^{R} \lambda_i^* = 1$ and $\lambda_i^*, \mu_j^* \geq 0$. Then

$$c^T x^* = \sum_{i=1}^{R} \lambda_i^* c^T v_i + \sum_{j=1}^{S} \mu_j^* c^T r_j$$
**Possible Outcomes of an LP (2)**

**Proof:** Assume $P$ feasible and with optimal solution $x^*$. Let us see we can find a basic feasible solution as good as $x^*$.

By Minkowski-Weyl theorem, we can write

$$x^* = \sum_{i=1}^{r} \lambda_i^* v_i + \sum_{j=1}^{s} \mu_j^* r_j$$

where $\sum_{i=1}^{r} \lambda_i^* = 1$ and $\lambda_i^*, \mu_j^* \geq 0$. Then

$$c^T x^* = \sum_{i=1}^{r} \lambda_i^* c^T v_i + \sum_{j=1}^{s} \mu_j^* c^T r_j$$

- If there is $j$ such that $c^T r_j < 0$ then objective value can be decreased by taking $\mu_j^*$ larger. **Contradiction!**
Possible Outcomes of an LP (2)

**Proof:** Assume $P$ feasible and with optimal solution $x^*$. Let us see we can find a basic feasible solution as good as $x^*$.

By Minkowski-Weyl theorem, we can write

$$x^* = \sum_{i=1}^{r} \lambda_i^* v_i + \sum_{j=1}^{s} \mu_j^* r_j$$

where $\sum_{i=1}^{r} \lambda_i^* = 1$ and $\lambda_i^*, \mu_j^* \geq 0$. Then

$$c^T x^* = \sum_{i=1}^{r} \lambda_i^* c^T v_i + \sum_{j=1}^{s} \mu_j^* c^T r_j$$

- If there is $j$ such that $c^T r_j < 0$ then objective value can be decreased by taking $\mu_j^*$ larger. **Contradiction!**

- Otherwise $c^T r_j \geq 0$ for all $j$. Assume $c^T x^* < c^T v_i$ for all $i$.

$$c^T x^* \geq \sum_{i=1}^{r} \lambda_i^* c^T v_i > \sum_{i=1}^{r} \lambda_i^* c^T x^* = c^T x^* \sum_{i=1}^{r} \lambda_i^* = c^T x^*$$
Possible Outcomes of an LP (2)

Proof: Assume $P$ feasible and with optimal solution $x^*$. Let us see we can find a basic feasible solution as good as $x^*$. By Minkowski-Weyl theorem, we can write

$$x^* = \sum_{i=1}^{R} \lambda_i^* v_i + \sum_{j=1}^{S} \mu_j^* r_j$$

where $\sum_{i=1}^{R} \lambda_i^* = 1$ and $\lambda_i^*, \mu_j^* \geq 0$. Then

$$c^T x^* = \sum_{i=1}^{R} \lambda_i^* c^T v_i + \sum_{j=1}^{S} \mu_j^* c^T r_j$$

- If there is $j$ such that $c^T r_j < 0$ then objective value can be decreased by taking $\mu_j^*$ larger. **Contradiction!**

- Otherwise $c^T r_j \geq 0$ for all $j$. Assume $c^T x^* < c^T v_i$ for all $i$.

$$c^T x^* \geq \sum_{i=1}^{R} \lambda_i^* c^T v_i > \sum_{i=1}^{R} \lambda_i^* c^T x^* = c^T x^* \sum_{i=1}^{R} \lambda_i^* = c^T x^*$$

**Contradiction!** Thus there is $i$ such that $c^T x^* \geq c^T v_i$; in fact, $c^T x^* = c^T v_i$ by the optimality of $x^*$.