Combinatorial Problem Solving (CPS)

In this session we revisit the Queens Problem: given $n \geq 4$, place $n$ queens on an $n \times n$ chess board, so that they do not attack each other (a queen attacks pieces on the same row, on the same column, and on the same diagonals).

1. Write a program `queens` that, given $n$ from the command line, outputs a solution as follows:

```
$ queens 8
.......X
...X....
X.......  
..X.....  
.....X..  
.X......  
......X.  
.....X...
```

The program should internally write a CNF that encodes the problem, feed it to the SAT solver `kissat` and then interpret its output.

Which is the least $n$ for which your program takes more than 5 secs. in your computer? How long does it take to solve the queens problem for this $n$?

Use the `run` script below, which is also available at the website of CPS (http://www.cs.upc.edu/~erodri/cps.html):

```
#!/bin/bash

for i in $(seq 4 1000); do
echo "i:$i"
time ./$1 $i
done
```

**Hint:** Given a set of variables $\{x_1,\ldots,x_n\}$, you can encode that at most one of them can be true by adding the clauses: $\neg x_i \lor \neg x_j$ for $1 \leq i < j \leq n$.

2. Two very common kinds of constraints are:

- **At Least One (ALO)** constraints: among a set of propositional variables, at least one of them has to be true.
- **At Most One (AMO)** constraints: among a set of propositional variables, at most one of them has to be true.

In the queens problem we have ALO and AMO constraints: for example, the constraint that states that at each row (or column) there is exactly one queen is the conjunction of an ALO and an AMO constraint. There are also other AMO constraints for the diagonals.

ALO constraints admit a very good encoding into SAT: each such constraint corresponds to a single clause. On the other hand, for AMO constraints there exist several encodings. The one suggested in the hint above is usually called the quadratic encoding of AMO.

Another encoding for AMO is the so-called logarithmic one. Let us consider the constraint $x_0 + \ldots + x_{n-1} \leq 1$ and define $m = \lceil \log_2 n \rceil$. The logarithmic encoding consists in adding the $m$ variables $y_0, y_1, \ldots, y_{m-1}$ and, for all $i$ and $j$ with $0 \leq i < n$, $0 \leq j < m$, the clause $x_i \lor y_j$.
if the $j$-th digit in the binary representation of $i$ is 1, or $\overline{x} \lor \overline{y}$ otherwise. This needs $O(\log n)$ auxiliary variables (hence the name of the encoding) and $O(n \log n)$ clauses.

Write a program `queens` that, given $n$ from the command line, solves the queens problem with the logarithmic encoding for the AMO constraints.

Which is the least $n$ for which your program takes more than 5 secs. in your computer?

3. Yet another encoding for AMO is the Heule encoding. If $n \leq 3$, the encoding is the quadratic encoding. If $n > 4$, it consists in introducing an auxiliary variable $y$ and encoding (recursively with the Heule encoding) the AMO constraints $x_1 + x_2 + y \leq 1$ and $x_3 + x_4 + \cdots + x_n + \overline{y} \leq 1$. This encoding needs about $O(n)$ auxiliary variables and $O(n)$ clauses.

Write a program `queens` that, given $n$ from the command line, solves the queens problem with the Heule encoding for the AMO constraints.

Which is the least $n$ for which your program takes more than 5 secs. in your computer?

4. By generalizing the idea of the Heule encoding, given an AMO constraint $x_1 + \ldots + x_n \leq 1$, we may introduce an auxiliary variable $y$ and decompose it into two AMO constraints $x_1 + \ldots + x_k + y \leq 1$ and $\overline{y} + x_{k+1} + x_{k+2} + \cdots + x_n \leq 1$ for a certain $1 \leq k \leq n$. Then we can choose whether to encode each of these new AMO constraints with the quadratic encoding, the logarithmic one, or decompose recursively. This leads to a family of hybrid encodings with different numbers of auxiliary variables `num_aux_vars` and of clauses `num_clauses`.

Write a program that, for a fixed $0 \leq \omega \leq 1$, finds an encoding for AMO constraints that minimizes

$$\omega \cdot \text{num\_aux\_vars} + (1 - \omega) \cdot \text{num\_clauses}.$$ 

Then write a program that, for a fixed $\omega$, solves the $n$-queens problem with the $\omega$-optimal encoding for the AMO constraints.

For the best possible value of $\omega$, which is the least $n$ for which your program takes more than 5 secs. in your computer?