Introduction to Constraint Programming

Combinatorial Problem Solving (CPS)

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Constraint Satisfaction Problem

- A constraint satisfaction problem (CSP) is a tuple \((X, D, C)\) where:
  
  - \(X = \{x_1, x_2, \ldots, x_n\}\) is the set of variables
  - \(D = \{d_1, d_2, \ldots, d_n\}\) is the set of domains
    \((d_i\) is a finite set of potential values for \(x_i\))
  - \(C = \{c_1, c_2, \ldots, c_m\}\) is a set of constraints

- For example: \(x, y, z \in \{0, 1\}, x + y = z\) is a CSP where:
  
  - Variables are: \(x, y, z\)
  - Domains are: \(d_x = d_y = d_z = \{0, 1\}\)
  - There is a single constraint: \(x + y = z\)
A constraint \( C \) is a pair \((S, R)\) where:

- \( S = (x_{i_1}, ..., x_{i_k}) \) are the variables of \( C \) (scope)
- \( R \subseteq d_{i_1} \times ... \times d_{i_k} \) are the tuples satisfying \( C \) (relation)

According to this definition: \( x + y = z \) in the CSP \( x, y, z \in \{0, 1\} \), \( x + y = z \) is short for

\[
((x, y, z), \{(0, 0, 0), (1, 0, 1), (0, 1, 1)\})
\]

A tuple \( \tau \in d_{i_1} \times ... \times d_{i_k} \) satisfies \( C \) iff \( \tau \in R \)

The arity of a constraint is the size of its scope

- Arity 1: unary constraint (usually embedded in domains)
- Arity 2: binary constraint
- Arity 3: ternary constraint
- ...

This is called the extensional representation of constraints
Constraints

- But constraints are usually described more compactly: intensional representation
- A constraint with scope $S$ is determined by a function

$$\prod_{x_i \in S} d_i \rightarrow \{\text{true}, \text{false}\}$$

- Satisfying tuples are exactly those that give true
- In the example: $x + y = z$
- Unless otherwise stated, we will assume that evaluating a constraint takes time linear in the arity
- This is usually, but not always, true
Solution

- Given a CSP with variables $X = \{x_1, x_2, \ldots, x_n\}$, domains $D = \{d_1, d_2, \ldots, d_n\}$ and constraints $C$, a solution is an assignment of values $(x_1 \mapsto \nu_1, \cdots, x_n \mapsto \nu_n)$ such that:
  - Domains are respected: $\nu_i \in d_i$
  - The assignment satisfies all constraints in $C$

- Solving a CSP consists in finding a solution to it

- Other related problems:
  - Finding all solutions
  - Finding a best solution wrt. an objective function (then we talk of a Constraint Optimization Problem)
Examples (I): Prop. Satisfiability

- Given a formula $F$ in propositional logic, is $F$ satisfiable?
- Variables are the atoms of the formula
- Variables have all domain \{true, false\}
- A single constraint: the evaluation of $F$ must be 1

- Let $F$ be $(p \lor q) \land (p \lor \neg q) \land (\neg p \lor q)$:
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- Domains are $d_p = d_q = \{\text{true, false}\}$
- Constraint is $(p \lor q) \land (p \lor \neg q) \land (\neg p \lor q) = \text{true}$
Examples (II): Graph Coloring

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- **Variables** are $\{c_v \mid v \in V\}$, the color for each vertex
- **Domains** are $\{1, 2, \ldots, K\}$, the available colors
- **Constraints** are: for each $(u, v) \in E$, $c_u \neq c_v$
Examples (III): Knapsack

- Given:
  - $n$ items with weights $w_i$ and values $v_i$
  - a capacity $W$
  - a number $V$,

is there a subset $S$ of the items s.t. $\sum_{i \in S} w_i \leq W$ and $\sum_{i \in S} v_i \geq V$?
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- **Variables:** $n$ variables $x_i$ meaning “item $i$ is selected”
- **Domains:** $d_i = \{0, 1\}$
- **Constraints:** $\sum_{i=1}^{n} w_i x_i \leq W$, $\sum_{i=1}^{n} v_i x_i \geq V$
Theorem. Solving a CSP is an NP-complete problem

