# Introduction: Combinatorial Problems

## Combinatorial Problem Solving (CPS)

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February 20, 2025

# Combinatorial Problems

- A combinatorial problem consists in finding, among a finite set of objects, one that satisfies a set of constraints

- Several variations:
  - Find one solution
  - Find all solutions
  - Find best solution according to an objective function

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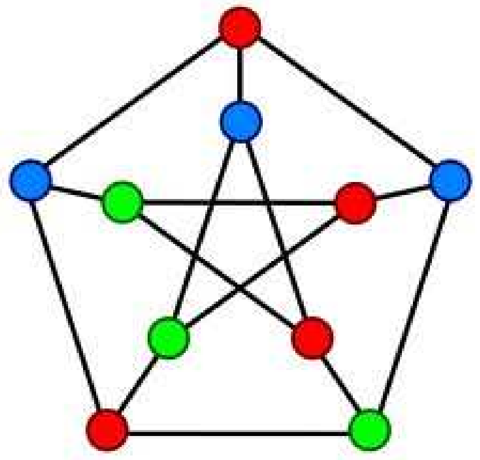
- Arises in:
  - Hardware verification
  - Circuit optimization
  - ...

# Examples (II): Graph Coloring

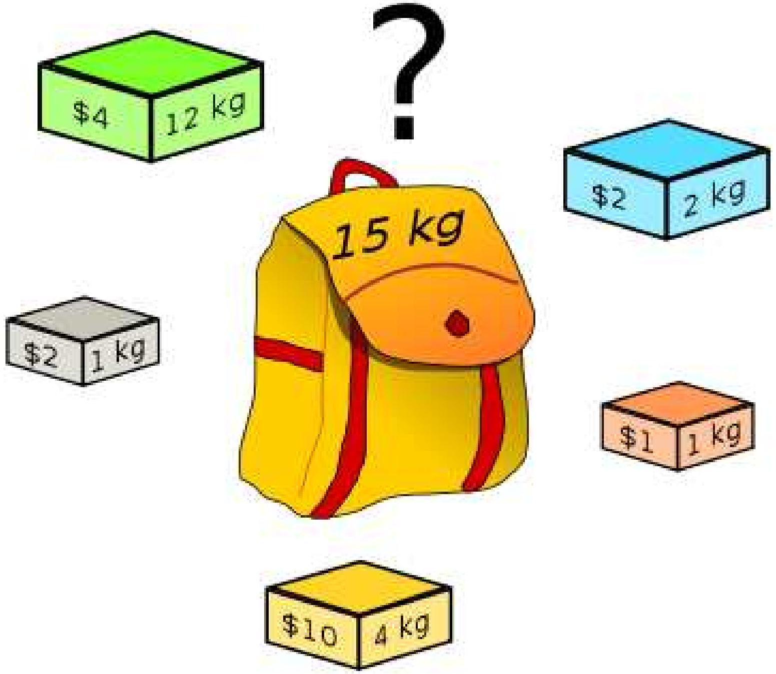
- Given a graph and a number of colors, can vertices be painted so that neighbors have different colors?



- Arises in:
  - Frequency assignment
  - Register allocation
  - ...

# Examples (III): Knapsack

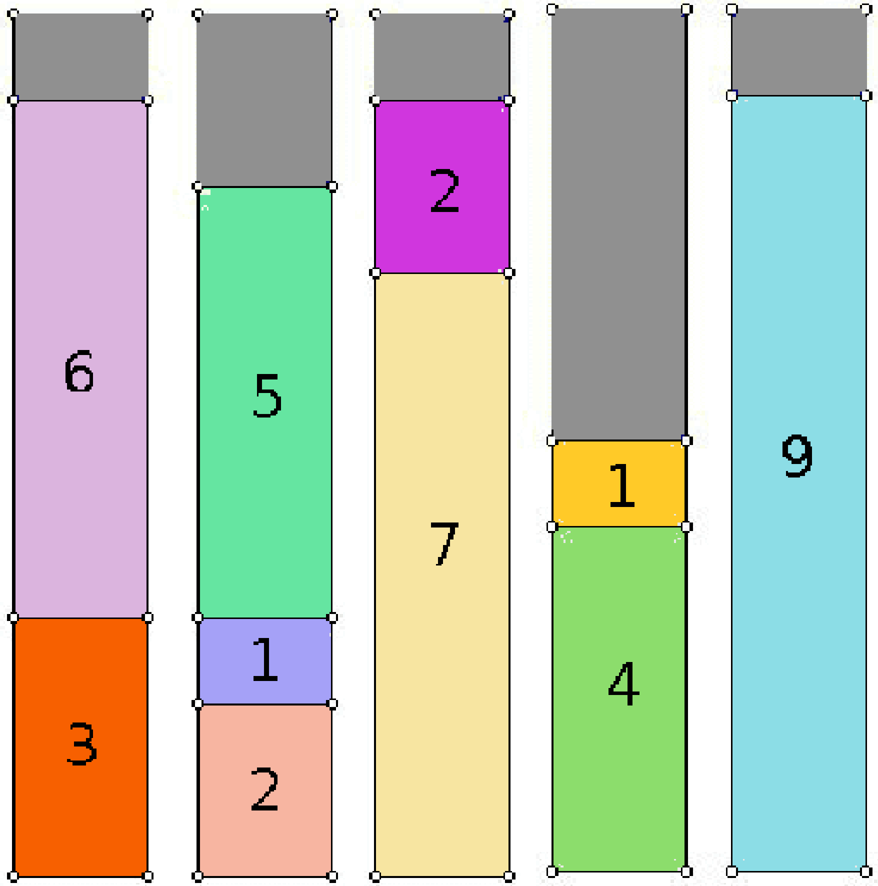
- Given $n$ items with weights $w_i$ and values $v_i$, a capacity $W$ and a number $V$, is there a subset $S$ of the items such that $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i \geq V$?




- Arises in:
  - Selection of capital investments
  - Cutting stock problems
  - ...

# Examples (IV): Bin Packing

- Given $n$ items with volumes $v_i$ and $k$ identical bins with capacity $V$, is it possible to place all items in bins?




- Arises in:
  - Logistics
  - ...

# A Note on Complexity

- All previous examples are NP-complete
  - No known polynomial algorithm (likely none exists)
  - Available algorithms have worst-case exp behavior: there will be small instances that are hard to solve
  - In real-world problems there is a lot of structure, which can hopefully be exploited

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- Other combinatorial problems solvable in P-time, e.g.
  - Bipartite matching: given a set of boys and girls and their compatibilities, can we marry all of them?
  - Shortest paths: given a graph and two vertices, which is the shortest way to go from one to the other?

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- Our focus will be on hard ($=$ NP-complete) problems

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- Pros of Declarative methodology
  - Specification of the problem is all we need to solve it!
  - Fast development and easy maintenance
  - Often better performance than ad-hoc techniques

# About CPS

- Problem solving frameworks
  - Constraint Programming (CP)
  - Linear Programming (LP)
  - Propositional Satisfiability (SAT)
- For each of these frameworks
  - Modeling techniques
  - Inner workings of solvers