Proof:

- It is in NP, because one can check a solution in polynomial time
- It is NP-hard, as there is a reduction e.g. from Prop. Satisfiability (which is known to be NP-complete)

For any CSP, there are instances that require exp time
Can we solve real life instances in reasonable time?
Constraint Programming

- **Constraint programming (CP)** is a general framework for modeling and solving CSP's:
  
  - Offers the user many kinds of constraints, which makes modeling easy and natural

    Check out the Global Constraint Catalogue at https://sofdem.github.io/gccat/gccat/sec5.html with more than 400 different types of constraints!

  - Provides solving engines for those constraints (CP toolkits: in this course, Gecode http://www.gecode.org)
Generate and Test

■ How can we solve CSP’s?
■ 1st naïf approach: Generate and Test (aka Brute Force)
  ◆ Generate all possible candidate solutions
    (assignments of values from domains to variables)
  ◆ Test whether any of these is a true solution indeed
Generate and Test

- Example: Queens Problem. Given $n \geq 4$, put $n$ queens on an $n \times n$ chessboard so that they don’t attack each other.
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Wlog, we can place one queen per row so that no two are in the same column or diagonal.

- **Variables:** $c_i$, column of the queen of row $i$
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Basic Backtracking

- Generate and Test is very inefficient
- 2nd approach to solving CSP’s: Basic Backtracking
- The algorithm maintains a partial assignment that is consistent with the constraints whose variables are all assigned:
  - Start with an empty assignment
  - At each step choose a var and a value in its domain
  - Whenever we detect a partial assignment that cannot be extended to a solution, backtrack: undo last decision
Basic Backtracking

- We can solve the problem by calling `backtrack(x1)`:

```plaintext
function backtrack(variable X) returns bool
    for all a in domain(X) do
        val(X) := a
        if compatible(X, assigned)
            assigned := assigned ∪ {X}
        if no next(X) then return TRUE
        else if backtrack(next(X)) then return TRUE
        else assigned := assigned - {X}
    return FALSE

function compatible(variable X, set A) returns bool
    for all constraint C with scope in A∪{X} and not in A do
        // Let A be {Y1, ..., Yn}
        if (val(X), val(Y1),..., val(Yn)) don’t satisfy C then
            return FALSE
    return TRUE
```
Basic Backtracking

Using Backtracking to Solve 4-Queens

The orange squares represent the placement of queens, based off of a diagram in [5]
Basic Backtracking

The set of all possible partial assignments forms a search tree:

- The root corresponds to the empty assignment
- Each edge corresponds to assigning a value to a variable
- For each node, there are as many children as values in the domain of the chosen variable
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- Generate and Test corresponds to visiting each of the leaves until a solution is found
- Complexity: $O(m^n \cdot e \cdot r)$

- $n =$ no. of variables
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- **Basic Backtracking** performs a depth-first traversal
- Complexity: the same, as in the worst case we need to visit all leaves
- But in practice it works much better than Generate and Test
Basic Backtracking

- Problems with backtracking
  - Inconsistencies may be found late, after a lot of useless work

If $x_1 \mapsto a$ is incompatible with $x_n \mapsto$ anything, then BT explores the subtree rooted at $x_1 \mapsto a$ (if $x_1$ can take $m$ values, this subtree is $\frac{1}{m}$ of the whole search tree!) to realize that no solution can be found
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    to realize that no solution can be found

  - The right backtracking point may not be the last decision
Basic Backtracking
Basic Backtracking

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The diagram shows a grid with letters 'Q', 'X', and 'X' arranged in a specific pattern, indicating a backtracking process.
Propagation

- CP approach: prune search tree a priori by removing values from the domains that can’t appear in any solution

  while (solution not found) do
    assign values to some of the variables
    propagate with constraints to prune other domains
    if (found inconsistency) undo last decision

- Smaller search tree, at the cost of more time per node
- There exist different kinds of propagation with different tradeoffs between pruning power and cost in time
## Propagation

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Propagation

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